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Independence for Characterizing Axioms of the Pre–Nucleolus

by

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Abstract

The pre–nucleolus for cooperative side–payment games with finitely many players in an infinite universe can be characterized by single valuedness, covariance under strategic equivalence, the reduced game property, and anonymity ([10]). In [6] it is shown that the last axiom can be replaced by the equal treatment property. In the present paper the logical independence of both systems of axioms is proved and it is deduced that the assumption on the cardinality of the universe of players can be dropped as a prerequisite of neither Sobolev's nor Orshan's theorem.

1. *Introduction, Notation, and Well-Known Results*

The paper is organized as follows. In this section the necessary notation is given and the well-known independence of three of a system of four axioms characterizing the pre-nucleolus (see [10]) is recalled.

Section 2 provides a proof for the independence of the remaining property.

In Section 3 it turns out that Sobolev's theorem (Theorem 1.6) remains valid for a finite universe U of players, if and only if the cardinality of this set does not exceed three. Finally it is shown that the second system of axioms which arises from the first one by interchanging anonymity and the equal treatment property (see [6]) is no longer equivalent, if U has any finite cardinality larger than three.

A **cooperative game with transferable utility** – a **game** – is a pair (N,v), where N is a finite nonvoid set and

$$v : \mathcal{P}(N) \to \mathbb{R},\ v(\emptyset) = 0$$

is a mapping. Here $\mathcal{P}(N) = \{S \subseteq N\}$ is the set of **coalitions** of (N,v).
If (N,v) is a game, then N is the **grand** coalition or the set of **players** and v is called **characteristic** (or **coalition**) **function** of (N,v). Since the nature of N is determined by the characteristic function, v is called game as well.
The set of **feasible payoff vectors** of a game (N,v) is denoted

$$X^*(N,v) := X^*(v) := \{x \in \mathbb{R}^N \mid x(N) \leq v(N)\},$$

whereas

$$X(N,v) := X(v) := \{x \in \mathbb{R}^N \mid x(N) = v(N)\}$$

is the set of **pre-imputations** of (N,v) (also called set of **Pareto optimal** feasible payoffs of (N,v)). Here

$$x(S) := \sum_{i \in S} x_i \quad (x(\emptyset) = 0)$$

for each $x \in \mathbb{R}^N$ and $S \subseteq N$.
A **solution concept** σ **on** a set Γ of games is a mapping

$$\sigma : \Gamma \to \bigcup_{v \in \Gamma} \mathcal{P}(X^*(v)),\ \sigma(v) \subseteq X^*(v).$$

If $\bar{\Gamma}$ is a subset of Γ, then the canonical restriction of a solution concept σ on Γ is a solution concept on $\bar{\Gamma}$. We say that σ is a solution concept on $\bar{\Gamma}$, too. If Γ is not specified, then σ is a solution concept on every set of games.

Definition 1.1:

(i) For a set U let $\Gamma_U = \{(N,v) \mid N \subseteq U\}$ denote the set of games with player set contained in U.

(ii) Let (N,v) be a game, $x \in \mathbb{R}^N$, and $\bar{S}$ be a nonvoid coalition of N. The game $(\bar{S}, v^{\bar{S},x})$, where

$$v^{\bar{S},x}(S) = \begin{cases} v(N) - x(N \setminus \bar{S}), & \text{if } S = \bar{S} \\ 0, & \text{if } S = \emptyset, \\ \max\ \{v(S \cup Q) - x(Q) \mid Q \subseteq N \setminus \bar{S}\}, & \text{otherwise} \end{cases}$$

is the **reduced game** of v w.r.t. x and $\bar{S}$ (see [1]).

Some more notation will be needed. Let (N,v) be a game and $x \in \mathbb{R}^N$. The **excess** of a coalition $S \subseteq N$ **at** x is the real number

$$e(S,x,v) := e(S,x) := v(S) - x(S).$$

The nucleolus of a game was introduced by Schmeidler ([9]). The **nucleolus** of v w.r.t. X, where $X \subseteq \mathbb{R}^N$, is the set

$$\mathcal{N}(X,v) := \{x \in X \mid \Theta(x,v) \leq_{lex} \Theta(y,v) \text{ for all } y \in X\},$$

where $\Theta(x,v)$ is the vector of all excesses at x w.r.t. v in a non-increasing order.
Schmeidler ([9]) formulated and proved the following

Lemma 1.2: Let (N,v) be a game and $X \subseteq \mathbb{R}^N$.

