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THE MODIFIED NUCLEOLUS AS CANONICAL REPRESENTATION OF WEIGHTED MAJORITY GAMES

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A new solution concept for cooperative transferable utility games is introduced, which is strongly related to the nucleolus and therefore called modified nucleolus. It has many properties in common with the prenucleolus and can be considered as the canonical restriction of the prenucleolus of a certain replicated game. For weighted majority games this solution concept induces a representation of the game. In the special case of weighted majority constant-sum games and and homogeneous games respectively the nucleolus and the minimal integer representation respectively are adequate candidates for a canonical representation (see Peleg 1968 and Ostman 1987a). Fortunately the modified nucleolus coincides with the just mentioned solutions in these special cases and can, therefore, be seen as a canonical representation in the general weighted majority case.

0. Introduction. A new solution concept, the modified nucleolus, for cooperative side payment games with a finite set of players is proposed in this paper. The expression "modified nucleolus" refers to the strong relationship of this solution to the (pre)nucleolus introduced by Schmeidler (1966).

An imputation belongs to the nucleolus of a game, if it successively minimizes the maximal excesses, i.e. the differences of the worths of coalitions and the aggregated weight of these coalitions with respect to (w.r.t.) the imputation, and the number of coalitions attaining them. For the precise definition see §1. By regarding the excesses as a measure of dissatisfaction, which should be minimized, the nucleolus obtains an intuitive meaning as pointed out by Maschler, Peleg, and Shapley (1979). If the excess of a coalition can be decreased without increasing larger excesses, this process will also increase some kind of "stability," they argued. Nevertheless, Maschler (1992) asked: "What is more 'stable', a situation in which a few coalitions of highest excess have it as low as possible, or one where such coalitions have a slightly higher excess, but the excess of many other coalitions is substantially lowered?" Anyone, like the present author, who is not convinced by the first or latter, may try to search for a completely different solution concept.

The solution introduced in the present paper constitutes an attempt to treat all coalitions equally as far as this is possible. This means that, instead of minimizing the highest dissatisfaction, the range of dissatisfaction is minimized by the modified nucleolus. To be more precise, the nucleolus is the lexicographical minimizer of the nonincreasingly ordered vector of excesses, whereas the modified nucleolus lexicographically minimizes the nonincreasingly ordered vector of differences of excesses. This means that the absolute value of dissatisfaction of a coalition is replaced by the envy between coalitions, i.e., by the difference of excesses of these coalitions. This leads to the following intuitive definition. A preimputation belongs to the modified nucleolus $\Psi(v)$ of a game v, if it successively minimizes the maximal differences of excesses and the number of coalition pairs attaining them. Therefore the modified

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nucleolus takes into account both the "power," i.e. the worth, and the "blocking power" of a coalition, i.e. the amount which the coalition cannot be prevented from by the complement coalition (see Remark 1.2). Like the prenucleolus, the modified nucleolus is a singleton.

To give an example look at the two-sided market game with one seller (**1**) and two buyers (**2, 3**) defined as follows. The worth of a coalition is 1 unit if the seller and at least one buyer are members of the coalition. In all other cases the worth of the coalition is zero. If a coalition has positive worth, then the seller is a member of the coalition. This means that player **1** is a veto player possessing, in some sense, all of the power. Indeed, it is well-known that the core of this game, i.e. the set of preimputations such that all excesses are nonpositive, is the singleton $\{x\}$, where $x = (1, 0, 0)$. The nucleolus is a "core selector" and, thus, it coincides with x. On the other hand the buyers together can prevent the seller from any positive amount by forming a "syndicate." Therefore the buyers together have the same "blocking" power as the seller. The modified nucleolus takes care of this fact and assigns $\frac{1}{2}$ to the seller and $\frac{1}{4}$ to each of the buyers (see Example 2.9). This example shows that the modified nucleolus, unlike the nucleolus, is not necessarily contained in a nonempty core. Fortunately in Sudhölter (1993) it is shown that the modified nucleolus is contained in the core for convex games.

If, instead of successively minimizing excesses, only the highest excess is minimized, then this procedure results in the least core. The analogous modified solution concept arises from minimizing the highest difference of excesses and is called modified least core. The relation of the modified solutions is similar to the relation of the classical solutions. The modified nucleolus is a member of the modified least core being a compact convex polyhedron.

Section 1 presents the precise definition of the modified solution concepts. The dual game v^* of a game v assigns to each coalition the real number which can be given to it if the worth of the grand coalition is shared and the complement coalition obtains its worth. By looking at complements it turns out that the modified nucleoli of v and v^* coincide, this also being a characteristic of the Shapley value. A certain replication of a game is defined which allows one to reformulate many assertions concerning the prenucleolus for the modified nucleolus. The dual cover of a game arises from a game v with player set N by taking the union of two disjoint copies of N to be the new player set and assigning to a coalition S the maximum of the sums of the worths of the intersections of S with the first copy w.r.t. v and the second copy w.r.t. v^* or, conversely, the first copy w.r.t. v^* and the second w.r.t. v. Hence both, the game and its dual, are totally symmetric ingredients of the dual cover. A main result of this section, Proposition 1.4, states a strong relationship between the prenucleolus of the dual cover and the modified nucleolus of the initial game. One solution concept arises from the other by the canonical replication or restriction respectively. Therefore, e.g., the modified nucleolus can be computed by each of the well-known algorithms for the calculation of the prenucleolus (see, e.g., Kopelowitz 1967 or Sankaran 1992) applied to the dual cover.

Section 2 starts applying Kohlberg's characterization of the (pre)nucleolus by balanced collections of coalitions (see Kohlberg (1971)) to the modified nucleolus with the help of Proposition 1.4. It turns out that Ψ can be characterized similarly by balanced collections of coalition pairs (see Theorem 2.2). The coincidence of the pre- and modified nucleolus whenever possible w.r.t. duality, i.e. whenever the prenucleoli of the game and its dual cannot be distinguished, is the content of Theorem 2.3 and a consequence of Theorem 2.2. Additionally, it is shown that Ψ satisfies the dummy property (a dummy is any player whose component of the characteristic function behaves additively), weakly respects desirability between players in the sense of

Maschler and Peleg (1966), and is reasonable in the sense of Milnor (1952). In view of these facts the modified nucleolus reflects structure of the game. In the next section it turns out that this solution concept completely reveals the structure of each game in the remarkable class of weighted majority games with a fixed number of winning coalitions.

In §3 the behavior of the modified nucleolus in the weighted majority case is discussed. It turns out in Theorems 3.1 and 3.3 that $\Psi(v)$ induces a representation of v, if and only if v is representable. Moreover, the unique element of $\Psi(v)$ coincides with the normalized vector of weights of the unique minimal integer representation in the homogeneous case. Peleg (1968) showed the same results for weighted majority constant-sum games and the nucleolus. In view of the coincidence of the pre- and modified nucleolus in the constant-sum case (Corollary 2.4) and the coincidence of the prenucleolus and the nucleolus in the weighted majority constant-sum case, this result of the present paper is a generalization of Peleg's. Moreover, the modified nucleolus generalizes the concept of the minimal integer representation in the homogeneous case. Hence this solution can be seen as a canonical representation in the general weighted majority case.

Coincidence of the solution concepts on constant-sum games together with a "modified" constant-sum extension property are the basic properties of a characterization of Ψ on the class of weighted majority games. The dual constant-sum extension of a game v has two additional players and assigns the worth of each coalition w.r.t. v and v^* respectively to the coalition together with the first and second additional player respectively, the worth of the grand coalition to each coalition containing both additional players, and zero to each other coalition. Consequently, the dual constant-sum extensions of a game and its dual coincide up to renaming the additional players. The modified nucleolus satisfies the dual constant-sum extension property in the weighted majority case, i.e. the modified nucleoli of the game and of the dual constant-sum extension arise from each other in a canonical way (see Theorem 3.5). In Proposition 3.8 it turns out that the modified nucleolus on the set of weighted majority games with player set contained in an infinite universe is uniquely characterized by Pareto optimality, coincidence with the nucleolus on constant-sum games, and the dual constant-sum extension property. Moreover, all axioms including the infinity assumption on the universe of players turn out to be logically independent.

1. Notation and definitions. A *cooperative game with transferable utility*—a *game* —is a pair $G = (N, v)$, where N is a finite nonvoid set and

$$v: 2^N \to \mathbb{R}, \qquad v(\emptyset) = 0$$

is a mapping. Here $2^N = \{S \subseteq N\}$ is the set of *coalitions* of G. If $G = (N, v)$ is a game, then N is the *grand* coalition or the set of *players* and v is called *characteristic* (or *coalitional*) *function* of G. Since the nature of the game is determined by the characteristic function, v is called a game as well.

If $G = (N, v)$ is a game, then the *dual* game (N, v^*) of G is defined by $v^*(S) = v(N) - v(N \setminus S)$ for all coalitions S. The set of *feasible payoff vectors* of G is denoted

$$X^*(N, v) := X^*(v) := \{x \in \mathbb{R}^N | x(N) \le v(N)\},$$

whereas

$$X(N, v) := X(v) := \{x \in \mathbb{R}^N | x(N) = v(N)\}$$

is the set of *preimputations* of G (also called set of *Pareto optimal* feasible payoffs of G). Here

$$x(S) := \sum_{i \in S} x_i \qquad (x(\emptyset) = 0)$$

for each $x \in \mathbb{R}^N$ and $S \subseteq N$. Additionally let x_S denote the restriction of x to S, i.e. $x_S = (x_i)_{i \in S} \in \mathbb{R}^S$, whereas $A_S := \{x_S | x \in A\}$ for $A \subseteq \mathbb{R}^N$.

