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Equal Treatment for Both Sides of Assignment Games in the Modified Least Core

by

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Abstract: In contrast to the least core and the nucleolus, the modified least core and the modified nucleolus take into account both, the "power", i.e. the worth, and the "blocking power" of a coalition, i.e. the amount which the coalition cannot be prevented from by the complement coalition. The least core and nucleolus respectively minimizes the highest excess and successively minimizes the nonincreasingly ordered vector of excesses respectively. The modified solutions arise from an analogous procedure in which excesses are replaced by differences of excesses. In this paper it is shown that both sides of an assignment game are treated equally from every element of the modified least core. Moreover, a similar property is shown for games which can be written as a minimum of two additive games and are called M2-games. Both classes of games share a common property. Every assignment and every M2-game is a member of the class of complementary concave games. With the help of complementary convexity it is shown that the modified least core of an assignment and M2-game is contained in the least core of its dual game.

1. *Introduction*

Two-sided matching markets are frequently modelled as cooperative transferable utility games which are called assignment games. (For an extensive discussion and applications of two-sided matching and economic applications see Roth and Sotomayor (1990).) An assignment game describes a two-sided market with, let us say, potential sellers $i \in P$ on the one hand and potential buyers $j \in Q$ on the other hand. The sellers are assumed to supply indivisible objects of trade. Associated with every pair (i, j) of one seller and one buyer is a nonnegative real number, the worth of the coalition $\{i, j\}$, which is frequently interpreted as the difference of the value of the object for the buyer and the seller. The worth of an arbitrary coalition is defined to be the maximal aggregate worth of all pairwise combinations of sellers and buyers within this coalition.

To give an example look at the glove game (N, v), for which $P \subset N$ is the nonvoid subset of left-hand glove owners whereas $Q = N \setminus P$ is the nonvoid set of right-hand glove owners. The worth $v(S)$ of a coalition S is defined to be the number of pairs of gloves owned by the members of S. Players of one side (P or Q) are interchangeable. In

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case the number of left-hand and right-hand gloves in the market coincides, the groups P and Q are interchangeable. Therefore both, the Shapley value and nucleolus, which satisfy symmetry in the sense of Shapley (1953), coincide and assign the same amount to every player. If the cardinalities of P and Q differ, let us say $|P| < |Q|$, then it is well-known that both solution concepts assign a higher aggregate amount to the smaller side (P) of the market. The nucleolus (and any other element of the core) distributes the total worth of the grand coalition equally among the players of P. However, the following observation indicates that both groups P and Q possess the same "blocking power". Indeed, P and Q are able to prevent the opposite side from any positive amount by forming a "syndicate". In view of this fact it can be argued that the Shapley value and the nucleolus underestimate the blocking power of the coalitions.

The modified nucleolus and the modified least core introduced by Sudhölter (1996, 1997) take into account both, the "power", i.e. the worth, and the "blocking power" of every coalition, i.e. the amount which the coalition cannot be prevented from by the complement coalition in case the total worth $v(N)$ is distributed. A measure for the blocking power of S is given by the number $v^*(S) = v(N) - v(N \setminus S)$, i.e. the worth of S with respect to the dual game. In the above mentioned papers it is proved that the modified nucleolus has many properties in common with the prenucleolus, possesses an axiomatization, and behaves adequately for, e.g., weighted majority games.

In this paper it is shown that every element of the modified least core which, of course, contains the modified nucleolus assigns the same amount to P and Q in any assignment game. This paper is organized as follows:

In Section 2 several definitions of cooperative game theory are recalled and some necessary notation is introduced. Moreover, the two solution concepts mentioned above are described. They are, in some sense, related to the prenucleolus and the least core. The modified nucleolus successively minimizes highest differences of excesses - the classical prenucleolus successively minimizes highest excesses, whereas the modified least core minimizes the highest difference of excesses - the classical least core minimizes the highest excess. The modified nucleolus is a singleton contained in the modified least core, which is a convex compact polyhedron as shown in Sudhölter (1996). Both modified solutions satisfy duality, i.e. coincide for the game and its dual. Examples which illustrate the results of Section 4 are included.

In Section 3 it is shown that assignment games and M2-games (games which can be written as the minimum of two additive games) belong to the larger class of complementary concave (c-concave) games with respect to (P, Q). For M2-games, P is the set of players for which the first additive game dominates the second one. Moreover, it turns out that the least core of any c-convex game treats P and Q equally with respect to excesses. To be more precise, Lemma 3.5 postulates that P and Q are coalitions of maximal excess at any element in the least core of a c-convex game with respect to (P, Q). For completeness it is shown that weighted majority games are not c-convex at all, unless they consist of

at most one winning player and a "unanimity component".

Section 4 presents the main results in Theorems 4.5 and 4.6: The modified least core of every M2-game and every assignment game is contained in the least core of the corresponding dual game.

2. Notation, definitions, and examples

A cooperative game with transferable utility - a *game* - is a pair (N, v), where N is a finite nonvoid set and

$$v : 2^N \to \mathbb{R},\ v(\emptyset) = 0$$

is a mapping. Here $2^N = \{S \mid S \subseteq N\}$ denotes the set of *coalitions* and v is the *coalitional function* of (N, v). Since the nature of the *player set* N is determined by the coalitional function, v is called game as well.

The *dual* game (N, v^*) of v is given by

$$v^*(S) = v(N) - v(N \setminus S).$$

The set of *preimputations* of (N, v) is denoted by

$$X(N, v) = X(v) = \{x \in \mathbb{R}^N \mid x(N) = v(N)\},$$

where

$$x(S) = \sum_{i \in S} x_i \text{ for } S \subseteq N,\ x \in \mathbb{R}^N.$$

For $x \in \mathbb{R}^N$, $S \subseteq N$ the *excess* of S at x (with respect to v) is the real number

$$e(S, x, v) = v(S) - x(S).$$

Moreover, let

$$\mu_0(x, v) = \max\{e(S, x, v) \mid \emptyset \neq S \subset N\}$$

denote the *maximal nontrivial*[1] *excess* at x.

The *least core* $\mathcal{LC}(v)$ of v is the set of preimputations which minimize the highest nontrivial excess, i.e.

$$\mathcal{LC}(v) = \{x \in X(v) \mid e(S, x, v) \leq \mu_0(y, v) \text{ for } y \in X(v),\ \emptyset \neq S \subset N\}.$$

[1]The notion "$\subset$" means proper subset, whereas "$\subseteq$" includes "=" as one possibility.

The least core of v is a nonvoid convex polytope containing the prenucleolus (see, e.g., Maschler, Peleg, and Shapley (1979)). It is a subset of the core whenever the core is nonempty. Recall that the *prenucleolus* $\mathcal{PN}(v)$ of v lexicographically minimizes the excesses within the set of preimputations. Formally, it is defined to be the set

$$\mathcal{PN}(v) = \{x \in X(v) \mid \vartheta(x,v) \leq_{\text{lex}} \vartheta(y,v) \text{ for } y \in X(v)\},$$

where $\vartheta(x,v) = (e(S,x,v))_{S \subseteq N}$ is the vector of excesses in a nonincreasing order. The prenucleolus of v is a singleton (see Schmeidler (1969)) and its unique element is abbreviated by $\nu(v)$. Maschler, Peleg, and Shapley (1979) tried to give an intuitive meaning to the prenucleolus by regarding the excess of a coalition as a measure of dissatisfaction which should be minimized. Indeed, the prenucleolus can be reached by minimizing the highest excess, then minimizing the number of coalitions attaining highest excess, then minimizing the second highest excess, and so on.

If the "envy" between two coalitions S and T, i.e. $e(S,x,v) - e(T,x,v)$, is regarded as a measure of dissatisfaction of the pair (S,T), then it is natural to consider the set of preimputations that lexicographically minimize the nonincreasingly ordered vector of envies. This leads to a procedure in which the values of excesses are replaced by the values of differences of excesses.