(i) If X is a nonvoid compact set, then $\mathcal{N}(X,v)$ is nonvoid.

(ii) If X is convex, then $\mathcal{N}(X,v)$ contains at most one vector.

(iii) If X is a nonvoid, closed convex subset of $X^*(v)$, then $\mathcal{N}(X,v)$ is a singleton.

The **pre-nucleolus** of (N,v) is defined to be the nucleolus w.r.t. the set of feasible payoff vectors and denoted $\mathcal{PN}(v)$, i.e.,

$$\mathcal{PN}(v) = \mathcal{N}(X^*(v),v).$$

By Lemma 1.2(iii) the pre-nucleolus of a game is a singleton, and, clearly, the pre-nucleolus is Pareto optimal by definition.
The unique element $\nu(v)$ of $\mathcal{PN}(v)$ is again called pre-nucleolus (point).
Now Kohlberg's ([2]) characterization of the pre-nucleolus by balanced collections of coalitions is recalled. It should be remarked that his assumptions on the considered games can be deleted without destroying the proofs (see [8]). Some notation is needed.

Let (N,v) be a game, $\bar{x} \in \mathbb{R}^N$, and X be a finite subset of $\mathbb{R}^N$. The set X is balanced, if there are real numbers $\delta_x > 0$ for all $x \in X$ such that

$$\sum_{x \in X} \delta_x x = 1_N,$$

where 1_S is the indicator function of a coalition S, considered as vector of $\mathbb{R}^{|N|}$. In this case the sequence $(\delta_x)_{x \in X}$ is a sequence of balancing coefficients for X. A subset $\mathscr{D}$ of $\mathscr{P}(N)$ is called **balanced**, if $\{1_S \mid S \in \mathscr{D}\}$ is balanced. For each $\alpha \in \mathbb{R}$ let

$$\mathscr{D}(\bar{x},\alpha,v) := \{S \subseteq N \mid e(S,\bar{x},v) \geq \alpha\}$$

denote the set of coalitions with excess not less than α.

Lemma 1.3 ([2]): Let (N,v) be a game and $x \in X(v)$ be a pre-imputation. Then x coincides with the pre-nucleolus of v, iff $\mathscr{D}(x,\alpha,v)$ is balanced for each α with $\mathscr{D}(x,\alpha,v) \neq \emptyset$.

Remark 1.4: Let N be a finite set.

If $\mathscr{D}$ is a balanced collection of coalitions in N and $\bar{S} \subseteq N$ is a coalition in the linear span of $\mathscr{D}$ – i.e. $\sum_{T \in \mathscr{D}} \alpha_T 1_T = 1_{\bar{S}}$ for some $\alpha_T \in \mathbb{R}$, $T \in \mathscr{D}$ – then $\mathscr{D} \cup \{\bar{S}\}$ is balanced.

Some convenient and well-known properties of a solution concept σ on a set Γ of games are as follows.

Definition 1.5: Let σ be a solution concept on a set Γ of games.

(i) σ is **single valued** (satisfies SIVA), if $|\sigma(v)| = 1$ for $v \in \Gamma$.

(ii) σ is **covariant under strategic equivalence** (satisfies COV), if for (N,v), $(N,w) \in \Gamma$ with $w = \alpha v + \beta$ for some $\alpha > 0$, $\beta \in \mathbb{R}^N$

$$\sigma(N,w) = \alpha\,\sigma(N,v) + \beta$$

holds. The games v and w are called **strategically equivalent**.

(iii) A solution concept σ on a set Γ of games satisfies the **reduced game property** (RGP) if $(N,v) \in \Gamma$, $x \in \sigma(v)$, $\emptyset \neq \bar{S} \neq N$ implies $(\bar{S},v^{\bar{S},x}) \in \Gamma$ and $x_{|\bar{S}} \in \sigma(\bar{S},v^{\bar{S},x})$.

(iv) σ is **anonymous** (satisfies AN), if for each $(N,v) \in \Gamma$ and each injective mapping $\tau : N \to N'$ with $(\tau N, \tau v) \in \Gamma$

$$\sigma(\tau N, \tau v) = \tau(\sigma(N,v))$$

holds (where $(\tau v)(T) = v(\tau^{-1}(T))$, $\tau_j(x) = x_{\tau^{-1}j}$ $(x \in \mathbb{R}^N, j \in \tau N, T \subseteq \tau N)$).

