On Assignment Games

T. E. S. Raghavan*
Department of Mathematics, Statistics and Computer Science
University of Illinois at Chicago
Chicago, IL, U.S.A.
ter@uic.edu

Peter Sudhölter
Department of Economics
University of Southern Denmark at Odense
Odense, Denmark
psu@sam.sdu.dk

Abstract

This chapter surveys recent developments on some basic solution concepts, like stable sets, the core, the nucleolus and the modiclus for a very special class of cooperative games, namely assignment games with transferable utility. The existence of a stable set for assignment games is still an open problem.

1 Introduction

In this survey we concentrate on a subclass of transferable utility (TU) games called *assignment games* and on some properties of their solutions. Assignment games with side payments were introduced by Shapley and Shubik [23]. These games are models of two-sided markets. Players on one side, called sellers, supply exactly one unit of some indivisible good, say, a house in exchange for money, with players from the other side, called buyers. Each buyer has a demand for exactly one house. When a transaction between seller i and buyer j takes place, a certain profit $a_{ij} \geq 0$ accrues. The worth of a coalition is given by an assignment of the players within the coalition which maximizes the total profit of the assigned pairs. Therefore the characteristic function is fully determined by the profits of the mixed pairs.

Assignment games have a nonempty core and are simply dual optimal solutions to the associated optimal assignment problem. It is known that prices which competitively balance supply and demand correspond to elements in the core. The nucleolus, lying in the lexicographic center of the nonempty core, has

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the additional property that it satisfies each coalition as much as possible. The corresponding prices favor neither the sellers nor the buyers, and hence provide some stability for the market.

To find the nucleolus for any general cooperative game, Kohlberg [11] proposed a weighted sum minimization approach leading to a single, but extremely large linear program (LP). In order to ensure that the highest excess gets the largest weight, the second highest excess gets the second largest weight, and so on, coefficients from a very wide range must appear in the constraints (causing serious numerical accuracy problems even for 3-person games). Since all possible permutations of the coalitions must be present among the constraints, the approach enlarges the size of the LP enormously. In Owen's [16] improved version, although one has to solve a somewhat simpler minimization problem, even then the constraints grow exponential in terms of the number of players. Indeed for an n-person game he reduces the problem to a linear programming problem in $2^{n+1}+n$ variables with 4^n+1 constraints, where n is the number of players.

When solving an optimal assignment problem, the ordinary primal simplex method encounters high levels of degeneracy. It is clearly outperformed by specifically designed algorithms, such as Kuhn's [12] well-known Hungarian algorithm. Also for assignment games a method based on general linear programming is not well suited, since the combinatorial structure of the characteristic function cannot be effectively translated into a continuous problem formulation. In the spirit of combinatorial techniques for assignment problems we apply graph-theoretic techniques to replace linear programming for locating the nucleolus for assignment games.

We will survey properties that are unique for the core of assignment games. Besides the nucleolus, as a point solution concept there is yet another point solution concept for all TU cooperative games, called the modiclus. However, it should be remarked that for assignment games another classical solution concept, namely the bargaining set $\mathcal{M}_1^{(i)}$, introduced by Davis and Maschler [5] (see also [1]) simply coincides with the core for assignment games [25] (also see [7]). While the nucleolus is defined by ranking imputations lexicographically via excesses, the modiclus is defined by lexicographical ranking bi-excesses (for definitions see the next section).