A preimputation belongs to the *modified nucleolus* $\Psi(v)$ of a game v if it minimizes the highest difference of excesses, then minimizes the number of pairs of coalitions attaining the highest difference of excesses, then minimizes the second highest difference of excesses, and so on. The modified least core arises from the modified nucleolus in the same way as the least core arises from the prenucleolus; by only proceeding along the first step of the minimizing procedure. The formal notation is given in

Definition 2.1: Let (N,v) be a game and $x \in \mathbb{R}^N$. Let

$$\Theta(x,v) = (e(S,x,v) - e(T,x,v))_{(S,T) \in 2^n \times 2^N}$$

denote the vector of differences of excesses in a nonincreasing order. Then

$$\Psi(v) = \{x \in X(v) \mid \Theta(x,v) \leq_{\text{lex}} \Theta(y,v) \text{ for } y \in X(v)\}$$

is the *modified nucleolus* of v, whereas

$$\mathcal{MLC}(v) = \{x \in X(v) \mid \tilde{\mu}(x,v) \leq \tilde{\mu}(y,v) \text{ for } y \in X(v)\},$$

where $\tilde{\mu}(x,v) = \max\{e(S,x,v) - e(T,x,v) \mid S,T \subseteq N\}$ denotes[2] the maximal difference of excesses of v at x, is the *modified least core* of v.

Remark 2.2:

1. The straightforward observation that $e(S,x,v) = -e(N \setminus S, x, v^*)$ is valid for $S \subseteq N$ and every preimputation $x \in X(v)$ shows that $\Theta(x,v)$ in Definition 2.1 can be replaced by the nonincreasing vector

$$(e(S,x,v) + e(T,x,v^*))_{(S,T) \in 2^N \times 2^N}$$

of sums of excesses with respect to v and v^*. Hence the modified least core can be rewritten as

$$\mathcal{MLC}(v) = \{x \in X(v) \mid \mu(x,v) + \mu(x,v^*) \leq \mu(y,v) + \mu(y,v^*) \text{ for } y \in X(v)\},$$

where $\mu(x,v) = \max\{e(S,x,v) \mid S \subseteq N\}$ denotes the maximal excess of v at x, i.e., the modified least core of v consists of all preimputations minimizing the sum of maximal excesses with respect to v and v^*.
In view of these facts both the modified nucleolus and the modified least core are self dual, i.e. $\Psi(v) = \Psi(v^*)$ and $\mathcal{MLC}(v) = \mathcal{MLC}(v^*)$. If the set of preimputations $X(v)$ is replaced by the set of imputations, i.e., the set of individually rational preimputations, then the resulting variants of the modified solution concepts do not satisfy self duality. However, $\Psi(v)$ and $\mathcal{MLC}(v)$ are individually rational (see Sudhölter (1996,1997)), if the game (N,v) is weakly superadditive (i.e., if $v(S \cup \{i\}) - v(S) \geq v(\{i\})$ for $S \subseteq N \setminus \{i\}$ and $i \in N$).

2. In the definition of the least core only nontrivial coalitions S (i.e. $S \neq \emptyset, N$) play a rôle. Analogously the modified least core remains unchanged if only sums of excesses of pairs of nontrivial coalitions (S,T), i.e., $\{S,T\} \not\subseteq \{\emptyset, N\}$ are considered. This can easily be verified by observing that $\mu(x,v) = \mu(x,v^*)$ $(= 0)$ can only hold for some $x \in X(v)$, if v is inessential (additive), i.e., if there exists a vector $m \in \mathbb{R}^N$ (take $m = x$ in this case) such that $v(S) = m(S)$.

3. Trivially the modified nucleolus is contained in the modified least core by definition. Moreover the modified nucleolus is a singleton (see Sudhölter (1996,1997)). The unique point $\psi(v)$ of $\Psi(v)$ is again called *modified nucleolus* (point).

Elements of the least core or core are, vaguely formulated, determined by only looking at the worth of coalitions ($v(S)$, $S \subseteq N$), whereas the "blocking power" of a coalition S, $v(N) - v(N \setminus S) = v^*(S)$, is not taken into consideration. In the modified solutions both the "power" of a coalition, $v(S)$, and the blocking power, $v^*(S)$, play a totally symmetric rôle in general.

A characterization of the modified solutions which is analog to that of the (pre)nucleo-

[2] Note that the maximum is taken over all pairs of coalitions including the empty and the grand coalition.

lus by balanced collections of coalitions due to Kohlberg (1971) can be obtained as follows. A finite nonvoid set $X \subseteq \mathbb{R}^N$ is *weakly balanced* (*balanced*), if X possesses a vector of *weakly balancing* (*balancing*) *coefficients* $(\delta_x)_{x \in X}$, i.e.

$$\sum_{x \in X} \delta_x x = 1_N \text{ and } \delta_x \geq 0 \ \ (\delta_x > 0) \text{ for } x \in X.$$

Here 1_S is the indicator function of S, considered as vector of $\mathbb{R}^N$. A nonvoid subset of D of coalitions or $\tilde{D}$ of pairs of coalitions is (weakly) *balanced*, if

$$\{1_S \mid S \in D\} \text{ or } \{1_S + 1_T \mid (S,T) \in \tilde{D}\} \text{ respectively}$$

is (weakly) balanced. For $x \in \mathbb{R}^N$, $\alpha \in \mathbb{R}$ define

$$D(x,\alpha,v) = \{S \subseteq N \mid e(S,x,v) \geq \alpha\}$$

and

$$\tilde{D}(x,\alpha,v) = \{(S,T) \in 2^N \times 2^N \mid e(S,x,v) + e(T,x,v^*) \geq \alpha\}.$$

Lemma 2.3: Let (N,v) be a game, $\alpha \in \mathbb{R}$, and $x \in X(v)$.

1. $x = \nu(v)$, iff each nonvoid $D(x,\alpha,v)$ is balanced.
2. $x \in \mathcal{LC}(v)$, iff $D(x,\mu_0(x,v),v)$ is weakly balanced or empty (i.e., $|N| = 1$).
3. $x = \psi(v)$, iff each nonvoid $\tilde{D}(x,\alpha,v)$ is balanced.
4. $x \in \mathcal{MLC}(v)$, iff $D(x,\mu(x,v) + \mu(x,v^*),v)$ is weakly balanced.

For a proof of assertions 1 and 2 Kohlberg (1971) is referred to, whereas assertions 3 and 4 of Lemma 2.3 are proved in Sudhölter (1996,1997).

We proceed by introducing some special classes of games.

Definition 2.4: Let (N,v) be a game and (P,Q) be a partition of N, such that $P + Q = N$ (where $A + B = A \cup B$, iff A, B are disjoint sets).

1. (N,v) is an *assignment game* (with respect to (P,Q)), if there is a nonnegative $P \times Q$ matrix $A = (a_{ij})_{i \in P, j \in Q}$ such that

$$v(S) = \max \left\{ \sum_{i \in S \cap P} \sum_{j \in S \cap Q} a_{ij} x_{ij} \;\middle|\; \begin{array}{l} \sum_{i \in S \cap P} x_{ij} \leq 1, \sum_{j \in S \cap Q} x_{ij} \leq 1, \\ x_{ij} \geq 0 \ \forall i \in S \cap P, j \in S \cap Q \end{array} \right\}.$$

(Note that x_{ij} can be chosen to be 0 or 1 (see Shapley and Shubik (1972).)

2. (N,v) is a *minimum of two additive games (an M2-game)*, if there are $m^1, m^2 \in \mathbb{R}^N$ such that

$$v(S) = \min\{m^1(S), m^2(S)\}.$$

Note that assignment games as well as M2-games are weakly superadditive. Indeed, if (N,v) is an assignment game, then $v(S + \{i\}) - v(S) \geq 0 = v(\{i\})$ for $S \subseteq N \setminus \{i\}$ and $i \in N$. If (N,v) is an M2-game corresponding to $m^1, m^2 \in \mathbb{R}^N$, then

$$\min\{m^1(S), m^2(S)\} + \min\{m^1(\{i\}), m^2(\{i\})\} \leq \min\{m^1(S + \{i\}), m^2(S + \{i\})\}$$

for $S \subseteq N \setminus \{i\}$ and $i \in N$. Therefore the modified least core as well as the "classical" least core are individually rational in both cases.