A *solution concept* σ *on* a set Γ of games is a mapping

$$\sigma : \Gamma \to \bigcup_{v \in \Gamma} 2^{X^*(v)}, \qquad \sigma(v) \subseteq X^*(v).$$

If $\bar{\Gamma}$ is a subset of Γ, then the canonical restriction of a solution concept σ on Γ is a solution concept on $\bar{\Gamma}$. We say that σ is a solution concept on $\bar{\Gamma}$, too. If Γ is not specified, then σ is a solution concept on every set of games.

Some convenient and well-known properties of a solution concept σ on a set Γ of games are as follows:

σ is *anonymous* (satisfies AN), if for each $(N, v) \in \Gamma$ and each bijective mapping $\tau : N \to N'$ with $(N', \tau v) \in \Gamma$, $\sigma(N', \tau v) = \tau(\sigma(N, v))$ holds (where $(\tau v)(T) = v(\tau^{-1}(T))$, $\tau_j(x) = x_{\tau^{-1}j} (x \in \mathbb{R}^N, j \in N', T \subseteq N')$). In this case v and τv are *equivalent* games. σ is *covariant under strategic equivalence* (satisfies COV), if for $(N, v), (N, w) \in \Gamma$ with $w = \alpha v + \beta$ for some $\alpha > 0$, $\beta \in \mathbb{R}^N$

$$\sigma(N, w) = \alpha \sigma(N, v) + \beta$$

holds. The games v and w are called *strategically equivalent.*

σ is *single valued* (satisfies SIVA), if $|\sigma(v)| = 1$ for $v \in \Gamma$.

σ satisfies *nonemptiness* (NE), if $\sigma(v) \neq \emptyset$ for $v \in \Gamma$.

σ is *Pareto optimal* (satisfies PO), if $\sigma(v) \subseteq X(v)$ for $v \in \Gamma$.

Note that both equivalence and strategical equivalence commute with duality, i.e.,

$$(\tau v)^* = \tau(v^*), \qquad (\alpha v + \beta)^* = \alpha v^* + \beta,$$

where τ, α, β are chosen according to the definitions given above. It should be remarked (see Shapley (1953)) that the Shapley value φ—to be more precise the solution concept σ given by $\sigma(v) = \{\varphi(v)\}$—satisfies all above properties.

More notation will be needed. Let (N, v) be a game and $x \in \mathbb{R}^N$. The *excess of* a coalition $S \subseteq N$ *at* x is the real number

$$e(S, x, v) := e(S, x) := v(S) - x(S).$$

Let $\mu(x, v) := \mu(x)$ be the *maximal excess at* x, i.e.,

$$\mu(x, v) := \max\{e(S, x) | S \subseteq N\}.$$

The nucleolus of a game was introduced by Schmeidler (1966). Some corresponding definitions and results are recalled: Let $\vartheta : \bigcup_{n \in \mathbb{N}} \mathbb{R}^n \to \bigcup_{n \in \mathbb{N}} \mathbb{R}^n$ be defined by $\vartheta(x) = y \in \mathbb{R}^n (x \in \mathbb{R}^n)$, where y is the vector which arises from x by arranging the components of x in a nonincreasing order. The *nucleolus of* v *w.r.t.* X, where $X \subseteq \mathbb{R}^N$, is the set

$$\mathcal{N}(X, v) := \left\{x \in X | \vartheta\left((e(S, x, v))_{S \subseteq N}\right) \underset{lex}{\leq} \vartheta\left((e(S, y, v))_{S \subseteq N}\right) \text{ for all } y \in X\right\}.$$

Schmeidler (1966) formulated and proved the following three assertions:

(1) If X is a nonvoid compact set, then $\mathscr{N}(X, v)$ is nonvoid.

(2) If X is convex, then $\mathscr{N}(X, v)$ contains at most one vector.

(3) If X is a nonvoid, closed convex subset of $X^*(v)$, then $\mathscr{N}(X, v)$ is a singleton.

The *prenucleolus of* (N, v) is defined to be the nucleolus w.r.t. the set of feasible payoff vectors and denoted $\mathscr{PN}(v)$, i.e., $\mathscr{PN}(v) = \mathscr{N}(X^*(v), v)$. By (3) the prenucleolus of a game is a singleton, and, clearly, the prenucleolus is Pareto optimal. The unique element $\nu(v)$ of $\mathscr{PN}(v)$ is again called *prenucleolus* (*point*). Let $\mathscr{LC}(v)$ denote *the least core* of v, i.e. the set of all preimputations of the game that minimize the maximal excess of nontrivial coalitions (see Maschler, Peleg and Shapley (1979)). Since the prenucleolus successively minimizes excesses and is Pareto optimal, the least core of v is given by

$$\mathscr{LC}(v) = \{x \in X(v) | e(S, x, v) \leq \max\{e(T, \nu(v), v) | \varnothing \neq T \neq N\}, \varnothing \neq S \neq N\}.$$

If N is no singleton, then it is not necessary to presume Pareto optimality, because this property can easily be deduced, i.e.

$$\mathscr{LC}(v) = \{x \in X^*(v) | e(S, x, v) \leq \max\{e(T, \nu(v), v) | \varnothing \neq T \neq N\}, \varnothing \neq S \neq N\}$$

in this case. The least core of a game is a convex compact polyhedron containing the prenucleolus.

For completeness reasons we recall that the nucleolus of (N, v) is the set $\mathscr{N}(X, v)$, where $X = \{x \in X(v) | x_i \geq v(\{i\})\}$ is the set of *imputations* of v.

The solution concept which will be introduced in this paper is defined as follows:

DEFINITION 1.1. Let (N, v) be a game. For each $x \in \mathbb{R}^N$ define

$$\tilde{\Theta}(x, v) := \vartheta\left((e(S, x, v) - e(T, x, v))_{(S,T) \in 2^N \times 2^N}\right) \in \mathbb{R}^{2^{2|N|}}.$$

The *modified nucleolus of* v is the set

$$\Psi(v) := \left\{x \in X(v) | \tilde{\Theta}(x, v) \underset{lex}{\leq} \tilde{\Theta}(y, v) \text{ for all } y \in X(v)\right\}.$$

Compared to the (pre)nucleolus which lexicographically minimizes the "dissatisfaction vector" of the coalitions, where the dissatisfaction of a coalition w.r.t. some (pre)imputation x is its excess at x, the modified nucleolus lexicographically minimizes the "envy vector" of pairs of coalitions, where coalition S envies coalition T at the preimputation x by α if the excess difference of S and T at x coincides with α.

REMARK 1.2. Let (N, v) be a game.

(i) If x is any preimputation of the game v, then the following equality holds by definition:

$$\begin{aligned} e(T, x, v^*) = v^*(T) - x(T) &= v(N) - v(N \setminus T) - x(N) + x(N \setminus T) \\ &= x(N \setminus T) - v(N \setminus T) \quad \text{(by Pareto optimality of } x) \\ &= -e((N \setminus T), x, v). \end{aligned}$$

(ii) With

$$\overline{\Theta}(y,v) := \vartheta\big((e(S,y,v) + e(T,y,v^*))_{(S,T)\in 2^N\times 2^N}\big)$$

for $y \in \mathbb{R}^N$ Remark 1.2 (i) directly implies for $x \in X(v)$ that $\overline{\Theta}(x,v) = \tilde{\Theta}(x,v)$ holds true. Note that x has to be Pareto optimal for this equation. Nevertheless the modified nucleolus can be redefined as

$$\Psi(v) = \left\{x \in X^*(v) | \overline{\Theta}(x,v) \underset{1ex}{\leq} \overline{\Theta}(y,v) \text{ for all } y \in X^*(v)\right\}, \tag{4}$$

since Pareto optimality is, now, automatically satisfied. Indeed, this property can be verified by observing that for every nonvoid coalition both, the excess w.r.t. v and w.r.t. v^*, strictly decrease if all components of a feasible payoff vector can be strictly increased.

(iii) The alternate definition of $\Psi(v)$ in the last assertion (see (4)) directly shows that Ψ satisfies *duality*, i.e. $\Psi(v) = \Psi(v^*)$ holds. If $v(S)$ measures the power of coalition S, then $v^*(S)$ can be regarded as a measure of the blocking power of the coalition. In this sense both the power and the blocking power of all coalitions play a totally symmetric role in this characterization of the modified nucleolus. Note that the Shapley value also satisfies duality.

(iv) It is straightforward to verify that Ψ satisfies both, anonymity and covariance.

With the help of the next definition and proposition we obtain a relationship between the modified nucleolus of v and the prenucleolus of a game called dual cover of the game. Additionally it turns out that the modified nucleolus is a singleton.

DEFINITION 1.3. Let (N,v) be a game and $\overline{N} = N \times \{0,1\}$. We identify $N \times \{0\}$ with N and $N \times \{1\}$ with N^* in the canonical way, thus $\overline{N} = N \dot{\cup} N^*$. The game $(N \dot{\cup} N^*, \tilde{v})$, defined by

$$\tilde{v}(S \dot{\cup} T^*) = \max\{v(S) + v^*(T), v(T) + v^*(S)\}$$

for all $S, T \subseteq N$ is the *dual cover of* v.