2 Preliminaries

A (cooperative TU) *game* is a pair (N,v) such that $\emptyset \neq N$ is finite and $v: 2^N \to \mathbb{R}$, $v(\emptyset)=0$. A *coalition* is a nonempty subset of N and v is the *coalition function* of (N,v). A *Pareto optimal payoff vector* (*pre-imputation*) is a vector $x \in \mathbb{R}^N$ such that $x(N)=v(N)$, where $x(S)=\sum_{i\in S} x_i$ ($x(\emptyset)=0$) for every $S \subseteq N$ and every $x \in \mathbb{R}^N$ with $\sum_{i\in\emptyset} x_i = 0$ by convention. A pre-imputation x is an *imputation* if it is *individually rational*, that is, if $x_i \geq v(\{i\})$ for all $i \in N$. A game v is normalized if for any $S \subseteq T \subseteq N, v(S) \leq v(T)$. Let $X(N,v)$ and

$\mathcal{I}(N,v)$ denote the set of pre-imputations and imputations, respectively. Thus, $X(N,v)$ is a nonempty polyhedral set and $\mathcal{I}(N,v)$ is a polytope. Moreover, $\mathcal{I}(N,v) \neq \emptyset$, if and only if $\sum_{i\in N} v(\{i\}) \leq v(N)$. The *core* of (N,v), $\mathcal{C}(N,v)$, is the set of all imputations x such that

$$x(S) \geq v(S) \text{ for all } S \subseteq N. \tag{1}$$

Note that the core is always a polytope, but it may be empty even for games that have imputations. We will assume that $\mathcal{I}(N,v)$ is nonempty.

2.1 The Nucleolus and the Modiclus

Let (N,v) be a game. If $H = (h^k)_{k\in D}$ is a finite family of real-valued functions on $X = X(N,v)$ (the family of *dissatisfaction* functions) and $x \in X$, then let $\theta^H(x) \in \mathbb{R}^d$ (where $d = |D|$ denotes the cardinality of D) be the vector whose components are the numbers $h^k(x)$, $k \in D$, arranged in a nonincreasing order, that is,

$$\theta^H_t(x) = \max_{T\subseteq D, |T|=t} \min_{k\in T} h^k(x) \text{ for all } t = 1,\ldots,d.$$

Let $\geq_{lex}$ denote the lexicographical ordering on $\mathbb{R}^d$; that is, $x \geq_{lex} y$, where $x, y \in \mathbb{R}^d$, if $x = y$ or if there exists $1 \leq t \leq d$ such that $x_k = y_k$ for all $1 \leq k < t$ and $x_t > y_t$. The *nucleolus* of H, $\mathcal{N}(H)$ is defined (see [9]) by

$$\mathcal{N}(H) = \{x \in X \mid \theta^H(x) \geq_{lex} \theta^H(y) \text{ for all } y \in X\}. \tag{2}$$

Let the class H be taken to be the dissatisfactions of coalitions at any $x \in \mathbb{R}^N$ measured by $e(S,x,v) = v(S) - x(S)$ called the *excess* of S at x. Now the *pre-nucleolus* of (N,v) is defined to be the set $\mathcal{N}\left((e(S,\cdot,v))_{S\subseteq N}\right)$. Indeed the pre-nucleolus [19], $\mathcal{N}\left((e(S,\cdot,v))_{S\subseteq N}\right)$, is a singleton, abbreviated by $\nu(N,v)$. It is also called the *nucleolus* if the domain of excesses is restricted to the imputation set.

In order to define the modiclus of (N,v) we proceed similarly. Instead of the ordered vector of excesses we take the nonincreasingly ordered vector of bi-excesses. (Here the *bi-excess* of a pair (S,T), $S,T \subseteq N$, at x is the number $e^b(S,T,x,v) = e(S,x,v) - e(T,x,v)$.) The *bi-excess* can be seen as the level of *envy* of S against T at x. The *modiclus* of (N,v) is the set $\mathcal{N}\left((e^b(S,T,\cdot,v))_{S,T\subseteq N}\right)$. The modiclus denoted by $\psi(N,v)$ is a singleton [28].

Recall that the *dual* game of (N,v), is defined by $v^*(S) = v(N) - v(N\setminus S)$ for all $S \subseteq N$. Also, recall that (N,v) is

- *constant-sum* if $v(S) + v(N\setminus S)v(N)$ for all $S \subseteq N$;
- *convex* if $v(S) + v(T) \leq v(S\cap T) + v(S\cup T)$ for all $S,T \subseteq N$;
- *zero-monotonic* if $v(S\cup\{i\}) \geq v(S) + v(\{i\})$ for all $i \in N$ and all $S \subseteq N\setminus\{i\}$.