Parts 2 and 3 of Example 5.2 show that there are assignment games and M2-games such that the imputation sets of the dual games are empty. This observation implies that the dual games of assignment games or M2-games are not necessarily weakly superadditive. Thus the classes of assignment games and of M2-games are *not* closed under duality.

It is the main aim of this paper to show that in the assignment game case both groups P and Q are treated equally, i.e. get the same aggregate amounts, from each preimputation of the modified least core. This shows in which way the modified least core takes care of the blocking power of P and Q. Indeed, if P (or Q) form a "syndicate" in an assignment game, then P (or Q) can prevent the opposite group Q (or P) from any positive amount.

M2-games as well as assignment games are *linear production* games in the sense of Owen (1975) (see also Rosenmüller (1982)) and thus possess nonempty cores. However, the following examples show that the modified least core frequently does not intersect the core.

Example 2.5:

1. Let $N = \{1,\ldots,n\}$ with $n \in \mathbb{N}$, $n \geq 2$, and define $P = \{1\}$, $Q = N \setminus P$ and (N,v) via

$$v(S) = \begin{cases} 1, & \text{if } P \subseteq S \text{ or } Q \subseteq S \\ 0, & \text{otherwise} \end{cases}$$

Then v is a weighted majority game (cf. Section 3) with representation $(n-1; n-1, 1, \ldots, 1)$, whereas v^* is a weighted majority game with representation $(n; n-1, 1, \ldots, 1)$. Moreover, v^* is an assignment game (with respect to (P, Q)), defined by the $P \times Q$ matrix $(1, \ldots, 1)$, and an M2-game defined by the vectors $(1, 0, \ldots, 0)$, $(0, 1, \ldots, 1) \in \mathbb{R}^N$. Note that v^* is the (P, Q) glove game. A glove game with respect to disjoint finite nonvoid sets P, Q is the assignment game defined by the $P \times Q$ matrix A with $a_{ij} = 1$ for $i \in P, j \in Q$. It coincides with the M2-game defined by

$$m^1, m^2 \in \mathbb{R}^{P+Q},\; m_i^1 = \begin{cases} 1, & i \in P \\ 0, & i \in Q \end{cases},\; m_i^2 = \begin{cases} 1, & i \in Q \\ 0, & i \in P \end{cases}.$$

In this situation the nucleoli can be computed as

$$\nu(v^*) = (1, 0, \ldots, 0) \text{ and } \psi(v) = \psi(v^*) = (n-1, 1, \ldots, 1)/(2n-2).$$

Moreover, Lemma 2.3 implies $\mathcal{MLC}(v^*) = \mathcal{MLC}(v) = \{\psi(v)\}$ and $\mathcal{LC}(v^*) = \{\nu(v^*)\}$. The modified least core highly evaluates the blocking power (with respect to v^*) of the coalitions $\{1\}$ and $\{2, \ldots, n\}$, whereas the "classical" least core $\mathcal{LC}(v^*)$ does not. In view of the fact that none of the players $2, \ldots, n$ is a nullplayer in v^* it seems hard to imagine $\nu(v^*)$ as an "acceptable" proposal of how to share $v^*(N)$. Besides it should be noted that the Shapley value $\varphi(v) = \varphi(v^*) = ((n-1)^2, 1, \ldots, 1)/(n-1)n$ is a convex combination of the nucleolus and the modified nucleolus in the present case.

2. Let P, Q be two disjoint nonvoid finite sets and let $(P+Q, v)$ be the glove game with respect to (P, Q). Without loss of generality let the cardinality of P (p, q denote the cardinalities of P, Q, respectively) be smaller than or equal to the cardinality of Q. Then, as long as $p = q$, the nucleoli are given by

$$\nu(v) = \nu(v^*) = \psi(v) = (1, \ldots, 1)/2 = \varphi(v),$$

where φ again denotes the Shapley value. Note that $v^*(i) = 1$ for $i \in N$, thus the imputation set $\{x \in X(v^*) \mid x_i \geq v^*(\{i\}),\ i \in N\}$ is empty.
If $p < q$, then it can be verified with the help of Lemma 2.3 that

$$\nu_i(v) = \begin{cases} 1, & i \in P \\ 0, & i \in Q \end{cases},\; \nu_i(v^*) = \psi_i(v) = \begin{cases} 1/2, & i \in P \\ p/(2q), & i \in Q \end{cases} \quad \text{for } i \in N = P + Q,$$

whereas $\varphi(v)$ is a proper convex combination of $\nu(v)$ and $\psi(v)$ (see Shapley and Shubik

(1969)). Therefore the Shapley value can be seen, in some sense, as a compromise between the modified nucleolus and the (pre)nucleolus in this case. The modified nucleolus highly evaluates the blocking power of the groups P and Q, whereas the nucleolus does not. Note that the core of v is a singleton (consisting of the nucleolus) in this case, thus neither the modified least core nor the least core of the dual game intersect the core.

3. Let $m^1 = (5, 10, 6), m^2 = (2, 4, 10) \in \mathbb{R}^3$, and (N, v) with $N = \{1, 2, 3\}$ be the corresponding M2-game. The (modified) nucleoli can be computed as

$$\nu(v) = (2, 4.5, 9.5),\; \nu(v^*) = (3, 5, 8),\; \psi(v) = \psi(v^*) = \psi = (2.5, 5.5, 8).$$

By Theorem 4.6 (see below) the maximal excesses of the nucleoli with respect to v^* have to coincide. Indeed,

$$\mu(\nu(v^*), v^*) = \mu(\psi, v^*) = 2$$

holds true. Moreover, $\nu(v)$ is and has to be a member of the core of v, because v is a linear production game and $\nu(v)$ is a core selector for balanced games (games with nonvoid core). The maximal excesses can be computed as

$$\mu(\nu(v), v^*) = 3.5 > 2,\; \mu(\nu(v^*), v) = 1 > 0,\; \mu(\psi, v) = 0.5.$$

Therefore

$$\mathcal{LC}(v) \cap \mathcal{MLC}(v) = \text{Core}(v) \cap \mathcal{MLC}(v) = \emptyset.$$

Moreover, $\nu(v^*) \notin \mathcal{MLC}(v)$, because

$$\mu(\nu(v^*), v) = 1 > 0.5 = \mu(\psi, v).$$

The observations $v^*(\{1\}) = 2$, $v^*(\{2\}) = 5$, $v^*(\{3\}) = 10$, and $v^*(N) = 16$, show that v^* does not possess any imputation.

4. Let (N, v) be the 4-person assignment game given by the matrix $A = \begin{pmatrix} 32 & 40 \\ 4 & 8 \end{pmatrix}$.
Here P is the set $\{1, 2\}$ of "row-players" and Q is the set $\{3, 4\}$ of "column-players". Lemma 2.3 can be used to check that

$$\nu(v) = (32, 2, 2, 8),\; \nu(v^*) = (18, 4, 11, 11),\; \text{and } \psi(v) = (22, 0, 7, 15)$$

holds true. Moreover, a straightforward computation shows that

$$\mu(\nu(v),v)=0,\ \mu(\nu(v^*),v)=11, \mu(\psi(v),v)=3, \text{ and}$$
$$\mu(\nu(v^*),v^*)=\mu(\psi(v),v^*)=22,$$

hence the modified least core is a proper subset of the least core of the dual game and it does not intersect the nonempty core of the game.

3. *C-convex games*

In this section a new class of cooperative games called "complementary convex (c-convex)" is introduced. Assignment games as well as M2-games turn out to be c-concave. Moreover, it is shown that each element of the least core treats a certain coalition in the same manner as its complement, i.e., the aggregate amounts which are assigned to both coalitions behave in such a way that the resulting excesses coincide.

Definition 3.1: Let (P,Q) be a partition of N. The game (N,v) is *complementary convex (c-convex)* with respect to (P,Q), iff

(1) $$v(S)+v(T)\le v\big((S\cap T)\cap P+(S\cup T)\cap Q\big)+v\big((S\cup T)\cap P+(S\cap T)\cap Q\big),$$

for $S,T\subseteq N$.