PROPOSITION 1.4. *The modified nucleolus of a game (N,v) is the restriction of the prenucleolus of $(N \cup N^*, \tilde{v})$ to N; formally $\Psi(v) = \mathcal{PN}(\tilde{v})_N$. Moreover $\nu_i(\tilde{v}) = \nu_{i^*}(\tilde{v})$ for $i \in N$.*

PROOF. By the well-known anonymity of the prenucleolus the second assertion is true. Let X be the set of *symmetric* feasible payoff vectors of $\tilde{v}$, i.e.,

$$X = \{x \in X^*(\tilde{v}) | x_i = x_{i^*} \text{ for } i \in N\}.$$

Then—by symmetry of the prenucleolus—we come up with

$$\mathcal{PN}(\tilde{v}) = \left\{x \in X | \vartheta\big((e(S,x,\tilde{v}))_{S \in N\cup N^*}\big) \underset{1ex}{\leq} \vartheta\big((e(S,y,\tilde{v}))_{S\in N\cup N^*}\big) \text{ for } y \in X\right\}.$$

For each

$$A = \{S \cup T^*, T \cup S^*\} \in D = \{\{S \cup T^*, T \cup S^*\} | S, T \in N\}$$

let $S(A)$ be defined by

$$S(A) = \begin{cases} S \cup T^*, & \text{if } v(S) + v^*(T) \geq v(T) + v^*(S), \\ T \cup S^*, & \text{otherwise,} \end{cases}$$

and $\tilde{D} = \{S(A) | A \in D\}$. Observe that

$$\mathscr{PN}(\tilde{v}) = \left\{ x \in X | \vartheta((e(S,x,\tilde{v}))_{S \in \tilde{D}}) \underset{1ex}{\leq} \vartheta((e(S,y,\tilde{v}))_{S \in \tilde{D}}) \text{ for } y \in X \right\}$$

holds true. On the other hand—for $x, y \in \mathbb{R}^N$—

$$\overline{\Theta}(x,v) \underset{1ex}{\leq} \overline{\Theta}(y,v) \quad \text{iff}$$

$$\vartheta((e(S,x,v) + e(T,x,v^*))_{S \cup T^* \in \tilde{D}}) \underset{1ex}{\leq} \vartheta((e(S,y,v) + e(T,y,v^*))_{S \cup T^* \in \tilde{D}}),$$

hence the proposition is proved. Q.E.D.

In view of Proposition 1.4 the modified nucleolus of a game v is a singleton denoted by $\psi(v)$, i.e. $\{\psi(v)\} = \Psi(v)$. The unique point $\psi(v)$ of $\Psi(v)$ is again called *modified nucleolus (point)*. For the sake of completeness it is shown that the dual cover of a game uniquely determines the game up to duality.

LEMMA 1.5. *Let (N, v) be a game. If $\tilde{v} = \tilde{w}$ for some game (N, w), then $w \in \{v, v^*\}$ is true.*

PROOF. Let $\tilde{w} = \tilde{v}$, $w \neq v$.

(5) *Claim*: $\{w(S), w^*(S)\} = \{v(S), v^*(S)\}$ for $S \subseteq N$.

By definition of $\tilde{}$ we have

(6)

$$\max\{w(S) + w^*(\emptyset), w^*(S) + w(\emptyset)\} = \max\{v(S) + v^*(\emptyset), v^*(S) + v(\emptyset)\},$$

and, e.g.,

$$w(S) + w^*(S) = v(S) + v^*(S). \tag{7}$$

Using (7) equality (6) directly implies (5). Now the proof can be completed. Take any $S \subseteq N$ with $w(\bar{S}) \neq v(\bar{S})$, i.e. $w(\bar{S}) = v^*(\bar{S}) \neq v(\bar{S})$ by (5). Take any T with $v(T) \neq v^*(T)$. By definition of $\sim$ and (5) we conclude

$$\begin{aligned} \max\{w(T) + w^*(\bar{S}), w(\bar{S}) + w^*(T)\} &= \max\{w(T) + v(\bar{S}), v^*(\bar{S}) + w^*(T)\} \\ &= \max\{v(T) + v^*(\bar{S}), v(\bar{S}) + v^*(T)\} \\ &\neq \max\{v(T) + v(\bar{S}), v^*(\bar{S}) + v^*(T)\}, \end{aligned}$$

thus $w(T) = v^*(T)$ again by (7). Q.E.D.

A further solution concept satisfying duality is useful. The modified least core arises from the modified nucleolus in the same way as the least core arises from the prenucleolus; by only minimizing the highest sum of excesses w.r.t. the game and its dual. The formal notation is given in

DEFINITION 1.6. Let (N, v) be a game. The *modified least core of* v is the set

$$\mathscr{MLC}(v) = \{x \in X^*(v) | \mu(x,v) + \mu(x,v^*) \leq \mu(y,v) + \mu(y,v^*) \text{ for } y \in X^*(v)\}.$$

LEMMA 1.7. *Let (N, v) be a game. Then*

(i) $\psi(v) \in \mathscr{MLC}(v)$,

(ii) $\{x \in \mathscr{LC}(\tilde{v}) | x_i = x_{i^*}$ *for* $i \in \mathbb{N}\} = \{x \in X^*(\tilde{v}) | x_i = x_{i^*}$ *for* $i \in \mathbb{N}$ *and* $x_N \in \mathscr{MLC}(v)\}$.

PROOF. By anonymity and Proposition 1.4 the second assertion implies the first one.

ad(ii): Let $x = \nu(\tilde{v})_N$. Using Remark 1.2 (i) we come up with

$$e(S,x,v) + e(T,x,v^*) = -e(N \setminus T, x, v) - e(N \setminus S, x, v^*)$$

for all $S, T \subseteq N$. This equality together with the definition of the dual cover $\tilde{v}$ directly implies that $\nu(\tilde{v})$ cannot be a member of the core of $\tilde{v}$, i.e. $C(\tilde{v}) = \emptyset$, or all excesses vanish, i.e. v is an additive game. These considerations imply (ii), since $\mu(\nu(\tilde{v}), \tilde{v}) = \mu(x,v) + \mu(x,v^*)$, whereas $\mu(y,v) + \mu(y,v^*) \geq \mu(\nu(\tilde{v}), \tilde{v})$ for $y \in X^*(v)$. Q.E.D.

Note that the modified least core of the game is a compact convex subset of the set of preimputations, since the same is true for the least core and, thus, for the *symmetric* least core of the dual cover (i.e. the set given in (ii) of Lemma 1.7).

2. Properties of the modified solutions. At first Kohlberg's (1971) characterization of the (pre)nucleolus by balanced collections of coalitions is recalled and applied to the modified nucleolus. It should be remarked that his assumption of zero-normalization—i.e. $v(\{i\}) = 0$ for all players i—can be deleted without destroying the proofs. Moreover, the original results were stated for the nucleolus, but it is easy to formulate analogous properties for the prenucleolus (see Peleg (1988, 1989)). Some notation is needed.

A finite nonvoid set $X \subseteq \mathbb{R}^N$ is *weakly balanced* (*balanced*), if X possesses a vector of *weakly balancing* (*balancing*) *coefficients* $(\delta_x)_{x \in X}$, i.e.

$$\sum_{x \in X} \delta_x x = 1_N \quad \text{and} \quad \delta_x \geq 0 (\delta_x > 0) \text{ for } x \in X.$$

Here 1_S is the indicator function of S, considered as vector of $\mathbb{R}^N$. A nonvoid subset D of coalitions or $\tilde{D}$ of pairs of coalitions is (weakly) *balanced* if

$$\{1_S | S \in D\} \quad \text{or} \quad \{1_S + 1_T | (S,T) \in \tilde{D}\} \text{ respectively}$$

is (weakly) balanced. We say that S and (S, T) respectively is in the *span* of D and $\tilde{D}$ respectively if 1_S and $1_S + 1_T$ is in the span of $\{1_S | S \in D\}$ and $\{1_S + 1_T | (S,T) \in \tilde{D}\}$ resp. For $x \in \mathbb{R}^N$, $\alpha \in \mathbb{R}$ define

$$D(x, \alpha, v) = \{S \subseteq N | e(S,x,v) \geq \alpha\},$$

$$\tilde{D}(x, \alpha, v) = \{(S,T) \in 2^N \times 2^N | e(S,x,v) + e(T,x,v^*) \geq \alpha\}.$$

THEOREM 2.1 (KOHLBERG). *Let* (N, v) *be a game,* $\alpha \in \mathbb{R}$, *and* $x \in X(v)$.
(i) $x = \nu(v)$ *iff each nonvoid* $D(x, \alpha, v)$ *is balanced.*
(ii) $x \in \mathscr{LC}(v)$ *iff* $D(x, \mu_0(x, v), v)$ *is weakly balanced, where* $\mu_0(x, v)$ *is the highest nontrivial excess at* x, *i.e.,* $\mu_0(x, v) = \max\{e(S, x, v) | \emptyset \neq S \subsetneqq N\}$.

For a proof of this theorem see Kohlberg (1971). The analogon for the modified solutions is

THEOREM 2.2. (i) $x = \psi(v)$ *iff each nonvoid* $\tilde{D}(x, \alpha, v)$ *is balanced.*
(ii) $x \in \mathscr{MLC}(v)$ *iff* $\tilde{D}(x, \mu(x, v) + \mu(x, v^*), v)$ *is weakly balanced.*

PROOF. The corresponding assertions of Theorem 2.1 together with Proposition 1.4 and Lemma 1.7 (ii) imply (i) and (ii) respectively. A proof of the latter assertion is similar to a part of the proof of assertion (i) and thus skipped. In order to verify (i) let $z := (x, x^*) \in \mathbb{R}^{N \cup N^*}$, thus z is a preimputation of the dual cover $\tilde{v}$ of v (see Definition 1.3). By the same definition we come up with the following two assertions:
if $(S, T) \in \tilde{D}(x, \alpha, v) =: \tilde{D}$, then $S \cup T^* \in D(z, \alpha, \tilde{v}) := D \ni T \cup S^*$,
if $S \cup T^* \in D$, then $(S, T) \in \tilde{D}$ or $(T, S) \in \tilde{D}$.
Particularly, $D \neq \emptyset$ iff $\tilde{D} \neq \emptyset$. Assume that $\tilde{D}$ is nonvoid.