In this case v and τv are equivalent games.

(v) σ satisfies the **equal treatment property** (ETP), if for $x \in \sigma(v)$, $v \in \Gamma$, and interchangeable players i,j (i.e. $v(S \cup \{i\}) = v(S \cup \{j\})$ for $S \subseteq N \setminus \{i,j\}$) $x_i = x_j$ holds.

(vi) σ satisfies **non emptiness** (NE), if $\sigma(v) \neq \emptyset$ for $v \in \Gamma$.

(vii) σ is **Pareto optimal** (satisfies PO), if $\sigma(v) \subseteq X(v)$ for $v \in \Gamma$.

(viii) σ satisfies the **converse reduced game property** (CRGP), if for $(N,v) \in \Gamma$, $x \in X^*(v)$ the following is true: $(S,v^{S,x}) \in \Gamma$ for $S \subseteq N$ with $|S| = 2$ and if $x_{|S} \in \sigma(v^{S,x})$ for all $S \subseteq N$ with $|S| = 2$, then $x \in \sigma(v)$.

Theorem 1.6 ([10]): If U is an infinite set, then there exists a unique solution concept on Γ_U satisfying SIVA, COV, RGP, AN; and this is the pre-nucleolus.

Theorem 1.7 ([6]): If U is an infinite set, then there exists a unique solution concept on Γ_U satisfying SIVA, COV, RGP, ETP; and this is the pre-nucleolus.

Next three examples of solution concepts are given which show the independence of the axioms in the theorems with the exception of AN and ETP respectively.

The pre-kernel satisfies COV, RGP (see [7]), ETP, AN, but clearly is not single valued as four-player examples show.

The "equal treatment solution", i.e. $\sigma(v) = \{x \in X(v) \mid x_i = x_j \text{ for } i,j \in N\}$, satisfies SIVA, RGP, AN, ETP, but not COV.

The Shapley value satisfies SIVA, COV, AN, ETP, but not RGP.

2. *Independence of Anonymity or the Equal Treatment Property*

During this section let U be an infinite set. To prove the main result of this paper the following definition and lemma is useful.

Definition 2.1: For each $\epsilon \in \mathbb{R}$ the solution $\tilde{\sigma}^\epsilon$ on Γ_U is given by

$$\tilde{\sigma}^\epsilon(N,v) = \{x \in X(N,v) \mid e(S,x,v) \le \max\{\epsilon, e(S,\nu(v),v)\} \text{ for } S \subseteq N\}$$

for $(N,v) \in \Gamma_U$.

Note that $\tilde{\sigma}^\epsilon(N,v)$ is a nonvoid convex polytope containing the pre–nucleolus. If, e.g., ϵ is a lower bound for the excesses of non–trivial coalitions (the grand and empty are the trivial coalitions) w.r.t. $\nu(v)$, then $\tilde{\sigma}^\epsilon(N,v)$ coincides with the pre–nucleolus. On the other hand, if the ϵ–core of v – as defined in [5] – is nonvoid, then these solution concepts coincide. Therefore $\tilde{\sigma}^\epsilon(N,v)$ can be vaguely interpreted as consisting of all vectors, which successively minimize excesses and the number of coalitions attaining them as long as the excesses are larger than ϵ. Clearly $\tilde{\sigma}^\epsilon$ satisfies NE, AN, but not ETP. This solution concept also satisfies RGP as stated in the following

Lemma 2.2: Let $\epsilon \in \mathbb{R}$.

(i) $\tilde{\sigma}^\epsilon$ satisfies RGP.

(ii) If $x \in \tilde{\sigma}^\epsilon(N,v)$ for a game (N,v), $y \in \tilde{\sigma}^\epsilon(S,v^{S,x})$ for some reduced game $(S,v^{S,x})$ of (N,v) w.r.t. x and if $z \in \mathbb{R}^N$ is defined by $z_i = \begin{cases} x_i, & i \in N\setminus S \\ y_i, & i \in S \end{cases}$, then $z \in \tilde{\sigma}^\epsilon(N,v)$.