Like in the definition of classical convexity the c-convexity property can be expressed in terms of increasing marginal contributions of players. This characterization enables us to show that M2-games as well as assignment games are c-concave, meaning that the opposite inequality of (1) holds true, with respect to a natural partition of the player set. The details are formulated in the following Lemma.

Lemma 3.2: Let (N,v) be a game and (P,Q) be a partition of N. Then the following properties are equivalent.

1. v is c-convex with respect to (P,Q).
2.

(2) $$v(S+\{i\})-v(S)\le v(T+\{i\})-v(T) \text{ for } i\in P\setminus T,$$

(3) $$v(T+\{j\})-v(T)\le v(S+\{j\})-v(S) \text{ for } j\in Q\setminus S,$$

hold true for $S,T\subseteq N$ with $S\cap P\subseteq T,\ T\cap Q\subseteq S$.

3. For $S\subseteq N,\ i,i_0\in P\setminus S,\ j,j_0\in Q\setminus S,\ i\ne i_0,\ j\ne j_0$ the following properties hold:

(4) $$v(S+\{i,j\})-v(S+\{j\})\le v(S+\{i\})-v(S),$$

(5) $$v(S+\{i,j\})-v(S+\{i\})\le v(S+\{j\})-v(S),$$

(6) $$v(S+\{i,i_0\})-v(S+\{i\})\ge v(S+\{i_0\})-v(S),$$

(7) $$v(S+\{j,j_0\})-v(S+\{j\})\ge v(S+\{j_0\})-v(S).$$

***Proof*:** To verify that 1 implies 2 and 2 implies 3 is straightforward and therefore skipped.

3 implies 2: Let $S\cap P\subseteq T,\ T\cap Q\subseteq S,\ i\in P\setminus T,\ j\in Q\setminus S$. Therefore there are nonnegative integers k,r and $i_1,\ldots,i_k\in P,\ j_1,\ldots,j_r\in Q$ such that $S+\{i_1,\ldots,i_k\}=T+\{j_1,\ldots,j_r\}=S\cup T$. Inequalities (6) and (4) directly imply

$$\begin{aligned}
& v(S+\{i\})-v(S)\\
\le\ & v(S+\{i,i_1\})-v(S+\{i_1\}) \quad \text{(by (6))}\\
\le\ & \cdots\\
\le\ & v(S+\{i,i_1,\ldots,i_k\})-v(S+\{i_1,\ldots,i_k\}) \quad \text{(by (6))}\\
\le\ & v\big((S\setminus\{j_1\}+\{i,i_1,\ldots,i_k\}\big)-v\big((S\setminus\{j_1\})+\{i_1,\ldots,i_k\}\big) \quad \text{(by (4))}\\
\le\ & \cdots\\
\le\ & v(T+\{i\})-v(T), \quad \text{(by (4))}
\end{aligned}$$

hence 2 is verified. Analogously, inequalities (5) and (7) directly imply 3.

2 implies 1: Let $\tilde S,\tilde T\subseteq N$ satisfy $\tilde S\cap P\subseteq\tilde T,\ \tilde T\cap Q\subseteq\tilde S$ and take $\tilde P\subseteq P\setminus\tilde T,\ \tilde Q\subseteq Q\setminus\tilde S$. Successive application of (2) and (3) respectively show that

(8) $$v(\tilde S+\tilde P)-v(\tilde S)\le v(\tilde T+\tilde P)-v(\tilde T),$$

(9) $$v(\tilde T+\tilde Q)-v(\tilde T)\le v(\tilde S+\tilde Q)-v(\tilde S)$$

hold. Adding (8) and (9) yields

(10) $$v(\tilde S+\tilde P)+v(\tilde T+\tilde Q)\le v(\tilde S+\tilde Q)+v(\tilde T+\tilde P).$$

Take $S,T\subseteq N$ and define

$$\tilde S=(S\cap T)\cap P+S\cap Q,\quad \tilde P=(S\setminus T)\cap P,$$
$$\tilde T=T\cap P+(S\cap T)\cap Q,\quad \tilde Q=(T\setminus S)\cap Q.$$

Indeed, $\tilde{S} \cap P \subseteq \tilde{T},\ \tilde{T} \cap Q \subseteq \tilde{S},\ \tilde{P} \subseteq P \setminus \tilde{T},\ \tilde{Q} \subseteq Q \setminus \tilde{S}$, hence

$$\begin{aligned} v(S) + v(T) &= v(\tilde{S} + \tilde{P}) + v(\tilde{T} + \tilde{Q}) \quad \text{(by definition)} \\ &\leq v(\tilde{S} + \tilde{Q}) + v(\tilde{T} + \tilde{P}) \quad \text{(by (10))} \\ &= v\big((S \cap T) \cap P + (S \cup T) \cap Q\big) + v\big((S \cup T) \cap P + (S \cap T) \cap Q\big). \end{aligned}$$

Thus 1 is valid. *q.e.d.*

Lemma 3.3:
1. If (N,v) is an assignment game with respect to (P,Q), then v is c-concave with respect to (P,Q).
2. Let (N,v) be an M2-game defined by the vectors $m^1, m^2 \in \mathbb{R}^N$. Then v is c-concave with respect to any (P,Q) satisfying

$$\{i \in N \mid m_i^1 > m_i^2\} \subseteq P \subseteq \{i \in N \mid m_i^1 \geq m_i^2\}$$

and $Q = N \setminus P$.

Proof: ad 1: In Shapley (1962) it is shown that an assignment game v satisfies all inequalities opposite to (4) - (7). Hence v^* satisfies these inequalities and v^* is c-convex with respect to (P,Q) by Lemma 3.2.

ad 2: Let $S,T \subseteq N$. The inequalities

$$m^i(S) + m^i(T) = m^i\big((S \cap T) \cap P + (S \cup T) \cap Q\big) + m^i\big((S \cup T) \cap P + (S \cap T) \cap Q\big)$$

for $i = 1,2$ and

$$\begin{aligned} & m^1(S) + m^2(T) \\ = \;& m^1\big((S \cap T) \cap P + (S \cup T) \cap Q\big) + m^2\big((S \cup T) \cap P + (S \cap T) \cap Q\big) \\ & + \Big(m^1\big((S \setminus T) \cap P\big) - m^2\big((S \setminus T) \cap P\big)\Big) + \Big(m^2\big((T \setminus S) \cap Q\big) - m^1\big((T \setminus S) \cap Q\big)\Big) \\ \geq \;& m^1\big((S \cap T) \cap P + (S \cup T) \cap Q\big) + m^2\big((S \cup T) \cap P + (S \cap T) \cap Q\big) \\ & \text{(by definition of } P,Q) \end{aligned}$$

directly imply c-concavity with respect to (P,Q) for M2-games. *q.e.d.*

Remark 3.4:
1. A game (N,v) is (weakly) c-convex with respect to every partition (P,Q) of N, iff v is additive. Indeed, if v is additive, then inequality (1) is an equality. Conversely, assume that v is not additive, hence there are coalitions $S,T \subseteq N$ with $v(S) + v(T) \neq v(S \cap T) + v(S \cup T)$. Two cases may occur. If $v(S) + v(T) < v(S \cap T) + v(S \cup T)$, then v is not c-convex with respect to $(S \setminus T, (N \setminus S) \cup T)$. In case the opposite inequality holds, $P = S \cup T,\ Q = N \setminus P$ shows the assertion.
2. Any convex game (N,v), i.e., a game v satisfying

$$v(S) + v(T) \leq v(S \cap T) + v(S \cup T) \text{ for } S,T \subseteq N$$

is c-convex with respect to $(N, \emptyset)$.
3. It is easy to verify that the simple majority game with 3 persons cannot be c-convex with respect to any partition of the player set. At the end of this section we show that a weighted majority game satisfies c-convexity if and only if it is a "composition" of at most one winning player and a unanimity game.
4. Note that if (N,v) is c-convex with respect to (P,Q), then the dual game (N,v^*) is c-concave with respect to (P,Q).