Note that x coincides with $\psi(v)$ iff z coincides with $\nu(\tilde{v})$ by Proposition 1.4. Assume, now, $x = \psi(v)$ and take balancing coefficients $\delta_{S \cup T^*} > 0$ of D, i.e.,

$$\sum_{S \cup T^* \in D} \delta_{S \cup T^*} 1_{S \cup T^*} = 1_{N \cup N^*}.$$

For $(S, T) \in \tilde{D}$ define a real number

$$\delta_{(S,T)} = \begin{cases} \frac{1}{2}\delta_{S \cup T^*}, & \text{if } (T, S) \in \tilde{D}, \\ \frac{1}{2}(\delta_{S \cup T^*} + \delta_{T \cup S^*}), & \text{otherwise}. \end{cases}$$

Then

$$\begin{aligned} \sum_{(S,T) \in \tilde{D}} \delta_{(S,T)}(1_S + 1_T) &= \sum_{\substack{(S,T) \in \tilde{D} \\ \text{and } (T,S) \notin \tilde{D}}} \frac{1}{2}(\delta_{S \cup T^*} + \delta_{T \cup S^*})(1_S + 1_T) \\ &\quad + \sum_{\substack{(S,T) \in \tilde{D} \\ \text{and } (T,S) \in \tilde{D}}} \frac{1}{2}\delta_{S \cup T^*}(1_S + 1_T) \\ &= \sum_{S \cup T^* \in D} \frac{1}{2}\delta_{S \cup T^*}(1_S + 1_T) = 1_N \end{aligned}$$

holds true, thus $\tilde{D}$ is balanced. Conversely, if $\tilde{D}$ is balanced with balancing coefficient $\delta_{(S,T)} > 0$ for $(S, T) \in \tilde{D}$, then $(\delta_{S \cup T^*})_{S \cup T^* \in D}$, where

$$\delta_{S \cup T^*} = \begin{cases} \delta_{(S,T)} + \delta_{(T,S)}, & \text{if } (S, T) \in \tilde{D} \ni (T, S), \\ \delta_{(S,T)}, & \text{if } (T, S) \notin \tilde{D} \ni (S, T), \\ \delta_{(T,S)}, & \text{if } (S, T) \notin \tilde{D} \ni (T, S), \end{cases}$$

are balancing coefficients for D. Q.E.D.

By Remark 1.2 (iii) the modified nucleolus satisfies duality. Nevertheless, ψ coincides with the prenucleolus whenever this makes sense w.r.t. the duality property. To be more precise, all of $\psi(v)$, $\nu(v)$, and $\nu(v^*)$ coincide if the last two vectors coincide. Formally, this assertion is the contents of the next theorem.

THEOREM 2.3. *Let (N, v) be a game. If $\nu(v) = \nu(v^*)$ holds, then $\psi(v) = \nu(v)$ is also true.*

PROOF. In view of Theorem 2.2 it is sufficient to show that $\tilde{D} := \tilde{D}(x, \alpha, v)$ is balanced, whenever $\tilde{D}$ is nonvoid, where $x = \nu(v)$. Define

$$A := \{S \subseteq N | \text{there is } T \subseteq N \text{ with } (S, T) \in \tilde{D}\} \quad \text{and}$$

$$B := \{T \subseteq N | \text{there is } S \subseteq N \text{ with } (S, T) \in \tilde{D}\}$$

and take $S \in A$. Then $(S, T) \in \tilde{D}$, iff $e(T, x, v^*) \geq \alpha - e(S, x, v) =: \alpha(S)$. Since $\nu(v^*)$ coincides with x by assumption, Theorem 2.1 can be applied; hence the set $D_S^\alpha := D(x, \alpha(S), v^*)$ is balanced, let us say, with balancing coefficients $\delta^*_{(T,S)} > 0$, $T \in D_S^\alpha$. Let β be the maximal excess of coalitions in B w.r.t. v^*, i.e. $\beta = \max\{e(T, x, v^*) | T \in B\}$. Then, by definition, $A = D(x, \alpha - \beta, v)$ holds true, thus A is balanced—since $\nu(v) = x$—with balancing coefficients $(\delta_S)_{S \in A}$. With

$$c_S := \left(\sum_{T \in D_S^\alpha} \delta^*_{(T,S)}\right)^{-1} \quad \text{and} \quad c := \left(1 + \sum_{S \in A} c_S \delta_S\right)^{-1}$$

the following equation shows the balancedness of $\tilde{D}$:

$$\begin{aligned}
&\sum_{(S,T) \in \tilde{D}} c \cdot c_S \delta_S \delta^*_{(T,S)}(1 + 1_T) \\
&= c \cdot \sum_{S \in A} c_S \delta_S \sum_{T \in D_S^\alpha} \delta^*_{(T,S)}(1_S + 1_T) \\
&= c \cdot \sum_{S \in A} c_S \delta_S \left(1_N + \sum_{T \in D_S^\alpha} \delta^*_{(T,S)} 1_S\right) \\
&= c \cdot \left(1_N + \sum_{S \in A} c_S \delta_S 1_N\right) = 1_N. \quad \text{Q.E.D.}
\end{aligned}$$

Note that in case the least cores of the game and its dual coincide then they also coincide with the modified least core which can be proved similarly. Moreover, the prerequisite $\nu(v) = \nu(v^*)$ and $\mathscr{LC}(v) = \mathscr{LC}(v^*)$ is trivially satisfied for each constant-sum game. Recall that (N, v) is a *constant-sum game*, if $v(S) + v(N \setminus S) = v(N)$ for $S \subseteq N$. Therefore v is a constant-sum game, iff v coincides with the dual game v^*.

COROLLARY 2.4. *For each constant-sum game the pre- and the modified nucleolus coincide as well as the least core and the modified least core.*

This corollary can also be proved without using Theorem 2.3 by first observing that the dual cover $\tilde{v}$ of v is a constant-sum game and the prenucleolus of this game arises from the one of the game started with by replication, i.e., $\nu(\tilde{v}) = (\nu(v), \nu(v)^*)$. Again, similar considerations show that the symmetric least core of $\tilde{v}$ is the replication of the least core of v.

Up to the end of this section some properties of Ψ and $\mathscr{MLC}$ are formulated which directly arise from well-known properties of the prenucleolus and least core applied to the dual cover of the game. Moreover, one example is presented. Indeed, it is shown that the modified solutions are reasonable in the sense of Milnor (1952) and satisfy the dummy property. Moreover, the modified nucleolus weakly respects desirability. Some well-known definitions are recalled.

Let (N, v) be a game. Player $i \in N$ is a *dummy* of v, if $v(S \cup \{i\}) = v(S) + v(\{i\})$ for all $S \subseteq N \setminus \{i\}$. This player is a *null-player*, if additionally $v(\{i\}) = 0$ holds. Player $i \in N$ is *at least as desirable as* player $j \in N$, written $i \succcurlyeq_v j$, if $v(S \cup \{i\}) \geq v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$. The arising strict relation is abbreviated $\succ_v$: i is *more desirable than* j, if $i \succcurlyeq_v j$ and not $j \succcurlyeq_v i$. Players i and j are *interchangeable*—$i \sim_v j$—if $i \succcurlyeq_v j \succcurlyeq_v i$. This desirability relation between players was introduced in Maschler and Peleg (1966) and can be generalized to coalitions (see, e.g., Einy (1985)). In the first paper it was shown that the prenucleolus as an element of the prekernel (for the definition of the prekernel see Davis and Maschler (1965) and Maschler, Peleg and Shapley (1972)) respects desirability—i.e., $\nu_i(v) \geq \nu_j(v)$, if $i \succcurlyeq_v j$—and satisfies the dummy property—i.e., $\nu_i(v) = v(\{i\})$ for each dummy i of v. With the help of the following lemma that the same statement can be proved for the modified nucleolus.

LEMMA 2.5. *Let (N, v) be a game, $\tilde{v}$ be the dual cover of v, and $i, j \in N$.*
(i) *$i \succcurlyeq_v j$, iff $i \succcurlyeq_{\tilde{v}} j$.*
(ii) *i is a dummy of v, iff i is a dummy of $\tilde{v}$.*
(iii) $\min\{v(S \cup \{i\}) - v(S) | S \subseteq N \setminus \{i\}\} \leq$
$\min\{\tilde{v}(\tilde{S} \cup \{i\}) - \tilde{v}(\tilde{S}) | \tilde{S} \subseteq (N \setminus \{i\}) \cup N^*\}$.
(iv) $\max\{v(S \cup \{i\}) - v(S) | S \subseteq N \setminus \{i\}) \geq$
$\max\{\tilde{v}(\tilde{S} \cup \{i\}) - \tilde{v}(\tilde{S}) | \tilde{S} \subseteq (N \setminus \{i\}) \cup N^*\}$.