The following relationships between the modiclus and the pre-nucleolus are of interest. (See [14] and [28,29].)

Proposition 2.1. *Let (N,v) be a game.*

(1) *Let $^* : N \to N^*$ be a bijection such that $N \cap N^* = \emptyset$. If $(N \cup N^*, \widetilde{v})$ is defined by*

$$\widetilde{v}(S \cup T^*) = \max\{v(S) + v^*(T), v^*(S) + v(T)\} \text{ for all } S, T \subseteq N,$$

then $\psi_i(N,v) = \nu_i(N \cup N^*, \widetilde{v}) = \nu_{i^*}(N \cup N^*, \widetilde{v})$ for all $i \in N$.

(2) *If (N,v) is a constant-sum game, then $\psi(N,v) = \nu(N,v)$.*

(3) *If (N,v) is a convex game, then $\psi(N,v) \in \mathcal{C}(N,v)$.*

(4) *If (N,v) is zero-monotonic, then $\psi(N,v)$ and $\nu(N,v)$ are individually rational.*

If $\nu(N,v) \in \mathcal{I}(N,v)$, then $\nu(N,v)$ is the *nucleolus* of (N,v).

2.2 Assignment Games

Let $N = P \cup Q$ where P, Q is a partition of the player set N into two types of players called sellers and buyers respectively. The players could also be colleges and students or even men and women dating. From now on we will stick to calling them sellers and buyers. Neither sellers nor buyers have any interest in mutual cooperation among themselves. Suppose each seller owns an indivisible object, say, a house which he values as worth at least c_i to him. Each buyer j has a ceiling price b_{ij} for the house of seller i. For any $i \in P, j \in Q$ the coalitional worth of the seller-buyer pair $\{i,j\}$ is taken to be $v(\{i,j\}) = a_{ij} = \max(b_{ij} - c_i, 0)$. Any arbitrary coalition $S \subseteq N$ decomposes into sellers S_1 and buyers S_2. Here if $|S_1| \neq |S_2|$ then by introducing either dummy sellers or dummy buyers if necessary we can assume $|S_1| = |S_2|$. We will take $a_{ij} = 0$ if i or j is a dummy player, namely a dummy seller or a dummy buyer respectively. Thus assuming $|S_1| = |S_2|$, let σ_S denote any arbitrary bijection $\sigma_S : S_1 \to S_2$. Given the coalition S and matrix A for player set $N = P \cup Q$ we define the assignment game with characteristic function given by

$$v_A(S) = \max_{\sigma_S} \sum_{i \in S_1} a_{i\sigma_S(i)}.$$

If $S \subseteq T, v_A(S) \leq v_A(T)$ and $v_A(\{i\}) = 0$ for all $i \in N$. Thus the pre-nucleolus is the same as its nucleolus. The following theorem is due to Shapley and Shubik [23].

Theorem 2.1. *The game (N, v_A) has a nonempty core. The worth of the grand coalition N of v_A is given by the following linear program:*

$$\begin{aligned}
&\max \textstyle\sum_{k \in P} \sum_{\ell \in Q} a_{k\ell} x_{k\ell} \\
&\textit{subject to} \\
&\textstyle\sum_{\ell \in Q} x_{\widetilde{k}\ell} \leq 1, \\
&\textstyle\sum_{k \in P} x_{k\widetilde{\ell}} \leq 1, \\
&x_{\widetilde{k}\widetilde{\ell}} \geq 0, \quad \forall\, \widetilde{k} \in P, \widetilde{\ell} \in Q.
\end{aligned} \tag{3}$$

The core of the game consists of dual optimal solutions to this linear programming problem. The core of the subgame (S, v) (that is, defined by $v(T) = v_A(T)$ for all $T \subseteq S$) is the set of optimal solutions of the dual program. Hence, (N, v_A) is totally balanced*, that is, (N, v_A) and each of its subgames (S, v_A), $\emptyset \neq S \subseteq N$, have nonempty cores and thus $\nu(N, v_A) \in \mathcal{C}(N, v_A)$.*

The following observation was made by Sudhölter [30].