Let (N,v) be a c-convex game w.r.t. (P,Q) for some nonvoid disjoint sets P and Q. The following notation is frequently used:

$$\gamma(v) = \frac{v(P) + v(Q) - v(N)}{2},\ \alpha(v) = \frac{v(P) - v(Q) + v(N)}{2},\ \beta(v) = \frac{v(Q) - v(P) + v(N)}{2}.$$

Note that $\gamma(v)$ is a lower bound for the maximal excess of v at an arbitrary preimputation. Indeed, if $x \in X(v)$, then

$$\begin{aligned} 2\mu(x,v) &\geq \big(v(P) - x(P)\big) + \big(v(Q) - x(Q)\big) = v(P) + v(Q) - x(N) \\ &= v(P) + v(Q) - v(N) \qquad \text{(by } x \in X(v)) \\ &= 2\gamma(v). \end{aligned} \tag{11}$$

Moreover, (11) shows that there is a nontrivial coalition (P or Q) of a nonnegative excess, because $v(N) = v(N) + v(\emptyset) \leq v(P) + v(Q)$ (by c-convexity), hence $\gamma(v) \geq 0$. The next lemma shows that there are preimputations which guarantee that $\gamma(v)$ is the highest possible excess. This means that the excesses of both coalitions P and Q coincide at every element of the least core. Coalition P and its complement Q are treated equally and no other coalition possesses a higher excess.

Lemma 3.5: If $N,v)$ is c-convex, then $\mathcal{LC}(v) = \{x \in X(v) \mid \mu(x,v) = \gamma(v)\}$.

Proof: In view of (11) it is sufficient to show that $\{x \in X(v) \mid \mu(x,v) = \gamma(v)\} \neq \emptyset$. Define (P,u) by

$$\begin{aligned} u(S) &= \max\{v(S), v(S+Q) - \beta(v)\} - \gamma(v) \\ &= \max\{v(S) - \gamma(v), v(S+Q) - v(Q)\} \text{ for } S \subseteq P. \end{aligned}$$

Observe that $u(\emptyset) = \max\{-\gamma(v), 0\} = 0$, hence (P,u) is a game. Moreover,

$$\begin{aligned} u(P) &= \max\{v(P) - \gamma(v), v(N) - v(Q)\} \\ &= \max\{\alpha(v), v(N) - v(Q)\} = \alpha(v) \end{aligned}$$

holds true. The last equality is satisfied, because $\alpha(v) - \big(v(N) - v(Q)\big) = \gamma(v) \geq 0$.

Claim: u is convex. Take $S,T \subseteq P$ and distinguish the following 4 cases:

1. $u(S) = v(S) - \gamma(v)$, $u(T) = v(T) - \gamma(v)$. Then

$$\begin{aligned} u(S) + u(T) &= v(S) + v(T) - 2\gamma(v) \\ &\leq v(S \cap T) + v(S \cup T) - 2\gamma(v) \text{ (by c-convexity)} \\ &\leq u(S \cap T) + u(S \cup T). \end{aligned}$$

2. $u(S) = v(S+Q) - v(Q)$, $u(T) = v(T+Q) - v(Q)$. Then

$$\begin{aligned} & u(S) + u(T) \\ = \ & v(S+Q) + v(T+Q) - 2v(Q) \\ \leq \ & \Big(v\big((S \cap T) + Q\big) - v(Q)\Big) + \Big(v\big((S \cup T) + Q\big) - v(Q)\Big) \text{ (by c-convexity)} \\ \leq \ & u(S \cap T) + u(S \cup T). \end{aligned}$$

3. $u(S) = v(S) - \gamma(v)$, $u(T) = v(T+Q) - v(Q)$. Then

$$\begin{aligned} u(S) + u(T) &= v(S) + v(T+Q) - \gamma(v) - v(Q) \\ &\leq \Big(v\big((S \cap T) + Q\big) - v(Q)\Big) + \Big(v(S \cup T) - \gamma(v)\Big) \text{ (by c-convexity)} \\ &\leq u(S \cap T) + u(S \cup T). \end{aligned}$$

4. The remaining case $u(S) = v(S+Q) - v(Q)$, $u(T) = v(T) - \gamma(v)$ can be solved analogously to the third case by interchanging the rôles of S and T.

Take any $x \in \text{Core}(u) = \{x \in X(u) \mid e(S,x,u) \leq 0 \text{ for } S \subseteq P\}$. Such x exists by the convexity of u. Now define a game (Q,w) on Q, which depends on the choice of x, by $w(S) = \max\{v(R+S) - x(R) \mid R \subseteq P\} - \gamma(v)$ for $S \subseteq Q$. Indeed, $w(\emptyset) = \max_{R \subseteq P}(v(R) - x(R)) - \gamma(v) \leq \max_{R \subseteq P}(u(R) - x(R)) = 0$, because $x \in \text{Core}(u)$. On the other hand, $w(\emptyset) \geq v(P) - x(P) - \gamma(v) = 0$, thus $w(\emptyset) = 0$. Moreover,

$$\begin{aligned} w(Q) &= \max_{R \subseteq P} v(R+Q) - x(R) - \gamma(v) \leq \max_{R \subseteq P} u(R) - x(R) + \beta(v) \leq \beta(v), \\ w(Q) &\geq v(Q) - \gamma(v) = \beta(v), \end{aligned}$$

thus $w(Q) = \beta(v)$. Again, a verification of convexity of w is straightforward and skipped. Take any $y \in \text{Core}(w)$ and define $z \in \mathbb{R}^N$ by

$$z_i = \begin{cases} x_i, & i \in P \\ y_i, & i \in Q \end{cases}.$$

Then z is a preimputation, because

$$v(N) = \alpha(v) + \beta(v) = x(P) + y(Q) = z(N).$$

Moreover, observe that

$$\begin{aligned} v(S) - z(S) &\leq \max_{R \subseteq P} v\big(R + (S \cap Q)\big) - x(R) - y(S \cap Q) \\ &= w(S \cap Q) - y(S \cap Q) + \gamma(v) \leq \gamma(v) \quad \text{(by } y \in \text{Core}(w)) \end{aligned}$$

holds; thus $e(S,z,v) \leq \gamma(v)$ for $S \subseteq N$. *q.e.d.*

Remark 3.6: Almost all weighted majority games are not c-convex at all. Here a weighted majority game (N,v) is a simple game (i.e., $v(S) = 1$ or $v(S) = 0$ for all $S \subseteq N$) possessing a representation $(\lambda; m)$:

$$\lambda > 0,\ m \in \mathbb{R}^N,\ m \geq 0,\ m(N) \geq \lambda,$$

$$v(S) = \begin{cases} 1, & \text{if } m(S) \geq \lambda \\ 0, & \text{otherwise} \end{cases} \quad \text{for } S \subseteq N.$$

A weighted majority game is monotone, i.e. $v(S) \leq v(T)$ if $S \subseteq T \subseteq N$, hence v is determined by its set of minimal winning coalitions

$$W_v^m = \{S \subseteq N \mid v(S) = 1 \text{ and } v(T) = 0 \text{ for } T \subset S\}.$$

The set of nullplayers is denoted $D(v) = N \setminus \bigcup_{S \in W_v^m} S$. A weighted majority game (N, v) is c-convex, iff v is a composition of at most one winning player i_0 and a unanimity game, i.e. $N \setminus D(v) \in W_v^m$ or there is a "winning" player $i_0 \in N$ such that $\{i_0\} \in W_v^m$ and $N \setminus (\{i_0\} \cup D(v)) \in W_v^m$.

In the first case v is c-convex with respect to any partition (P, Q) satisfying $P \subseteq D(v)$ and in the second case with respect to any partition (P, Q) satisfying $\{i_0\} \subseteq P \subseteq \{i_0\} \cup D(v)$.