PROOF. Observe that v^* has the same "desirability structure," the same dummies and so on. To be more precise the exact formulations are as follows—a proof is a straightforward consequence of the definition of v^*:

$i \succcurlyeq_v j$ iff $i \succcurlyeq_{v^*} j$;

i is a dummy of v iff i is one of v^* and $v(\{i\}) = v^*(\{i\})$;

$a := \min\{v(S \cup \{i\}) - v(S) | S \subseteq N \setminus \{i\}\} = \min\{v^*(S \cup \{i\}) - v^*(S) | S \subseteq N \setminus \{i\}\}$,

$b := \max\{v(S \cup \{i\}) - v(S) | S \subseteq N \setminus \{i\}\} = \max\{v^*(S \cup \{i\}) - v^*(S) | S \subseteq N \setminus \{i\}\}$.

ad(i): Let $i \succcurlyeq_{\tilde{v}} j$ for some $i, j \in N$ and $S \in N \setminus \{i, j\}$. Defining $\bar{S} = S \cup \{i\}$, $\bar{T} = S \cup \{j\}$, we come up with

$$v(\bar{T}) + v^*(\bar{S}) \leq \tilde{v}(\bar{T} \cup \bar{S}^*) \leq \tilde{v}(\bar{S} \cup \bar{S}^*) = v(\bar{S}) + v^*(\bar{S})$$

by definition and assumption, thus $i \succcurlyeq_v j$. Conversely, if $i \succcurlyeq_v j$ and $S \cup T^* \subseteq (N \setminus \{i, j\}) \cup N^*$, let w.l.o.g.

$$\tilde{v}(S \cup \{j\} \cup T^*) = v(S \cup \{j\}) + v^*(T),$$

otherwise exchange the roles of v and v^*. Hence

$$\tilde{v}(S \cup \{j\} \cup T^*) \leq v(S \cup \{i\}) + v^*(T) \leq \tilde{v}(S \cup \{i\} \cup T^*).$$

ad(ii): A proof of this assertion is straightforward using the above observation and therefore we skip it.

ad(iii): Take $S \subseteq N \setminus \{i\}$ and $T \subseteq N$. W.l.o.g. $\tilde{v}(S \cup T^*) = v(S) + v^*(T)$—otherwise exchange v and v^*. Then

$$\tilde{v}(S \cup \{i\} \cup T^*) - \tilde{v}(S \cup T^*) \geq v(S \cup \{i\}) - v(S) \geq a.$$

Finally, assertion (iv) can be proved analogously to (iii). Q.E.D.

This last lemma and the well-known properties of the prenucleolus and the least core together with Proposition 1.4 and Lemma 1.7 directly imply

COROLLARY 2.6. *Let (N, v) be a game and $x \in \mathscr{MLC}(v)$.*
(i) $\psi_i(v) \geq \psi_j(v)$, *if* $i \succcurlyeq_v j$;
(ii) $x_i = v(\{i\})$ *for each dummy* i *of* v;
(iii) $x_i \geq \min\{v(S \cup \{i\}) - v(S) | S \subseteq N \setminus \{i\}\}$ *for* $i \in N$;
(iv) $x_i \leq \max\{v(S \cup \{i\}) - v(S) | S \subseteq N \setminus \{i\}\}$ *for* $i \in N$.

Assertion (iv) is called *reasonableness* (in the sense of Milnor (1952)). A solution concept satisfying (iii) and (iv) for each of its elements is called *reasonable on both sides*. In §3 it will be deduced that ψ strongly respects desirability in the weighted majority case, i.e. ψ is generically "more sensitive" than ν in this case. The following example shows that the weak desirability property cannot be replaced by the strong one in general. That means $\geq$, $\succcurlyeq_v$ and $>$, $\succ_v$ cannot be exchanged in Corollary 2.6 (i). The only well-known solution concept strongly respecting desirability is the Shapley value.

EXAMPLE 2.7. Let $N = \{1, 2, \ldots, 9\}$ and

$$W^{\mathrm{sm}} = \{\{1,2\}, \{1,4,5\}, \{1,4,6,7\}, \{1,6,7,8,9\}, \{2,3,4,8,9\}, \{2,4,5,7,9\}\}.$$

As each coalition S can be identified with the indicator function 1_S—considered as $\{0,1\}$-vector of length $|N|$—each set of coalitions A can be identified with a matrix $I_A = (1_S)_{S \in A}$ with lexicographically ordered rows, e.g.

$$I_{W^{\mathrm{sm}}} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

The *shift relation* on 2^N is defined by $S \geq_{sh} T$, iff $|S \cap \{1, \ldots, k\}| \geq |T \cap \{1, \ldots, k\}|$ for $k \in N$, where $N = \{1, \ldots, n\}$. This means that $S \geq_{\mathrm{sh}} T$ holds, if the number of players of S with indices less than k is not smaller than the corresponding number w.r.t. T for each k. In our particular case let v be defined by

$$v(S) = \begin{cases} 1, & \text{if there is } T \in W^{\mathrm{sm}} \text{ with } S \geq_{\mathrm{sh}} T, \\ 0, & \text{otherwise}. \end{cases}$$

It should be noted that v is a directed game and $I_{W^{\mathrm{sm}}}$ is the shift-minimal matrix of v in the sense of Ostmann (1987b, 1989) and Krohn and Sudhölter (1995). (A *directed* game (N, v) is a game such that
(i) $N = \{1, \ldots, n\}$ for some $n \in \mathbb{N}$;
(ii) v is *monotone*, i.e. $v(S) \leq v(T)$ if $S \subseteq T \subseteq N$;
(iii) v is *simple*, i.e. $v(S) \in \{0, 1\}$ for $S \subseteq N$;
(iv) The desirability relation is complete and $1 \succcurlyeq_v 2 \succcurlyeq_v \cdots \succcurlyeq_v n$.)

In our special case it turns out that

$$1 \succ 2 \succ 3 \sim 4 \succ 5 \succ 6 \sim 7 \succ 8 \sim 9$$

holds true where the subscript v has been deleted for clearness reasons. Moreover, it is easy to check the constant-sum property of v.

Corollary 2.4 implies $\psi(v) = \nu(v)$. Since there are no *winning* players, i.e. $v(\{i\}) = 0$ for $i \in N$, the pre-nucleolus of v coincides with the nucleolus (see, e.g., Peleg, Rosenmüller and Sudhölter (1994)). Thus $\psi(v)$ can be computed by one of the well-known algorithms—see, e.g., Kopelowitz (1967) or Sankaran (1992). It turns out that

$$\psi(v) = \nu(v) = (7,5,2,2,2,1,1,1,1)/22.$$

Therefore $\psi_4(v) = \psi_5(v)$ and $4 \succ_v 5$, $\psi_7(v) = \psi_8(v)$ and $7 \succ_v 8$.

REMARK 2.8. For technical reasons the following assertion is needed. A proof which is straightforward is skipped.

Let N be a finite nonvoid set, D and $\tilde{D}$ be balanced collections of coalitions and pairs of coalitions respectively. Then every subset E and $\tilde{E}$ with $D \subseteq E \subseteq 2^N$, $\tilde{D} \subseteq \tilde{E} \subseteq 2^N \times 2^N$ such that E and $\tilde{E}$ are in the span of D and $\tilde{D}$ respectively are balanced.

EXAMPLE 2.9. Let (N, v) be the "one-seller-and-two-buyers-game" defined by $N = \{1,2,3\}$,

$$v(S) = \begin{cases} 1, & \text{if } 1 \in S \text{ and } |S| \geq 2, \\ 0, & \text{otherwise}. \end{cases}$$

To prove that $x = (2,1,1)/4$ coincides with the modified nucleolus it is sufficient to show that $\mu(x,v) + \mu(x,v^*) = 3/4 := \alpha$ and $\tilde{D}(x,\alpha,v)$ is balanced and spans 2^N by Theorem 2.2 and Remark 2.8. Indeed, $1/4$ and $1/2$ respectively is the maximal excess of v and v^* respectively at x, hence the first assertion is verified. Therefore

$$\tilde{D}(x,\alpha,v) = \{(S,T) | S \in \{\{1,2\},\{1,3\}\}, T \in \{\{1\},\{2,3\}\}\}, \quad \text{thus}$$

$$\left\{1_S + 1_T | (S,T) \in \tilde{D}\right\} = \{(2,1,0),(2,0,1),(1,2,1),(1,1,2)\}$$

hold true. The vector of coefficients $(1,1,3,3)/10$ shows that $\tilde{D}$ is balanced. Moreover, the missing property is obvious.

3. The modified solutions in the weighted majority case. A *simple game* (N, v) is a game satisfying $v(S) \in \{0,1\}$ for $S \subseteq N$. A simple game (N, v) is a *weighted majority game* if there is a vector of *weights*—a measure—$m \in \mathbb{R}^N_{\geq 0}$ and a *level* $\lambda > 0$ such that

$$v(S) = \begin{cases} 1, & \text{if } m(S) \geq \lambda, \\ 0, & \text{if } m(S) < \lambda. \end{cases}$$

To exclude the pathological games $m(N) \geq \lambda$ is always presumed. The tupel $(\lambda; m)$ is a *representation* of v. A simple game v is uniquely determined by its set of *winning coalitions*

$$W_v := \{S \subseteq N | v(S) = 1\},$$

whereas each monotone (see Example 2.7 for the definition of monotonicity) simple game is uniquely determined by its set of *minimal* winning coalitions

$$W_v^m := \{S \in W_v | v(T) = 0 \text{ for } T \subsetneqq S\}.$$

Note that a weighted majority game is monotone. A representation $(\lambda; m)$ of a simple game v is *homogeneous* if $m(S) = \lambda$ for $S \in W_v^m$. A simple game is *homogeneous* if it has a homogeneous representation. Simple games were introduced by von Neumann and Morgenstern (1944) and have many interesting applications (see, e.g., Shapley 1962). According to the structure of simple games it should also be referred to Isbell (1956, 1958, 1959) and, in the homogeneous case, to Ostmann (1987a), Rosenmüller (1982, 1984, 1987), and Sudhölter (1989).

Each weighted majority game $(N; v)$ has a—not necessarily unique—*minimal* representation $(\bar{\lambda}; \bar{m})$, i.e., an integer representation ($\bar{m} \in \mathbb{Z}^N_{\geq 0}$, $\bar{\lambda} \in \mathbb{N}$) such that there is no distinct integer representation $(\lambda; m)$ of v with $m \leq \bar{m}$ componentwise. A nonnegative vector $m \in \mathbb{R}^N$ *induces* a representation of a weighted majority game (N, v) if there is a level $\lambda > 0$ such that $(\lambda; m)$ represents v.