Proposition 2.2. *Given an assignment game (N, v_A) with sellers P and buyers Q the modiclus of (N, v_A) treats P and Q equally, that is, $\psi(P) = \psi(Q)$ where $\psi = \psi(P \cup Q, v_A)$.*

Example 2.1. [Glove Game] Let $P = \{1, \ldots, p\}$, $Q = \{1, \ldots, q\}$, $p \leq q$, let $A = (a_{k\ell})_{k \in P, \ell \in Q}$ be given by $a_{k\ell} = 1$, and let $v = v_A$. Then $v(S) = \min\{|S \cap P|, |S \cap Q|\}$ for all $S \subseteq N$. Moreover, let $\nu = \nu(N, v)$ and $\psi = \psi(N, v)$. If $p = q$, then $\nu_i = 1/2 = \psi_i$ for all $i \in N$. If $p < q$, then $\nu_k = 1$ for all $k \in P$ and $\nu_\ell = 0$ for all $\ell \in Q$. Proposition 2.2 and the well-known equal treatment property yield $\psi_k = \frac{1}{2}$ for $k \in P$ and $\psi_\ell = \frac{p}{2q}$ for $\ell \in Q$. Hence, $\psi \in \mathcal{C}(N, v)$ if and only if $p = q$.

Let $(P \cup Q, v_A)$ be an assignment game. As all our solution concepts satisfy the strong null-player property, we shall always assume that $|P| = |Q|$. Moreover, our solution concepts are anonymous. Hence, we shall always assume that

$$P = \{1, \ldots, p\}, Q = \{1', \ldots, p'\}, \text{ and } v_A(N) = \sum_{i=1}^{p} a_{ii'}, \tag{4}$$

that is, an optimal assignment for N is attained at the *diagonal* $\{\{i, i'\} \mid i = 1, \ldots, p\}$.

3 Core Stability and Related Concepts

It was von Neumann and Morgenstern [32] who first introduced the notion of a *stable set*. Stable sets are characterized by the notions of internal stability and external stability. The two definitions hinge on comparing pairs of imputations for a game (N, v). We say imputation x dominates imputation y via coalition $S (x \succ_S y)$ if $x_i > y_i, i \in S$, and $\sum_{i \in S} x_i \leq v(S)$. Intuitively players in coalition S object to their share according to y when they have a better share of the grand coalitional worth according to x which is not a dream, but is within their reach. A set $V \subseteq \mathcal{I}(N, v)$ is called *internally stable* if no imputations in V can dominate another imputation in V. Further the set V is *externally stable* if any imputation not in V is dominated by some imputation in V via a coalition. A set V is called *stable* for a game (N, v) if V is both internally and externally stable.

Since the core when it exists is a polyhedral set, the problem of existence is simply reduced to the existence of solution for a system of linear inequalities. Using the duality theorem, the so-called Bondareva [4] and Shapley [20] theorem sharpens the problem to the existence of balanced collections. Thus the existence is decidable via a simplex algorithm in a constructive fashion. Unfortunately, there is no such constructive approach to the existence of a stable set for an arbitrary game (N, v). In general games that are physically motivated have been found to have a plethora of stable sets. It was Lucas [13] who surprised game theorists by constructing a ten-person game with no stable set. In this connection we have the following.

Open Problem: Do all assignment games admit nonempty stable sets?