A proof of this assertion is as follows:

The proof that each composition of at most one winning player and a unanimity game is c-convex in the desired sense is straightforward and therefore omitted. Conversely, assume that (N, v) is a weighted majority game with representation $(\lambda; m)$ which is c-convex with respect to (P, Q). If $P \supseteq N \setminus D(v)$, then $S \subseteq P$ for $S \in W_v^m$, hence $\{P \setminus D(v)\} = W_v^m$ by c-convexity. In the remaining case, i.e., $P \cap (N \setminus D(v)) \neq \emptyset \neq Q \cap (N \setminus D(v))$, we proceed as follows:

(a) There is no $S \subseteq W_v^m$ with $S \cap P \neq \emptyset$ and $S \cap Q \neq \emptyset$. Conversely assume there is a minimal winning coalition S intersecting both P and Q. Then

$$1 = v(S) + v(\emptyset) > 0 = v(S \cap P) + v(S \cap Q),$$

a contradiction.

By (a) a minimal winning coalition is either contained in $S = P \setminus D(v)$ or in $T = Q \setminus D(v)$ and both S and T contain at least one minimal winning coalition by the assumption.

(b) $S, T \in W_v^m$. Conversely assume without loss of generality $S \notin W_v^m$. Hence there is $S^0 \subset S$ with $S^0 \in W_v^m$. Take $i \in S \setminus S^0$ and observe there is $S^1 \in W_v^m$ with $i \in S^1$, because $i \notin D(v)$. Therefore $S^1 \subseteq P$ and

$$\begin{aligned} 2 &= v(S^0) + v(S^1) > 1 = v(S^0 \cup S^1) \\ &= v\big((S^0 \cup S^1) \cap P + (S^0 \cap S^1) \cap Q\big) + v\big((S^0 \cap S^1) \cap P + (S^0 \cup S^1) \cap Q\big), \end{aligned}$$

a contradiction.

Assume without loss of generality $N = \{1, \ldots, n\}$, $m_1 \geq \cdots \geq m_n$, and $1 \in P$.

(c) There is $i_0 \in N$ such that $S = \{1, \ldots, i_0\}$. Conversely assume there is $i \in N \setminus S$ with $i + 1 \in S$. With $\tilde{S} = S \cap \{1, \ldots, i-1\}$ we have

$$m(\tilde{S}) < \lambda \leq m(\tilde{S} + \{i, \ldots, n\}).$$

Let r be minimal such that $\lambda \leq m(\tilde{S} + \{i, \ldots, r\})$ and observe that $\tilde{S} + \{i, \ldots, r\}$ intersects both, P and Q, and is a minimal winning coalition, hence a contradiction is established in this case.

(d) $i_0 = 1$. Conversely assume $i_0 > 1$. Again there is a minimal r with

$$m\big((S \setminus \{i_0\}) \cup \{i_0 + 1, \ldots, r\}\big) \geq \lambda,$$

hence $(S \setminus \{i_0\}) \cup \{i_0 + 1, \ldots, r\}$ intersects both, P and Q, and is a minimal winning coalition.

Summarizing we have shown that $v(\{1\}) = 1$ and $W_v^m = \{\{1\}, T\}$, hence v has the desired properties. *q.e.d.*

4. Equal treatment of P and Q for assignment and M2-games

It is the aim of this section to show that the modified least core of an assignment game or an M2-game is a subset of the least core of the corresponding dual game if both P and Q are nonempty. This means that all coalitions of *lowest* excess have it as high as possible. In view of Section 3 equal treatment of P and Q is a direct consequence.

If P or Q are empty, i.e., if the dual game is convex, then the modified least core is contained in the core of the dual game as shown in Sudhölter (1997). The following lemmata will be used in the proofs of Theorems 4.5 and 4.6. Let

$$\mathcal{D}(x, v) = D(x, \mu(x, v), v)$$

(for the definitions of $D(\cdot, \cdot, \cdot)$ and $\mu(\cdot, \cdot)$ we refer to Section 2) for a game (N, v) and $x \in \mathbb{R}^N$ denote the set of coalitions of maximal excess.

Lemma 4.1: Let (N, v) be a c-convex game w.r.t. (P, Q) and $x \in \mathbb{R}^N$. Then

1. $e(S, x, v) + e(T, x, v) \leq$
   $e\big((S \cap T) \cap P + (S \cup T) \cap Q, x, v\big) + e\big((S \cup T) \cap P + (S \cap T) \cap Q, x, v\big)$
   for $S, T \subseteq N$;
2. If $S, T \in \mathcal{D}(x, v)$, then $(S \cap T) \cap P + (S \cup T) \cap Q$ and $(S \cup T) \cap P + (S \cap T) \cap Q$ are members of $\mathcal{D}(x, v)$;
3. $$\begin{aligned} S^L &= \textstyle\bigcap_{S \in \mathcal{D}(x,v)} S \cap P + \bigcup_{S \in \mathcal{D}(x,v)} S \cap Q \in \mathcal{D}(x, v), \\ S^R &= \textstyle\bigcup_{S \in \mathcal{D}(x,v)} S \cap P + \bigcap_{S \in \mathcal{D}(x,v)} S \cap Q \in \mathcal{D}(x, v). \end{aligned}$$

For classical convex games, i.e., $P = \emptyset$ or $Q = \emptyset$, property 2 of Lemma 4.1 is the near-ring property (see Maschler, Peleg, and Shapley (1972)). Therefore a set of coalitions satisfying 2 of Lemma 4.1 is called *c-near-ring* here.

Proof: Assertion 1 is a direct consequence of the definition of c-convexity. Assertion 2 is directly implied by 1, whereas 2 implies 3. *q.e.d.*

Lemma 4.2: Let $x \in \mathcal{MLC}(v)$ for some c-convex game (N,v) w.r.t. (P,Q). Then

$$(P \subseteq S^R \text{ and } Q \subseteq S^L) \text{ or} \tag{12}$$

$$(P \supseteq S^R \text{ and } Q \supseteq S^L) \tag{13}$$

where S^R, S^L are defined as in Lemma 4.1.

Proof: Assume the contrary. By (12) we have $P \not\subseteq S^R$ or $Q \not\subseteq S^L$. Without loss of generality $P \not\subseteq S^R$ (otherwise interchange the rôles of P and Q). Two cases may occur.

Case 1: $S^R \subset P$. Then $S^L \cap P \neq \emptyset$ (see (13)), but $S^L \cap P \subseteq S \cap P \subseteq S^R$ for $S \in \mathcal{D}(x,v)$ by definition of S^L and S^R. Take $i \in S^L \cap P$, $j \in P \setminus S^R$, and a sequence $(\delta_{(S,T)})_{(S,T)\in\tilde{D}}$ of weakly balancing coefficients for $\tilde{D} = \tilde{D}(x, \mu(x,v) + \mu(x,v^*), v) = \mathcal{D}(x,v) \times \mathcal{D}(x,v^*)$, i.e.

$$\delta_{(S,T)} \geq 0 \text{ and } \sum_{(S,T)\in\tilde{D}} \delta_{(S,T)}(1_S + 1_T) = 1_N. \tag{14}$$

For the existence of a weakly balancing sequence see Lemma 2.3. For $S \in \mathcal{D}(x,v)$ and $T \in \mathcal{D}(x,v^*)$ define $\delta_S = \sum_{T\in\mathcal{D}(x,v^*)} \delta_{(S,T)}$, $\delta^*_T = \sum_{S\in\mathcal{D}(x,v)} \delta_{(S,T)}$, hence $\delta_S \geq 0 \leq \delta^*_T$. Thus (14) can be rewritten to

$$\sum_{S\in\mathcal{D}(x,v)} \delta_S 1_S + \sum_{T\in\mathcal{D}(x,v^*)} \delta^*_T 1_T = 1_N, \tag{15}$$

$$\sum_{S\in\mathcal{D}(x,v)} \delta_S = \sum_{T\in\mathcal{D}(x,v^*)} \delta^*_T. \tag{16}$$

Therefore (15), applied to i, and the fact $i \in S$ for $S \in \mathcal{D}(x,v)$ implies

$$\sum_{S\in\mathcal{D}(x,v)} \delta_S \leq 1, \tag{17}$$

whereas (15), applied to $j \notin S$ for $S \in \mathcal{D}(x,v)$, implies

$$\sum_{T\in\mathcal{D}(x,v^*)} \delta^*_T \geq 1. \tag{18}$$

Now (16),(17),(18) are simultaneously true, thus

$$\sum_{S\in\mathcal{D}(x,v)} \delta_S = \sum_{T\in\mathcal{D}(x,v^*)} \delta^*_T = 1. \tag{19}$$