Up to now this author does not know any "direct" method of generating or enumerating the class of weighted majority games—even recursively w.r.t. the number of players. It is true that only weighted majority constant-sum games have to be generated to obtain the general case (see, e.g., Krohn and Sudhölter 1995). Two recursive methods are known in this case, which are indirect in the following sense. They do not yield any recursive formula for the number of games and need comparisons or tests. The first one introduced in Isbell (1959) is strongly based on the comparison of games already constructed. The second one (see Krohn and Sudhölter 1995) generates a larger class of constant-sum games—the directed (see Example 2.7 for the corresponding definition) constant-sum games—and extracts those weighted majority games by testing for representability. Isbell suggests that it could be useful to assign a "canonical" unique vector of weights—inducing a representation—to each of these constant-sum games. Peleg (1968) showed that the nucleolus always induces a representation in this case, thus this solution concept yields a canonical vector of weights. Moreover, he concluded that the nucleolus is the normalized vector of weights of the unique minimal representation in the homogeneous case. Besides, it should be remarked that the testing procedure of Krohn and Sudhölter (1995) is based on this result. As seen in, e.g., Peleg and Rosenmüller (1992) and Rosenmüller and Sudhölter (1994) the nucleolus coincides with the prenucleolus for monotone simple constant-sum games. The last paper contains examples which show that the prenucleolus does not necessarily induce a representation if the constant-sum property is dropped.

There are procedures which generate homogeneous constant-sum games (see Isbell 1959 and Sudhölter 1988), but these methods, unfortunately, again require tests. Therefore no formula for the enumeration of this class of games can be deduced from the algorithms. Nevertheless, the larger class of all homogeneous n-person games—up to equivalence—can be generated and enumerated recursively w.r.t. the number of players as shown by Sudhölter (1989). Analogously it could perhaps be possible to generate or enumerate the class of all weighted majority games instead of assuming the constant-sum property. Again, following Isbell and Peleg, it could be useful to have unique weights—which are homogeneous if the game is—even in this case. The first aim of this section is to show that the modified nucleolus satisfies this condition.

In the constant-sum weighted majority case Peleg implicitly showed that each element of the least core and not only the nucleolus induces a representation. Moreover the least core is a singleton if the game is a homogeneous constant-sum game. The analogous assertions in the general weighted majority and homogeneous case respectively are valid. Note that the modified least core satisfies covariance, anonymity, and the dummy property.

THEOREM 3.1. *Each element of the modified least core of a weighted majority game induces a representation of the game.*

PROOF. Let $(\lambda; m)$ be a normalized representation of the weighted majority game (N, v), i.e. $m(N) = 1$. Then—by normalization—$m \in X(v)$. Let S be a coalition in N. By $m \geq 0$ and $0 < \lambda \leq 1$ we conclude

$$-\lambda < e(S, m, v) \leq 1 - \lambda, \quad \text{thus}$$

$$\mu(m, v) + \mu(m, v^*) < 1. \tag{1}$$

By reasonableness on both sides each element x of the modified least core has nonnegative components. It remains to show $x(T) < x(S)$ for each pair (S, T) with $T \notin W_v \ni S$. Assume, on the contrary, there is a pair (S, T) with $T \notin W_v \ni S$ and $x(T) \geq v(S)$. Then

$$\begin{aligned} e(S, x, v) + e(N \setminus T, x, v^*) &= v(S) + v^*(N \setminus T) - x(S) - x(N \setminus T) \\ &= 1 - x(S) + x(T) \quad (\text{by } x \in X(v)) \\ &\geq 1 \quad (\text{by assumption}). \end{aligned}$$

In view of Definition 1.6 the last inequality together with (1) establishes a contradiction. Q.E.D.

If (N, v) is a homogeneous game, then it has a unique minimal representation being automatically homogeneous itself (see, e.g., Ostmann 1987a). Before showing that the normalized vector of weights of this representation coincides with the unique element of the modified least core—hence with the modified nucleolus, an additional result concerning homogeneous games is needed. Though the following lemma can easily be extracted from Ostmann (1987a) or Sudhölter (1989), a proof is given for completeness reasons. Each weighted majority game (N, v) is equivalent to a directed game (N', v'). Indeed, let $(\lambda; m)$ be a representation of v, $N' = \{1, 2, \ldots, |N|\}$, and $(i_t)_{t \in N}$ be defined by

$$\{i_t | t \in N'\} = N, m_{i_1} \geq \cdots \geq m_{i_{|N|}}.$$

Define $v'(S) = v(\{i_t | t \in S\})$ for $S \subseteq N'$. Clearly, the set of normalized representations satisfies AN.

LEMMA 3.2. *Let $N = \{1, \ldots, n\}$ and $(\lambda; m)$ be the minimal representation of a directed homogeneous game (N, v) without null-players. For each player $i \in N$ there is a pair of coalitions (S^i, T^i) such that the following conditions are satisfied*:

(i) $S^i \ni i \notin T(i)$;

(ii) $m(S^i) = \lambda$, $m(T^i) = \lambda - 1$;

(iii) $(S^i \setminus \{i\}) \cap \{1, \ldots, \max S^i\} = T \cap \{1, \ldots, \max S^i\}$.

PROOF. Following Ostmann (1987a), $m_n = 1$. Choose any coalition $S^n \in W_v^m$ with $n \in S^n$, which is possible since n is no null-player. With $T^n = S^n \setminus \{n\}$ the pair (S^n, T^n) satisfies (i), (ii), (iii) for $i = n$. Assume the assertion is already shown for

$j > i$ and some $i \in \{1, \ldots, n-1\}$. The set

$$M = \{S \in W_v^m | i \in S \text{ and } m(\{i+1, \ldots, n\} \setminus S)$$
$$\geq m(\{i+1, \ldots, n\} \setminus T) \text{ for } i \in T \in W_v^m\}$$

contains a coalition S^i with $\{i, i+1, \ldots, \max S^i\} \subseteq S^i$ (see, e.g., Sudhölter 1989). If i is a step, i.e., $m(T) < m_i$—where $T = \{1 + \max S^i, \ldots, n\}$—then $m(T) = m_i - 1$ by Ostmann (1987a). With $T^i = S \setminus \{i\} \cup T$ the proof is completed in this case. If i is a sum, i.e., $m(T) \geq m_i$, then T contains a subset U with $m(U) = m_i$ by homogeneity (which can be constructed by dropping recursively players of the largest indices if necessary). Let j be the maximal index of U, thus $j > \max S^i$. Because of the assumption there are coalitions S^j and T^j with the desired properties. With

$$T^i = (S \setminus \{i\}) \cup (U \setminus \{j\}) \cup (T^j \setminus S^j)$$

it can easily be verified that (S^i, T^i) satisfies (i), (ii), (iii). Q.E.D.

It should be remarked that Lemma 3.2 remains true if the assumption of the absence of null-players is dropped and the expression "For each player i" is replaced by "For each nonnull-player i."

THEOREM 3.3. *Let $(\bar{\lambda}\,\bar{m})$ be the minimal representation of a homogeneous game (N, v) and $x \in \mathscr{MLC}(v)$. Then $x = \bar{m}/\bar{m}(N)$.*

PROOF. Assume w.l.o.g. that v is directed (by anonymity) and has no dummies (by Lemma 3.2, the preceding remark and the well-known fact that the normalized vector of weights of the minimal representation of a homogeneous game satisfies the dummy property).

Let $m = \bar{m}/\bar{m}(N)$ and $N = \{1, \ldots, n\}$. Analogously to (1) we have

$$\mu(m, v) + \mu(m, v^*) = 1 - 1/\bar{m}(N) \tag{2}$$

and thus

$$e(S^i, m, v) + e((N \setminus T^i), m, v^*) = 1 - 1/\bar{m}(N). \tag{3}$$

Here (S^i, T^i) are defined according to Lemma 3.2. The definition of the modified least core implies

$$x(S^i) - x(T^i) \geq 1/\bar{m}(N) = m_n \tag{4}$$

for $x \in \mathscr{MLC}(v)$ and $i \in N$. Starting with $i = n$ we obtain $x(S^n) - x(T^n) = x_n \geq m_n$. In general—by (4)—we come up with

$$x_i \geq m_n + x(T^i \setminus S^i) \geq m_n + m(T^i \setminus S^i) = m_i,$$

if $x_{i+1} \geq m_{i+1}, \ldots, x_n \geq m_n$ is already shown. Finally we conclude that $x \geq m$; but $x, m \in X(v)$, thus $x = m$. Q.E.D.

Summarizing Theorems 3.1 and 3.3, the modified nucleolus establishes a canonical representation of a weighted majority game which coincides—up to normalization—with the unique minimal (homogeneous) representation in the homogeneous case and with the nucleolus for constant-sum games. In what follows a strong relationship between the (pre)nucleolus and the modified nucleolus in the weighted majority case is presented. Some notation is needed.

DEFINITION 3.4. Let (N, v) be a game.

(i) Let $^* := (N, 0)$ and $\diamond := (N, 1)$ be two additional players. The game $(N \cup \{^*, \diamond\}, v^0\}$ defined by

$$v^0(S) = \begin{cases} v(S \setminus \{\diamond\}) & \text{if } ^* \notin S \ni \diamond, \\ v^*(S \setminus \{*\}) & \text{if } \diamond \notin S \ni *, \\ v(N) & \text{if } \diamond, ^* \in S, \\ 0 & \text{otherwise}, \end{cases}$$

is the *dual constant-sum extension of* v.

(ii) (N, v) is a *dual constant-sum extension*, if $|N| \geq 3$ and there are different players $i, j \in N$ $(i \neq j)$ and a game $(N \setminus \{i, j\}, w)$ such that v coincides—up to renaming i, j by $^*, \diamond$ respectively—with w^0.

Note that v^0, indeed, is a constant-sum game and that v^{*0} coincides—up to exchanging the additional new players—with v^0. The dual constant-sum extension is in contrast to the constant-sum extension as defined, e.g., in Einy and Lehrer (1989). It is true that the constant-sum extensions of a game and its dual generically do not coincide.