There are special classes of games for which the stable set exists and is unique. Perhaps the best-known such class is the class of convex games. In fact Shapley [21] proved that for convex games the core is the unique stable set. Since assignment games have a nonempty core, a natural question is to identify those assignment games whose core is also stable, and hence is the unique stable set. For assignment games we have two special imputations called the seller's corner and buyer's corner. In the seller's corner the seller takes away the full coalitional worth and the optimally matched mate receives nothing. In the buyer's corner, it is the buyer who takes away the coalitional worth, with the optimally matched mate receiving nothing. Since domination of an imputation by another imputation is possible only with buyer-seller coalitional pairs, the above two extreme imputations cannot be dominated by any imputation. Thus for the core to be a stable set, necessarily these two imputations must lie in the core. Interestingly, that condition is also sufficient for core stability [27].

Several other sufficient conditions for the stability of the core have been discussed in the literature.

Given an n-person game (N, v) with a nonempty core, the game admits a *Large core* if and only if for any n-vector x with $x(S) \geq v(S)$, $\forall\ S \subseteq N$, there exists a core element y such that $y \leq x$ coordinatewise. In an unpublished paper Kikuta and Shapley [10] investigated another condition, baptized to extendability of the game in the work of van Gellekom et al. [31]. For a totally balanced game (N, v) the *core is extendable* if and only if any core element x of any subgame (S, v), $S \subseteq N$ is simply the restriction of some core element $y \in \mathcal{C}(N, v)$ to the coordinates in S. The core of a game (N, v) is *exact* if and only if for any coalition S, there is some core element x such that $x(S) = v(S)$. Sharkey [23] and Biswas et al. [3] proved the following.

Theorem 3.1. *For any totally balanced game (N, v) we have the following: Core is Large $\Rightarrow$ Core is extendable $\Rightarrow$ Core is exact.*

A game (N, v) is called symmetric if for any two coalitions S, T with $|S| = |T|$, $v(S) = v(T)$. In fact Biswas et al. [3] proved the following.

Theorem 3.2. *For any totally balanced symmetric game or for games with* $|N| < 5$ *Core is exact* $\Rightarrow$ *Core is extendable* $\Rightarrow$ *core is Large.*

Unfortunately, given the data of the game, (N, v) we have no easy way to verify any of these conditions.

It turns out that for the class of assignment games, Largeness of the core, extendability and exactness of the game are all equivalent conditions, but are strictly stronger than the stability of the core. However for assignment games (v_A, N) many of these implications are equivalent and are easily verifiable via the matrix A defining the assignment game.

Let A be a nonnegative real matrix such that (4) is satisfied. We say that A has a *dominant diagonal* if $a_{ii'} \geq a_{ij'}$ and $a_{ii'} \geq a_{ji'}$ for all $i, j \in P$. Also, we say that A has a *doubly dominant diagonal* if $a_{ii'} + a_{jk'} \geq a_{ik'} + a_{ji'}$ for all $i, j, k \in P$. Now we are able to state the following characterization [27].

Theorem 3.3. *Let* $P = \{1, \ldots, p\}$, $Q = \{1', \ldots, p'\}$, *let* A *be a nonnegative real matrix on* $P \times Q$ *satisfying (4), let* $N = P \cup Q$, *and let* v_A *be the coalition function of the corresponding assignment game.*

(N, v_A) *has a stable core* $\Leftrightarrow$ A *has a dominant diagonal.*
(N, v_A) *has a Large core* $\Leftrightarrow$ (N, v_A) *has an extendable core* $\Leftrightarrow$ (N, v_A) *is exact* $\Leftrightarrow$ A *has a dominant and doubly dominant diagonal.*
(N, v_A) *is convex* $\Leftrightarrow$ A *is a diagonal matrix (that is,* $a_{ij'} \neq 0$ *implies* $j = i$*).*

Despite Example 2.1, from the above theorem we may deduce the following result for the modiclus [18].

Theorem 3.4. *The modiclus of an assignment game is in the core, provided the core is stable.*

The authors present a 15-person game which is exact and has a Large core and hence has a stable core and yet its modiclus is not a member of the core.