Define $\bar{D} := \{T \in \mathcal{D}(x,v^*) \mid \delta^*_T > 0\}$, hence $\bar{D} \neq \emptyset$ by (19). Equations (15),(19) together with $S \cap Q \in S^L$ for $S \in \mathcal{D}(x,v)$ imply

$$Q \setminus S^L \subseteq T \text{ for } T \in \bar{D}. \tag{20}$$

Claim:

$$T \cap (S^L \cap P) \neq \emptyset \text{ for } T \in \bar{D}. \tag{21}$$

If, on the contrary, (21) is not valid, then $P \cap S^L \subseteq U$, $U \cap Q \subseteq S^L \cap Q$, where $U = N \setminus T$ is a coalition of minimal excess at x with respect to v (see Section 2). Lemma 4.1 directly implies

$$e(S^L, x, v) + e(U \setminus (S^L \cap P), x, v) \leq e(S^L \cap Q, x, v) + e(U, x, v),$$

but $S^L \cap Q \notin \mathcal{D}(x,v)$ by definition of S^L, hence

$$e(U \setminus (S^L \cap P), x, v) < e(U, x, v),$$

a contradiction against the fact that the excess of U is minimal.

Take $T \in \bar{D}$ and $i \in T \cap (S^L \cap P)$. Then by (19) and (15), applied to player i, we come up with

$$1 \geq \sum_{S\in\mathcal{D}(x,v)} \delta_S + \delta_T = 1 + \delta_T > 1,$$

which is impossible.

Case 2: $S^R \cap Q \neq \emptyset$. Then $S^R \cap P \subset P$ by the assumption. Moreover, $S^R \cap Q \subseteq S$ for $S \in \mathcal{D}(x,v)$. The same procedure as in Case 1 establishes a contradiction. Indeed, using the notation of Case 1 each $T \in \bar{D}$ contains $P \setminus S^R$, hence intersects $S^R \cap Q$. *q.e.d.*

Up to the end of this section let P and Q be finite disjoint nonvoid sets. In what follows one interesting common property of many classical solution concepts for cooperative games is described.

Definition 4.3: Let $x \in \mathbb{R}^N$ and (N,v) be a game (not necessarily c-convex). Then x is said to be *reasonable (on both sides)*, if each component of x is bounded from below by the minimal and from above by the maximal marginal contribution of the corresponding player, i.e., if

$$\min_{S\subseteq N\setminus\{i\}} v(S+\{i\}) \le x_i \le \max_{S\subseteq N\setminus\{i\}} v(S+\{i\})$$

for $i \in N$.

It is well-known that, e.g., the Shapley value and each element of the least core of a game are reasonable. In Sudhölter (1996) it is verified that each element of the modified least core is reasonable, too. Nevertheless, a proof is given below.

Lemma 4.4: Let $x \in \mathcal{MLC}(v)$ for some game (N, v). Then x is reasonable.

Proof: Assume, on the contrary, there is $x \in \mathcal{MLC}(v)$ being not reasonable. For $S \subseteq N \setminus \{i\}$ we have

$$v(S+\{i\}) - v(S) = v^*(N\setminus S) - v^*\big((N\setminus S)\setminus\{i\}\big),$$

hence

$$g_i = \min_{S\subseteq N\setminus\{i\}} v(S+\{i\}) - v(S) = \min_{S\subseteq N\setminus\{i\}} v^*(S+\{i\}) - v^*(S)$$

and

$$h_i = \max_{S\subseteq N\setminus\{i\}} v(S+\{i\}) - v(S) = \max_{S\subseteq N\setminus\{i\}} v^*(S+\{i\}) - v^*(S)$$

for $i \in N$.
Case 1: $x_i > h_i$ for some $i \in N$. Then, for $S \subseteq N$ with $i \in S$,

$$e(S,x,v) < e(S\setminus\{i\},x,v),\ e(S,x,v^*) < e(S\setminus\{i\},x,v^*),$$

hence $i \notin S$ for $S \in \mathcal{D}(x,v) \cup \mathcal{D}(x,v^*)$. Therefore $\mathcal{D}(x,v) \times \mathcal{D}(x,v^*)$ cannot be weakly balanced, a contradiction.
Case 2: $x_i < g_i$ for some $i \in N$. A similar argument as in Case 1 shows $i \in S$ for $S \in \mathcal{D}(x,v) \cup \mathcal{D}(x,v^*)$. With weakly balancing coefficients $(\delta_{(S,T)})_{(S,T)\in\tilde{D}}$ of $\tilde{D} = \mathcal{D}(x,v) \times \mathcal{D}(x,v^*)$, i.e., $\delta_{(S,T)} \ge 0$ and $\sum_{(S,T)\in\tilde{D}} \delta_{(S,T)}(1_S + 1_T) = 1_N$, it turns out that $\sum_{(S,T)\in\tilde{D}} \delta_{(S,T)} = 1/2$ (by applying the last equality to player i), hence $N \in \mathcal{D}(x,v)$. On the other hand $e(\{i\},x,v) = v(\{i\}) - x_i > 0 = e(N,x,v)$, a contradiction. *q.e.d.*

Theorem 4.5: The modified least core of an assignment game with respect to (P,Q) is a subset of the least core of the dual game.

Proof: Let (N,u) be an assignment game with respect to (P,Q) and A the defining matrix (see 1 of Definition 2.4). Let $v = u^*$ be the corresponding dual game which is c-convex with respect to (P,Q) in view of Remark 3.4. By 1 of Remark 2.2 it suffices to show $\mathcal{MLC}(v) \subseteq \mathcal{LC}(v)$, because $\mathcal{MLC}(w) = \mathcal{MLC}(w^*)$ for arbitrary games w. Take $x \in \mathcal{MLC}(u)$ and assume, on the contrary, $x \notin \mathcal{LC}(v)$. Moreover, assume without loss of generality $x(Q) \ge \beta(v)$ (otherwise interchange the rôles of P and Q) - for the definition of $\beta(\cdot)$ see Section 3. By the assumptions $Q \notin \mathcal{D}(x,v)$, hence - in view of Lemma 4.2 - two cases may occur: $Q \subset S^L$ or $S^L \subset Q$.

1. Case: $Q \subset S^L$. Then $P \subseteq S^R$ by (12). By definition of an assignment game $u(T) = 0$ for $T \subseteq P$, i.e., $v(S) = v(N)$ for $S \supseteq Q$. Hence there is $i \in S^L \cap P$ such that $x_i < 0$, because $e(Q,x,v) < \mu(x,v) = e(S^L,x,v)$. On the other hand u, and thus v, is a monotonic game, implying

$$v(S+\{i\}) - v(S) \ge 0 \text{ for } S \subseteq N \setminus \{i\},$$

hence $x_i \ge 0$ by reasonableness of x (see Lemma 4.4). These considerations imply a contradiction in this case.

2. Case: $S^L \subset Q$. Hence $S^R \subseteq P$ (by (13)). For $S \subseteq N$, $i \in P \setminus S$, $j \in Q \setminus S$ it is well-known that

$$u(S+\{i,j\}) \ge u(S) + a_{ij} \tag{22}$$

holds true. Moreover, let $\sigma(S)$ denote the set of *assignments* of S, i.e.,

$$\sigma(S) = \{(i_k,j_k)_{k=1}^t \mid i_k \in S\cap P, j_k \in S \cap Q, \{i_k,j_k\} \cap \{i_r,j_r\} = \emptyset\ \forall k \ne r\},$$

where $t = \min\{|S\cap P|, |S \cap Q|\}$. Then

$$u(S) = \max_{(i_k,j_k)_{k=1}^t \in\sigma(S)} \sum_{k=1}^t a_{i_k j_k}.$$

For these properties see, e.g., Shapley and Shubik (1972) or Solymosi and Raghavan (1994).

Let $T = N \setminus S^L$, hence T has minimal excess at x with respect to u. Let $j \in T \cap Q$ with $x_j > 0$. Indeed, player j exists, because otherwise $e(S^L + \{j\}, x, v) \ge e(S^L, x, v)$ (by monotonicity of v) which is impossible.