In order to formulate the strong relationship between the modified nucleolus of a weighted majority game and the nucleolus of the dual constant-sum extension (Theorem 3.5) it is useful to deduce some relationship between the representations of v and v^0.

Let (λ, m) be a representation of the weighted majority game (N, v). It is well known (see, e.g., Krohn and Sudhölter 1995) that (λ^*, m) represents v^* for each real number λ^* satisfying

$$m(N) - \min\{m(S) | S \in W_v\} < \lambda^* \leq m(N) - \max\{m(T) | T \notin W_v\}.$$

In particular, the sum of the levels λ and λ^* is larger than $m(N)$. If two further components $m_\diamond := \lambda^*$, $m_* := \lambda$ are added, i.e. $(m_\diamond, m_*, m) \in \mathbb{R}^{N \cup \{\diamond, *\}}$, then this vector induces a representation of v^0. Indeed, $\lambda + \lambda^*$ is a level. To each vector m inducing a representation of v let $\lambda(m)$ and $\lambda^*(m)$ be the maximal levels respectively, i.e.,

$$\lambda(m) = \min\{m(S) | S \in W_v\}, \quad \lambda^*(m) = m(N) - \max\{m(T) | T \notin W_v\}.$$

The above considerations show that the vector $(\lambda^*(m), \lambda(m), m)$ induces a representation of the dual constant-sum extension of v and is therefore called *extended vector* of m.

THEOREM 3.5. *Let (N, v) be a weighted majority game. Then the prenucleolus $\nu(v^0)$ of the dual constant-sum extension of v coincides with the normalized extended vector of $\psi = \psi(v)$, formally written*

$$\nu(v^0) = (1 - \mu^*, 1 - \mu, \psi)/(3 - \mu - \mu^*), \quad \textit{where } \mu = \mu(\psi, v),\ \mu^* = \mu(\psi, v^*).$$

PROOF. Let $x \in \mathbb{R}^{N \cup \{\diamond, *\}}$ be defined by

$$x_i = \begin{cases} 1 - \mu^*, & i = \diamond, \\ 1 - \mu, & i = *, \\ \psi_i, & \text{otherwise}. \end{cases}$$

It remains to show that $\nu(v^0) = y := x/(3 - \mu - \mu^*)$ holds true (note that y is well defined by Theorem 3.1 and, thus, is normalized). Assume w.l.o.g. that v is superadditive, i.e. $\mu^* \geq \mu$ (otherwise exchange the roles of v and v^*), and that v has no null-players (the dual constant-sum extension conserves null-players and both, ψ and ν have the dummy property). By Theorem 2.1, Corollary 2.4, and Remark 2.8 it suffices to verify

(a) if $D_\alpha \neq 0$ and $\alpha \geq c$, then D_α is balanced, and

(b) the span of D_c contains every coalition $S \in N \cup \{\diamond, *\}$, i.e. D_c spans the Euclidean space $\mathbb{R}^{N \cup \{\diamond, *\}}$, where

$$c := \frac{1 - \mu}{3 - \mu - \mu^*}, \qquad D_\alpha := D(y, \alpha, v^0).$$

ad(b): It can easily be verified that D_c can be rewritten as

$$\begin{aligned} D_c = &\{S \cup \{*\} | S \subseteq N, 1 - \mu^* \leq \psi(S) \leq 1 - \mu^* + \mu\} \\ &\cup \{X \cup \{\diamond\} | S \subseteq N, 1 - \mu \leq \psi(S)\} \\ &\cup \{S \cup \{\diamond, *\} | S \subseteq N, \psi(S) \leq \mu\}. \end{aligned}$$

If $j \in N$ is no veto player of v, i.e. $j \notin \bigcap_{S \in W_v} S$, then $\psi_j < 1 - \mu^*$, since j is no winning player of v^* and $(1 - \mu^*, \psi)$ represents v^*. Moreover, $\psi(N \setminus \{j\}) \geq 1 - \mu$, since $N \setminus \{j\} \in W_v$, and thus $\psi_j \leq \mu$. Therefore both $S := \{\diamond, *\} \cup \{j\}$ and $T := \{\diamond, *\}$ are members of D_c, thus

(5) $$1_{\{j\}} = 1_S - 1_T.$$

If $i \in N$ is a veto player of v, then $\psi_i \geq 1 - \mu^*$, since $\{i\} \in W_{v^*}$. In Lemma 4.9 it turns out that $\psi_i = 1 - \mu^*$ in this case. We conclude

(6) $$\{*, i\} \in D_c.$$

Take any $S \in W_v$ with $e(S, \psi, v) = \mu$. Then $S \cup \{\diamond\} \in D_c$, thus the union of $\{\diamond\}$ and the set of veto players is in the span of D_c by (5). Hence, by (6), $\{*\}$ is in the span of D_c. But now it is straightforward to verify that $\{\diamond\}$ and $\{i\}$ (i is a veto player of v) are in the span of D_c. Up to now we have seen that each unit vector is in the span of D_c, thus D_c spans the Euclidean space.

ad(a): Let $\alpha_0 := c < \alpha_1 < \cdots < \alpha_r$ be defined via $\emptyset \neq D_{\alpha_r} \subsetneqq D_{\alpha_{r-1}} \subsetneqq \cdots \subsetneqq D_{\alpha_0}$ and let r, α_j be maximal, i.e. for each $\alpha \geq c$ with $D_\alpha \neq \emptyset$ there is $j \in \{0, \ldots, r\}$ such that $\alpha_j \geq \alpha$ and $D_{\alpha_j} = D_\alpha$. It remains to show that D_{α_j} is balanced for each j. This will be done inductively.

A straightforward computation shows for each $j > 0$ that $\emptyset \neq \tilde{D}_{\beta_j} \subseteq W_v \times W_{v^*}$, where

$$\beta_j = (3 - \mu - \mu^*)\alpha_j + \mu + \mu^* - 1, \qquad \tilde{D}_\beta = \tilde{D}(\beta, \psi, v).$$

Define $\overline{D}_\alpha := \{S \in D_\alpha | \{*, \diamond\} \subseteq S\} \cup \{\{\diamond, *\}\}$. Clearly, if $j > 0$, then

(7) $$(S, T) \in \tilde{D}_{\beta_j} \quad \text{implies} \quad S \cup \{\diamond\} \in \tilde{D}_{\alpha_j}, T \cup \{*\} \in D_{\alpha_j},$$

(8) $$S \subseteq N, S \cup \{\diamond\} \in D_{\alpha_j} \quad \text{implies the existence of} \quad (S, T) \in \tilde{D}_{\beta_j},$$

(9) $$T \subseteq N, T \cup \{*\} \in \tilde{D}_{\alpha_j} \quad \text{implies the existence of} \quad (S, T) \in \tilde{D}_{\beta_j}.$$

Theorem 2.3 guarantees that there are balancing coefficients $\alpha_{(S,T)}$ w.r.t. $\tilde{D}_{\beta_j}$, i.e.,

$$\sum_{(S,T)\in\tilde{D}_{\beta_j}} \alpha_{(S,T)}(1_S + 1_T) = 1_N. \tag{10}$$

The observation that—by integrating (10) w.r.t. ψ and using (1) applied to ψ—

$$d := \sum_{(S,T)\in\tilde{D}_{\beta_j}} \alpha_{(S,T)} < 1 \tag{11}$$

holds, directly implies $1 - d > 0$. Hence the equality

$$(1-d)1_{\{*,\diamond\}} + \sum_{(S,T)\in\tilde{D}_{\beta_j}} \alpha_{(S,T)}(1_{\bar{S}} + 1_{\bar{T}}) = 1_{N\cup\{\diamond,*\}},$$

where $\bar{S} = S \cup \{\diamond\}$, $\bar{T} = T \cup \{*\}$, proves that $\tilde{D}_{\alpha_j}$ is balanced for $j > 0$. Clearly $\overline{D}_{\alpha_r} = D_{\alpha_r}$ if $r \neq 0$. Assume D_{α_i} is balanced for each $i \in \{j+1, \ldots, r\}$ and some $j \geq 0$. If $j > 0$ we proceed by showing that each vector 1_R with $R \in D_{\alpha_j} \setminus \overline{D}_{\alpha_j}$ is a linear combination of vectors 1_Q, $Q \subseteq \overline{D}_{\alpha_j} \cup D_{\alpha_{j+1}}$. Indeed, take any such R and assume $R \notin \overline{D}_{\alpha_j} \cup D_{\alpha_{j+1}}$. Clearly, $R = \{*, \diamond\} \cup \tilde{R}$ for some $\emptyset \neq \tilde{R} \subseteq N$. If $|\tilde{R}| \geq 2$, then each $R^i = \{*, \diamond\} \cup \{i\}$, $i \in \tilde{R}$, is a member of $D_{\alpha_{j+1}}$, thus

$$1_R = \sum_{i\in\tilde{R}} 1_{R^i} - (|\tilde{R}| - 1)1_{\{*,\diamond\}}.$$

If $|\tilde{R}| = 1$, let us say $\tilde{R} = \{i\}$, then

$$\psi_i \leq 1 + (\mu + \mu^* - 3)\alpha_j \leq \mu < 1 - \mu^*,$$

thus i is no veto player of v. Take any $S \in D_{\alpha_r}$ with $\{*, \diamond\} \subseteq S$. Then $S \cup \{j\} \in \overline{D}_{\alpha_j}$, thus R is a linear combination of the preceding coalition, $\{\diamond, *\}$, and S. Hence R is in the span of $\overline{D}_{\alpha_j} \cup D_{\alpha j+1}$. An inductive argument and Remark 3.2 show that $D_{\alpha j}$ is balanced for $j > 0$.