4 The Geometric Shape of the Core for Assignment Games

While Shapley and Shubik characterized the core of assignment games as dual optimal solutions of (3), they made another key observation that given any two core elements $(u^1, v^1), (u^2, v^2)$, the elements $(u^1 \vee u^2, v^1 \wedge v^2)$, and $(u^1 \wedge u^2, v^1 \vee v^2)$ are also core elements where $\vee, \wedge$ are the usual lattice operations, namely for vectors $u^1, u^2, (u^1 \vee u^2) = \max(u^1, u^2)$ where max is taken coordinatewise.

Interestingly, the dual inequalities that are used for determining the core as the optimal dual allocations have a special geometric structure. The core is obtained by starting with a cube $b_i \leq u_i \leq e_i$, $i = 1, \ldots, p$ for some constants b_i, e_i $i = 1, \ldots, p$ and then chopping off the 45-45-90 degree triangular cylinders determined by inequalities of the type

$$u_i - u_k \geq d_{ik} \quad \forall\ i, k \in 1, \ldots, p;\ i \neq k$$

for some constants $\{d_{ik}\}$. In fact the converse is also true, namely Quint [17] proved the following.

Theorem 4.1. *Let P be a polytope with elements $(u_1, \dots, u_p) \in \mathbf{R}^p$ satisfying*

$$u_i - u_k \geq d_{ik} \quad \forall\ i, k \in 1, \dots, p; i \neq k$$

$$b_i \leq u_i \leq e_i$$

for some constants $\{d_{ik}\}, b_i \geq 0, e_i \geq 0 \quad i = 1, \dots, p$. Then we can always find an assignment game with p sellers whose u space core coincides with P.

The extreme points of the core of assignment games can also be nicely recognized by the following graph-theoretic technique of Balinsky and Gale [2]. Given any core element (u, v) we can associate with the core element a graph Γ_{uv} with vertices as $P \cup Q$ and with edges (p, q) where $u_p + v_q = a_{pq}$.

Theorem 4.2. *A core element (u, v) of the assignment game v_A is an extreme point if and only if the graph Γ_{uv} is connected.*

The extreme points of the cores of subgames of assignment games have the following extension property [2].

Theorem 4.3. *If $(\tilde{u}, \tilde{v})$ is an extreme point of the core of some subgame on $\tilde{P} \cup \tilde{Q}$ of an assignment game with sellers P and buyers Q and defining matrix A, then there is an extreme point (u, v) of the polyhedron*

$$X = \{(u, v) : u_p + v_q \geq a_{pq}, u_{p0} = 0,\ p \in P, q \in Q\},$$

where 0 denotes the dummy buyer such that (u, v) agrees with $(\tilde{u}, \tilde{v})$ on $\tilde{P} \cup \tilde{Q}$.

The cores of assignment games and convex games share the following common properties [8,15].

Property 4.1. In each extreme point of the core allocations of an assignment game (N, v) there is at least one player i who receives his marginal contribution $v(N) - v(N \setminus \{i\})$.

Property 4.2. Every marginal contribution for any player is attained at some core element.

5 An Algorithm to Compute the Nucleolus

Given a game (N, v) and an imputation x let $f(S, x) = -e(S, x, v)$ (see Section 2.1). Hence $f(S, x)$ is the *satisfaction of the coalition S* at imputation x. As we focus on assignment games, we shall henceforth always assume that (N, v)

is zero-monotonic. Hence the nucleolus of the game is just its pre-nucleolus. We now slightly modify our viewpoint. With $H = (-f(S,\cdot))_{S\subseteq N}$ the nucleolus of a zero-monotonic game (N,v) is the unique member of the set given by the right-hand side of (2) in which we may replace X by $\mathcal{I}(N,v)$ by Proposition 2.1. By the *lexicographic center* of a nonempty closed convex subset D of the imputation set, we mean the unique point $x^* \in D$ which lexicographically minimizes the vector $\theta^H(x)$ over D (that is, the set defined by the right-hand side of (2) with $X = D$ is a singleton as shown by Schmeidler [19]). Even though the determination of the nucleolus is quite difficult in general, it is possible to locate it efficiently for special subclasses of games. We will describe an algorithm [26] to locate the nucleolus for an assignment game. We will reinterpret the game slightly differently as follows.