Proof:

(i): Let $x \in X(N,v)$, $\emptyset \neq S \subsetneq N$, $y \in \tilde{\sigma}^\epsilon(S,v^{S,x})$, and z be defined as in (ii). It suffices to show that

$$x(T) < y(T) \tag{1}$$

and

$$e := e(T,x_{|S},v^{S,x}) > \epsilon \tag{2}$$

for some $\emptyset \subsetneq T \subsetneq S$ implies $x \notin \tilde{\sigma}^\epsilon(N,v)$. Let (1),(2) be satisfied by T and e be maximal. Then, by definition of $\tilde{\sigma}^\epsilon$ we come up with $x(R)=y(R)$ for all $R \subseteq S$ with $e(R,y,v^{S,x}) > e$. Let $P \subseteq N$ with $P \cap S \neq \emptyset$, $S \not\subseteq P$, and $e(P,z,v) > e$. Then $e(P \cap S,y,v^{S,x}) \ge e(P,z,v) > e$, hence $z(P) = x(P)$. Let $v^{S,x}(T) = v(T \cup Q) - x(Q)$

for some $Q \subseteq N\setminus S$. Then

$$\epsilon < e(T \cup Q,x,v) = e > e(T \cup Q,z,v),$$

hence $x \notin \tilde{\sigma}^\epsilon(N,v)$ by definition.

(ii): Using (i) it is obvious that

$$e(T,x_{|S},v^{S,x}) = e(T,y,v^{S,x})\ .$$

for $T \subseteq S$ with $e(T,y,v^{S,x}) > \epsilon$. The proof is finished by again noticing that

$$e(R \cap S,x_{|S},v^{S,x}) \ge e(R,x,v)$$

holds true for $R \subseteq N$ with $R \cap S \neq \emptyset$, $S \not\subseteq R$. q.e.d.

Let $\hat{U}$ be a completely ordered subset of U with order relation $\le$.

Theorem 2.3: If the cardinality of $\hat{U}$ is at least two, then there is a solution concept σ on Γ_U, satisfying SIVA, COV, RGP, and not ETP.

Proof:

Let (N,v) be a game in Γ_U and $\epsilon \in \mathbb{R}$. Define

$$\sigma(N,v) = \mathcal{N}(X,v),$$

where

$$X = \{x \in \tilde{\sigma}^\epsilon(N,v) \mid x_{|\hat{U}\cap N} \le_{lex} y_{|\hat{U}\cap N} \text{ for } y \in \tilde{\sigma}^\epsilon(N,v)\}.$$

Here $\le_{lex}$ is the lexicographic order induced by the relation $\le$ of $\hat{U}$ on each subset of $\hat{U}$.

Since $\tilde{\sigma}^\epsilon(N,v)$ is a convex compact polyhedron, the set X inherits this property. Lemma 1.2 implies the single valuedness of σ. Moreover, σ satisfies COV. Indeed, with the observation that "reducing commutes with strategic equivalence", i.e.

$$(\alpha v + \beta)^{S,\alpha x+\beta} = \alpha v^{S,x} + \beta_{|S}$$

for $\alpha > 0$, $\beta \in \mathbb{R}^N$, $\emptyset \neq S \neq N$, $x \in X(N,v)$, a proof of this property is straightforward.

Claim: σ satisfies RGP.

Let $x \in X$, $\emptyset \neq S \neq N$, $y \in \sigma(S,v^{S,x})$, and let z be defined according to Lemma 2.2(ii).

Applying this lemma we conclude

$$x_{|\tilde{U}\cap N} \underset{lex}{\leq} z_{|\tilde{U}\cap N} \text{ and } x_{|\tilde{U}\cap S} \underset{lex}{\geq} y_{|\tilde{U}\cap N},$$

hence

$$x_{|\tilde{U}\cap N} = z_{|\tilde{U}\cap N}.$$

It suffices to show that $x(T) < y(T)$ for some $\emptyset \neq T \subsetneq S$ implies $x \notin \sigma(N,v)$. This can be done completely analogous to the proof of Lemma 2.2(i) by dropping ϵ, whenever ϵ occurs.

To show that σ does not satisfy AN or ETP, an example is presented. Assume w.l.o.g. $\{1,2\} \subseteq \tilde{U}$ and $1 \leq 2$ w.r.t. the order relation on $\tilde{U}$. Define $(\{1,2\},v)$ by

$$v(S) = \begin{cases} \epsilon-1, & \text{if } |S| = 1 \\ 0, & \text{otherwise} \end{cases}.$$

Then $\tilde{\sigma}^{\epsilon}(v) =$ convex hull $\{(-1,1),(1,-1)\}$, hence $\sigma(v) = \{(-1,1)\}$. Clearly both players are interchangeable, thus, e.g., $\nu(v) = (0,0) \notin \sigma(v)$. q.e.d.