The case $j = 0$ can be treated analogously by observing that balancedness of $\tilde{D}_{\beta_0}$ directly implies balancedness of $\tilde{D}_{\beta_0} \cap (W_v \times W_{v^*})$. Q.E.D.

LEMMA 3.6. *With the notation of the last theorem* $\psi_i = 1 - \mu^*$ *for each veto player* i *holds true.*

PROOF. Assume w.l.o.g. that v has no null-players. If $\mu = 0$, then v is the unanimous game (i.e., $v(S) = 1$ iff $S = N$) and the assertion is trivially satisfied by AN. Moreover, assume there is a veto player i of v. Then the fact that i is a winning player of v^* implies $\mu^* \geq \mu$. Since v is not the unanimous game we come up with $\mu > 0$, thus

$$\tilde{D} := \tilde{D}(\mu + \mu^*, \psi, v) \subseteq W_v \times W_{v^*}.$$

Then

$$\sum_{(S,T)\in\tilde{D}} \alpha_{(S,T)} < 1 \quad \text{by (11)},$$

where $\alpha_{(S,T)}$ are balancing coefficients for $\tilde{D}$. Clearly, $i \in S$ for $(S,T) \in \tilde{D}$. Now the inequality

$$\sum_{(S,T)\in\tilde{D}} \alpha_{(S,T)}(1_S + 1_T)_i = 1 > \sum_{(S,T)\in\tilde{D}} \alpha_{(S,T)} = \sum_{(S,T)\in\tilde{D}} (1_S)_i$$

shows that $e(\{i\}, \psi, v^*) = \mu^*$, thus $\psi_i = 1 - \mu^*$. Q.E.D.

In order to combine Theorem 3.5 and Corollary 2.4 the following notation is useful.

DEFINITION 3.7. Let Γ be a set of weighted majority games. A solution concept σ on Γ satisfies the *dual constant-sum extension property* (*DCSE*), if for every dual constant-sum extension $(N, v) \in \Gamma$ the following holds: Let v be—up to renaming the additional players—the dual constant-sum extension of a game $(N \setminus \{i, j\}, w)$ and $w \in \Gamma$, then—for each $x \in \sigma(w)$—there is $c \in \mathbb{R}$ such that

$$c(1 - \mu(x, v^*), 1 - \mu(x, v), x) \in \sigma(v).$$

PROPOSITION 3.8. *Let* $\Gamma = \{(N, v) | N \subseteq U$ *and* v *is a weighted majority game*$\}$, *where* U *is an infinite set. There is a unique solution concept on* Γ, *which satisfies* NE, PO, DCSE, *and coincides with the nucleolus on constant-sum games, and it is the modified nucleolus.*

PROOF. The modified nucleolus satisfies NE, PO, DCSE, and coincides with the prenucleolus, thus with the nucleolus, on constant-sum games by Proposition 1.4, Corollary 2.4, Theorem 3.5, Definition 1.1 and anonymity. Conversely, each game $(N, v) \in \Gamma$ occurs as a game such that there is a game $(N \cup \{i, j\}, w) \in \Gamma$ which coincides—up to renaming the additional players—with the dual constant-sum extension of v. Thus the properties uniquely determine $\sigma(v)$. Q.E.D.

The following examples show that $\mu(\cdot, v)$ need not be constant on $\mathcal{MLC}(v)$, if the weighted majority property is deleted, and that an analogon to Theorem 3.5 cannot be valid if the dual constant-sum extension v^0 is replaced by the classical constant-sum extension of v as defined in, e.g., Einy and Lehrer (1989).

EXAMPLE 3.9. (i) Let $N = \{1, 2, 3, 4\}$, $S^1 = \{1\}$, $S^2 = \{1, 3, 4\}$, $T^1 = \{1, 4\}$, $T^2 = \{1, 3\}$, and (N, v) be defined by

$$v(S) = \begin{cases} 2, & \text{if } S = S^i \text{ for some } i = 1, 2, \\ -2, & \text{if } S = T^i \text{ for some } i = 1, 2, \\ 0, & \text{otherwise}. \end{cases}$$

For any $y \in X(v)$ we have

$$\sum_{i=1}^{2} e(S^i, y, v) + e(N \setminus T^i, y, v^*) = 8,$$

thus $\mu(y, v) + \mu(y, v^*) \geq 4$. With $\bar{x} = (-1, 1, 0, 0)$ and $y \in$ Convex Hull $\{\bar{x}, -\bar{x}\}$ we come up with $\mu(y, v) + \mu(y, v^*) = 4$, thus $y \in \mathcal{MLC}(v)$. Now $\mu(\bar{x}, v) = 3 \neq 1 = \mu(-\bar{x}, v)$ holds true, but both, $\bar{x}$ and $-\bar{x}$, are elements of $\mathcal{MLC}(v)$. Besides, $\psi(v)$, $\nu(v)$, $\nu(v^*)$, $\varphi(v)$ turn out to be $(0, 0, 0, 0)$, $-\bar{x}$, $\bar{x}$, and $(\frac{1}{3}, -\frac{1}{3}, 0, 0)$ respectively.

(ii) Let $N = \{1, \ldots, 8\}$ and (N, v) be the weighted majority game represented by $(\lambda; m)$, where $\lambda = 25$, $m = (13, 7, 6, 6, 4, 4, 4, 2)$. Recall that the constant-sum extension $(N \cup \{*\}, v_0)$ of v is defined by

$$v_0(S) = \begin{cases} v(S), & S \subseteq N, \\ v^*(S \setminus \{*\}), & * \in S. \end{cases}$$

It can be verified that v_0 can be represented by $(\lambda; m^i)$, $i = 1, 2$, where $m_j^i = m_i$ for $i = 1, \ldots, 7$; $m_8^1 = m_8$, $m_*^1 = 3$, $m_8^2 = m_*^2 = 2.5$. It should be remarked that v_0 is the constant-sum extension of a game with two minimal representations, introduced in Dubey and Shapley (1978). It turns out that $\nu(v_0) = \psi(v_0) = m^2/m^2(N \cup \{*\})$ (see Krohn and Sudhölter (1995): v_0 is one of the games listed in the appendix of this paper). The normalized restriction y of this vector m^2 to N, i.e.

$$y = (13, 7, 6, 6, 4, 4, 4, 2.5)/46.5$$

cannot be a member of the modified least core of v, since

$$\mu(y, v) + \mu(y, v^*) = \left(1 - \frac{25}{46.5}\right) + \left(1 - \frac{22.5}{46.5}\right) = 2 - \frac{47.5}{46.5} \quad \text{and}$$

$$\mu(z, v) + \mu(z, v^*) = \left(1 - \frac{25}{46}\right) + \left(1 - \frac{22}{46}\right) = 2 - \frac{47}{46},$$

where $z = m/m(N)$. Indeed, it can be verified that $z = \psi(v)$ holds true but the corresponding proof is skipped.

For the sake of completeness we present examples which show that all axioms of the last proposition as well as the infinitely assumption on the universe of players are logically independent.

REMARK 3.10. The axioms of Proposition 3.8, i.e., NE, PO, DCSE, and the coincidence with the nucleolus for a constant-sum game are logically independent on the set Γ of weighted majority games with a player set contained in the infinite set U. Indeed, let four solution concepts on Γ be defined by

$$\sigma^0(v) = \begin{cases} \mathscr{PN}(v), & \text{if } v \text{ is a constant-sum game,} \\ \varnothing, & \text{otherwise,} \end{cases}$$

$$\sigma^1(v) = \begin{cases} \mathscr{PN}(v), & \text{if } v \text{ is a constant-sum game,} \\ \{\frac{1}{2} \cdot \psi(v)\}, & \text{otherwise,} \end{cases}$$

$$\sigma^2(v) = \mathscr{PN}(v),$$

$$\sigma^3(v) = \{x \in X(v) | x \text{ induces a representation of } v\},$$

for each $v \in \Gamma$. Clearly σ^0 shows the independence of NE. The solution σ^1 satisfies DCSE and coincides with the nucleolus for constant-sum games but is not Pareto optimal. The solution concept σ^2 satisfies PO, but not DCSE, and coincides with the nucleolus on constant-sum games. The considerations following Definition 3.4 show that $(1 - \mu(m, v^*), 1 - \mu(m, v), m)$ induces a representation of the dual constant-sum extension of a given weighted majority game (N, v), if the normalized vector m induces a representation of v, i.e., $m(N) = 1$ and $(1 - \mu(m, v); m)$ is a representation of v. Therefore σ^3 satisfies PO and DCSE, but does not coincide with the modified nucleolus. The observation that the set of vectors inducing a representation of a weighted majority game is a full dimensional convex cone implies this last assertion.

Finally note that the assumption of the infinite cardinality of the universal player set U cannot be dropped in Proposition 3.7. Indeed, if U is finite and $|U| \geq 3$, the solution concept

$$\sigma(v) = \begin{cases} \psi(v), & \text{if } N \neq U, \\ \mathcal{PN}(v), & \text{if } N = U, \end{cases}$$

for each $(N, v) \in \Gamma$ satisfies PO, DCSE, and coincides with the nucleolus on constant-sum games. The set Γ contains a game which is equivalent to $(\{1, \ldots, |U|\}, v)$, where v is the homogeneous game represented by (λ, m), given by

$$m_i = \begin{cases} 2, & i = 1, \\ 1, & i \in \{2, 3\}, \; \lambda = 3, \\ 0, & \text{otherwise}. \end{cases}$$

It can easily be verified (see, e.g., Peleg, Rosenmüller, and Sudhölter (1994) or Rosenmüller and Sudhölter (1994)) that $\nu(v) = (1, 0, \ldots, 0)$, thus $\nu(v) \neq \psi(v)$.

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