Stable Real Estate Commissions: House owners $P = \{U_1, U_2, \dots, U_p\}$, each possessing one house, and house buyers $Q = \{V_1, V_2, \dots, V_q\}$, each wanting to buy one house, approach a common real estate agent. Not revealing the identity of the buyers and sellers, the agent wants an up-front commission $a_{ij} \geq 0$ if he links seller U_i to buyer V_j. The sellers and buyers prefer fixed commissions $u_1, u_2, \dots, u_p$ and $v_1, \dots, v_q$. The agent has no objection if they meet his expectation for every possible link. He guarantees their money's worth in his effort and promises to take no commission from a seller (buyer) if he cannot find a suitable buyer (seller).

We define $P_0 = P \cup \{0\}$, $Q_0 = Q \cup \{0\}$, $a_{i0} = 0\ \forall\ i \in P_0$, $a_{0j} = 0\ \forall\ j \in Q_0$, $u_0 := 0$, $v_0 = 0$, and all the constraints in (1) reduce to

$$f_{ij}(u,v) = u_i + v_j - a_{ij} \geq 0 \quad \forall\ (i,j) \in P_0 \times Q_0). \tag{5}$$

If $\sigma \subseteq P \cup Q$ is an optimal assignment and D is the core we get

$$\{i,j\} \in \sigma \Rightarrow f_{ij}(u,v) = 0 \quad \forall\ (u,v) \in D. \tag{6}$$

With the convention that $(0,0) \in \sigma$ we write $(i,0) \in \sigma$ ($(0,j) \in \sigma$) if in σ row $i \in P$ (column $j \in N$) is not assigned to any column $j \in N$ (row $i \in Q$). Here σ is extended to a subset of $P_0 \times Q_0$ so that (6) also expresses the fact that D lies in the hyperplane $u_i = 0$ (or $v_j = 0$) for any unassigned row i (column j). It is easily seen that

$$D = \{(u,v) : f_{ij}(u,v) = 0\ \forall\ (i,j) \in \sigma, f_{ij}(u,v) \geq 0\ \forall\ (i,j) \notin \sigma\}. \tag{7}$$

Here and from now on $(i,j) \notin \sigma$ is written instead of $(i,j) \in (P_0, Q_0) \setminus \sigma$.

Among many vectors of commissions $(u_0, u_1, \dots, u_m; v_0, v_1, \dots, v_n)$ in D for the agent, he wants to choose one that is "neutral" and "stable." The lexicographic center is a possible option that is neutral and stable for all pairs.

For every $(u,v) \in D$ the first $\max(p,q) + 1$ components (those coordinates $k = (i,j)$ corresponding to $(i,j) \in \sigma$) of $\theta^H(u,v)$ are equal to 0. Let

$$\alpha^1 = \max_{(u,v)\in D} \min_{(i,j)\notin\sigma} f_{ij}(u,v).$$

Let

$$D^1 = \left\{ (u,v) \in D : \min_{(i,j)\notin\sigma} f_{ij}(u,v) = \alpha^1 \right\}$$

Let

$$\sigma^1 = \{(i,j) : f_{ij}(u,v) = \text{constant on } D^1\}.$$

The set σ^1 can be regarded as an "assignment" between the equivalence classes of the relation $\sim^1$ defined on M_0 and N_0 by

$$\begin{aligned} i_1 \sim^1 i_2 &\text{ if and only if } u_{i_1} - u_{i_2} \text{ is constant on } D^1, \\ j_1 \sim^1 j_2 &\text{ if and only if } v_{j_1} - v_{j_2} \text{ is constant on } D^1, \end{aligned}$$

respectively.