Let $U \in \mathcal{D}(x,u)$ with $j \in U$. Such U exists because $\mathcal{D}(x,v) \times \mathcal{D}(x,u)$ is weakly balanced and $j \notin S$ for $S \in \mathcal{D}(x,v)$. Take an optimal assignment $(i_k,j_k)_{k=1}^t \in \sigma(U)$ for U, i.e., $u(U) = \sum_{k=1}^t a_{i_k j_k}$. If $j \notin \{j_k \mid k = 1,\dots,t\}$, then $u(U\setminus\{j\}) = u(U)$, hence $e(U,x,u) < e(U\setminus\{j\},x,u)$, a contradiction.

If $j = j_k$ for some $1 \le k \le t$, then put $i = i_k$. Obviously $u(U \setminus \{i,j\}) = u(U) - a_{ij}$ is valid; hence

(23) $$e(U,x,u) - e(U \setminus \{i,j\},x,u) = a_{ij} - x_i - x_j \geq 0.$$

Moreover, $u(T \setminus \{i,j\}) \leq u(T) - a_{ij}$ (by (22)), thus

$$\begin{aligned} e(T \setminus \{i,j\},x,u) &\leq u(T) - a_{ij} - x(T) + x_i + x_j \\ &= e(T,x,u) - (a_{ij} - x_i - x_j) \leq e(T,x,u) \text{ (by 23)}, \end{aligned}$$

hence all inequalities are equalities. Therefore $N \setminus (T \setminus \{i,j\}) = S^L + \{i,j\}$ has maximal excess with respect to v, which contradicts the definition of S^L. *q.e.d.*

Theorem 4.6: The modified least core of an M2-game is a subset of the least core of its dual game.

Proof: Let (N,u) be an M2-game defined by $u(S) = \min\{m^1(S), m^2(S)\}$ for some $m^1, m^2 \in \mathbb{R}^N$. If $\{i \in N \mid m_i^k \geq m_i^{3-k}\} = N$ for some $k = 1,2$, then $u(S) = m^{3-k}(S)$ for $S \subseteq N$, thus u is additive and the assertion is valid by reasonableness of each $x \in \mathcal{MLC}(u)$ (see Lemma 4.4) and each $y \in \mathcal{LC}(u)$ by $u = u^*$. Therefore choose any $P \subseteq N$ with

$$\{i \in N \mid m_i^1 > m_i^2\} \subseteq P \subseteq \{i \in N \mid m_i^1 \geq m_i^2\},$$

define $Q = N \setminus P$ and assume that $P \neq \emptyset \neq Q$. Moreover, $m^1(N) \leq m^2(N)$ can be assumed without loss of generality (otherwise interchange the rôles of m^1 and m^2). Let $v = u^*$ be the dual game. Then $\mathcal{MLC}(v) = \mathcal{MLC}(u)$. Moreover, v is c-convex with respect to (P,Q) by Remark 3.4. Assume, on the contrary, there is $x \in \mathcal{MLC}(v) \setminus \mathcal{LC}(v)$. With $\epsilon = m^1(N) - m^2(N)$ (≤ 0) it is easy to verify that

(24) $$v(S) = \max\{m^1(S), m^2(S) + \epsilon\} \text{ for } S \subseteq N.$$

Moreover, for every coalition S with $P \subseteq S$

$$\begin{aligned} m^2(S) + \epsilon &= m^2(S) + m^1(N) - m^2(N) \\ = m^1(N) - m^2(N \setminus S) &\leq m^1(S) \text{ (by the choice of } P,Q), \end{aligned}$$

hence

(25) $$v(S) = m^1(S) \text{ for } S \supseteq P.$$

The fact that $m^2(S) + m^1(N) - m^2(N) = m^1(N) - m^2(N \setminus S) \geq m^1(S)$ holds true for every $S \supseteq Q$ implies

(26) $$v(S) = m^2(S) + \epsilon \text{ for } S \supseteq Q.$$

For $i \in P$ the inequalities

$$\begin{aligned} m_i^2 &\leq \min\{v(S + \{i\}) - v(S) \mid S \subseteq N \setminus \{i\}\}, \\ m_i^1 &\geq \max\{v(S + \{i\}) - v(S) \mid S \subseteq N \setminus \{i\}\}, \end{aligned}$$

are direct consequences of (24) and the definition of P,Q. Thus - by reasonableness of x -

(27) $$m_i^2 \leq x_i \leq m_i^1 \text{ for } i \in P.$$

Analogously it turns out that

(28) $$m_j^1 \leq x_j \leq m_j^2 \text{ for } j \in Q.$$

Now two cases can be distinguished:
Case 1: $x(Q) \geq \beta(v)$. Then, by Lemma 4.2, $Q \subset S^L$ or $S^L \subset Q$.
1. $Q \subset S^L$. Take $i \in S^L \cap P$. Then

(29) $$v(S^L) = m^2(S^L) + \epsilon \text{ (by (26))},$$

$$\begin{aligned} v(S^l \setminus \{i\}) &= m^2(S^L \setminus \{i\}) + \epsilon \text{ (by (26))} \\ = v(S^L) - m_i^2 &\geq v(S^L) - x_i \text{ (by (27))} \end{aligned}$$

hold true, hence $e(S^L \setminus \{i\},x,v) \geq e(S^L,x,v)$, a contradiction.
2. $S^L \subset Q$. If $v(S^L) = m^2(S^L) + \epsilon$, take $j \in Q \setminus S^L$ and observe that

$$v(S^L + \{j\}) = v(S^L) + m_j^2 \geq v(S^L) + x_j \text{ (by (28))}.$$

Thus $e(S^L + \{j\},x,v) \geq e(S^L,x,v)$, which is impossible. If $v(S^L) = m^1(S^L)$, then $v(S^L) \leq x(S^L)$ (by (28)), hence $\mu(x,v) \leq 0 \leq \gamma(v)$, a contradiction.
Case 2: $x(Q) < \beta(v)$. Then, by Lemma 4.2, $P \subset S^R$ or $S^R \subset P$.
1. $P \subset S^R$. Hence $v(S^R) = m^1(S^R)$ (by (25)) and for $j \in S^R \cap Q$

$$\begin{aligned} v(S^R \setminus \{j\}) &= m^1(S^R) - m_j^1 \text{ (by (25))} \\ &\geq v(S^R) - x_j \text{ (by (28))}, \end{aligned}$$

thus $e(S^R \setminus \{j\},x,v) \geq e(S^R,x,v)$, a contradiction.

2. $S^R \subset P$. If $v(S^R) = m^1(S^R)$, take $i \in P \setminus S^R$ and observe that

$$v(S^R + \{i\}) \geq m^1(S^R) + m^1_i \geq v(S^R) + x_i \quad \text{(by (27))},$$

hence $e(S^R \mp \{i\}, x, v) \geq \overline{e(S^R, x, v)}$, a contradiction. If $v(S^R) = m^2(S^R) + \epsilon$, then

$$\begin{aligned} v(S^R) &\leq x(S^R) + \epsilon \quad \text{(by (27))} \\ &\leq x(S^R) \quad \text{(because } \epsilon \leq 0), \end{aligned}$$

thus $\mu(x, v) \leq 0 \leq \gamma(v)$, a contradiction. *q.e.d.*

Remark 4.7: Theorem 4.5 says that in the assignment game case both groups P and Q are treated equally from each preimputation of the modified least core. Moreover, P and Q are coalitions of minimal excess.

For M2-games Theorem 4.6 yields a similar assertion. If P and Q are defined as in Lemma 3.3, then both groups have the same (and minimal) excess at each element of the modified least core.

It is not known whether the modified least core of every c-convex game (N, v) with respect to a nontrivial partition of the player set is contained in the classical least core of the game. This author conjectures that the answer should be affirmative, i.e., $\mathcal{MLC}(v) \subseteq \mathcal{LC}(v)$, which is equivalent to the assertion that both (12) and (13) simultaneously hold under the conditions of Lemma 4.2.

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