3. *Sobolev's Theorem for Finite U*

Let U be a finite set during this section. The following examples show that Theorems 1.6 and 1.7 are no longer valid in case the cardinality of U strictly exceeds three. Finally it is proved that the theorems remain true in case $|U| < 4$.

Example 3.1:

(i) Let $U = \{1,2,...,n\}$ for some $n > 3$ and $n \in \mathbb{N}$. Let (U,v) be the weighted majority game defined by the representation (λ,m), where

$$m = (\underbrace{2,2,\ldots,2}_{n-2 \text{ times}},1,1)\ ,\ \lambda = 2n-5; \text{ i.e.}$$

$$v(S) = \begin{cases} 1, & \text{if } m(S) \geq \lambda \\ 0, & \text{otherwise} \end{cases}.$$

Put $x = (1,1,...,1,0,0)/n-2 \in \mathbb{R}^U$ and define

$$\sigma(w) = \begin{cases} \{\tau(\alpha x+\beta)\}, & \text{if } w = \tau(\alpha v+\beta) \text{ is equivalent to a game strategically equivalent to v.} \\ \mathcal{PN}(w), & \text{otherwise} \end{cases}$$

Clearly, σ satisfies SIVA, COV, AN, ETP by definition, and no game with player set U can occur as a pure reduced game of any game with player set contained in U. Let $z \in \sigma(w)$ for some game equivalent to a game strategically equivalent to v. It suffices to verify that $z_{|S} = \nu(w^{S,x})$ for the reduced game of w w.r.t. S and x. Since reducing commutes with strategic equivalence (see the proof of Theorem 2.3) and with equivalence (i.e. $(\tau v)^{\tau S,\tau x} = (\tau_{|S})v^{S,x}$), it suffices to show that $x_{|S}$ coincides with the pre-nucleolus of $v^{S,x}$. Using the "transitivity of reducing", i.e. $w^{T,y} = (w^{S,y})^{T,y_{|S}}$ for $\emptyset \subsetneq T \subseteq S \subseteq N$, for each game $(N,w) \in \Gamma_U$ and $x \in X(w)$, it suffices to verify the coincidence with the pre-nucleolus for reduced games w.r.t. coalitions of cardinality $|U|-1$. Players $1,...,n-2$ and players $n-1,n$ respectively are interchangeable, thus it suffices to restrict the attention to coalitions $S \in \{\{1,...,n-1\},\{1,...,n-3,n-1,n\}\}$. If S is the first coalition, i.e. $S = \{1,...,n-1\}$, then

$$v^{S,x}(T) = \begin{cases} 1, & \text{if } |T\cap\{1,...,n-2\}| \geq n-3 \\ 0, & \text{otherwise} \end{cases}.$$

Therefore n–1 is a null–player in $v^{S,x}$ and 1,...,n–2 are interchangeable. It is well–known that $\nu(v^{S,x})$ treats all interchangeable players equally (by the equal treatment property) and assigns 0 to null–players, hence $x_{|S} = \nu(v^{S,x})$ in this case.

In the other case ($S = \{1,...,n-3,n-1,n\}$), the characteristic function of the reduced game can be computed as

$$v^{S,x}(T)=\begin{cases}1 & \text{,if } |T| \geq n\text{-}2 \text{ and } \{1,\dots,n-3\} \subseteq T \\ \frac{n-3}{n-2} & \text{,if } T = \{1,\dots,n-3\} \text{ or } |T\cap\{1,\dots,n-3\}|=n-4<|T|. \\ 0 & \text{,otherwise}\end{cases}$$

Coalitions T with cardinality n–2 and $\{1,...,n-3\} \subseteq T$ and such coalitions R with the same cardinality and $\{n-1,n\} \subseteq R$ have maximal excess w.r.t. $x_{|S}$, span the Euclidean space, and form a balanced collection. Lemma 1.3 and Remark 1.4 imply the coincidence of $x_{|S}$ and $\nu(v^{S,x})$ in this case.

Therefore σ is a solution concept satisfying SIVA, COV, RGP, AN, ETP, which does not coincide with the pre–nucleolus. The last statement is true, since it can easily be verified that the pre–nucleolus of v coincides – up to normalization – with the vector m of weights.

In (ii) a variant of this solution concept is presented, which satisfies all axioms with the exception of AN.

(ii) Under the prerequisites of Example 3.1(i) define

$$\sigma(w) = \begin{cases}\{\alpha x+\beta\}, & \text{if } w = \alpha v+\beta \text{ is strategically equivalent to } v. \\ \mathcal{PN}(w), & \text{otherwise}\end{cases}$$

Clearly, this solution concept σ satisfies SIVA, COV, RGP, and ETP as before. Applying a non–trivial permutation τ of U to (U,v) directly shows that σ does not satisfy AN.

Theorem 3.2: Theorems 1.6 and 1.7 remain valid, if and only if $|U| \leq 3$.

Proof:

By Example 3.1 it suffices to show that the theorems are true, if $|U| < 4$.

It is well–known that SIVA, COV, RGP imply PO (see [8], [10]). The axioms SIVA and AN clearly imply ETP and NE. Moreover, if σ satisfies PO, NE, ETP, COV on Γ_U, then σ is a standard solution (see [7], [8]), i.e. $\sigma(N,v) = \mathcal{PN}(N,v)$ for a player set N of

cardinality two. These considerations complete the proof in case $|U| < 3$.

Assume $|U| = 3$ from now on. Let σ be a solution concept on Γ_U satisfying SIVA, COV, AN, and ETP. If x is the unique member of $\sigma(v)$ for some game (U,v), then $x_{|S}$ coincides with the pre–nucleolus for subsets S of U with cardinality two. The pre–kernel – for the definition [1] and [4] are referred to – is a standard solution. Therefore $x_{|S}$ is the unique element of the pre–kernel of the corresponding reduced game. Let $\mathcal{PK}(N,w)$ denote the pre–kernel of a game (N,w). The pre–kernel satisfies CRGP (see [7]), thus $x \in \mathcal{PK}(U,v)$. It is sufficient to show that the pre–kernel of a three–player game is a singleton. In order to complete the proof some notation is needed.

A subset $\mathcal{D}$ of coalitions of a finite set N is separating, if the existence of $T \in \mathcal{D}$ with $j \notin T \ni i$ for some $i,j \in N$ always implies the existence of $R \in \mathcal{D}$ with $i \notin R \ni j$. If $y \in \mathcal{PK}(N,w)$, then it is well–known that $\mathcal{D}(y,\alpha,w)$ is separating for $\alpha \in \mathbb{R}$ (see [3],[4]). Using Kohlberg's result (Section 1 is referred to), it is sufficient to verify that each non–trivial separating set of coalitions is balanced in case $|N| = 3$.

Therefore let $\mathcal{D}$ be a separating set of coalitions in U. Then we can assume w.l.o.g. that $\emptyset, U \notin \mathcal{D} \neq \emptyset$. First it can be observed that

$$\bigcap_{S\in\mathcal{D}} S = \emptyset \text{ and } \bigcup_{S\in\mathcal{D}} S = U. \qquad (1)$$

The set $\mathcal{D}$ contains at least two coalitions and if this is exactly true, then $\mathcal{D}$ consists of a coalition and its complement (by (1)), hence $\mathcal{D}$ is balanced in this case. Therefore it can be assumed that $\mathcal{D}$ contains at least three different coalitions. The following cases can be distinguished.

(i) All two–player or all one–player coalitions are elements of $\mathcal{D}$. These coalitions clearly form a balanced basis of $\mathbb{R}^U$, hence $\mathcal{D}$ is balanced by Remark 1.4.

(ii) Exactly two two–player coalitions are elements of $\mathcal{D}$, let us say (w.l.o.g.) $S^1 = \{1,2\}$ and $S^2 = \{1,3\}$, where – again w.l.o.g. – $U = \{1,2,3\}$. The separating property then directly implies $S^3 = \{2\}$ and $S^4 = \{3\}$ belong to $\mathcal{D}$. Now, $\{S^1,S^2,S^3,S^4\}$ is a balanced subset of $\mathcal{D}$ and spans the Euclidean space $\mathbb{R}^U$. Remark 1.4 directly yields the balancedness of $\mathcal{D}$.

(iii) The case $\mathcal{D}$ containing exactly two one–person coalitions can be treated analogously to (ii) by interchanging the roles of S^1,S^2 and S^3,S^4. q.e.d.

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