$$\begin{aligned} \alpha^2 &= \max_{(u,v)\in D^1} \min_{(i,j)\notin\sigma^1} f_{ij}(u,v), \\ D^2 &= \left\{ (u,v) \in D^1 : \min_{(i,j)\notin\sigma^1} f_{ij}(u,v) = \alpha^2 \right\}, \\ \sigma^2 &= \{(i,j) \in (M_0, N_0) : f_{ij}(u,v) \text{ is constant on } D^2\}. \end{aligned}$$

Let $i \sim^2 k$ if and only if $u_i - u_k$ is a constant on D^2. Observe that $\sigma^2 \supseteq \sigma^1 \supseteq \sigma$.

Therefore, after some $t \leq \min(m,n)$ rounds the process terminates with

$$\sigma^t = \{(i,j) = (M_0 \times N_0) : f_{ij}(u,v) \text{ is constant on } D^t\}.$$

This means that a subset of D is found that is parallel to all hyperplanes defining D. Since they include $u_i = 0$ for all $i \in M$ and $v_j = 0$ for all $j \in N$, this subset must consist of a single point. It can be proved [26] that this point is precisely the lexicographic center of D.

Next we illustrate by an example how to implement the procedure leading to the lexicographic center.

Example 5.1. We are given

$$A = \begin{bmatrix} 6 & 7 & 7 \\ 0 & 5 & 6 \\ 2 & 5 & 8 \end{bmatrix},$$

where $P = \{1,2,3\} = Q$. The unique optimal assignment for A is $\sigma = \{(1,1), (2,2), (3,3)\}$, i.e., the entries in the main diagonal. Starting with all commissions collected entirely from sellers, one could use the procedure to be described below to locate the u worst point $(u^1, v^1) = (0,6,4,6 : 0,0,1,2)$ in D. Further with rows numbered 0, 1, 2, 3 and columns numbered 0, 1, 2, 3 we can read off (u^1, v^1) from column 0, and row 0 of the matrix

$$[f_{ij}(u^1, v^1)] = \begin{bmatrix} 0 & 0^* & 1 & 2 \\ 6 & 0 & 0^* & 1 \\ 4 & 4 & 0 & 0^* \\ 6 & 4 & 2 & 0 \end{bmatrix}.$$

Even though the coordinates for the starred entries above are the next set with higher f_{ij} values in the lexicographic ranking, they are still 0. However, from now on there will be strict improvement with higher values when we follow the iteration. We want to move in a direction (s, t) inside D with one end at the extreme solution (u^1, v^1). Let the new point be $(u^2, v^2) = (u^1, v^1) + \beta \cdot (s, t)$ for some $\beta \geq 0$. Since the point (u^1, v^1) is the worst for all sellers in terms of commissions in D, they would like their commissions reduced.

Since (u^1, v^1) is the farthest from the hyperplanes indexed by $(0, 1), (1, 2)$ and $(2, 3)$ (indicated by a * in the above matrix) this translates to the requirements

$$s_0 + t_1 \geq 1, \quad s_1 + t_2 \geq 1, \quad s_2 + t_3 \geq 1 \tag{8}$$

with at least one inequality. Since we must remain in D we also have

$$s_i + t_i = 0, \quad i = 1, 2, 3. \tag{9}$$

Combining (8) and (9) gives

$$t_1 - t_0 \geq 1, \quad t_2 - t_1 \geq 1, \quad t_3 - t_2 \geq 1 \tag{10}$$

with at least one equality to hold. Thus the direction for improvement for sellers is $(s, t) = (0, -1, -2, -3 : 0, 1, 2, 3)$. Next we determine how far we can move along this direction inside D starting from the initial u worst corner of D. That is

$$[f_{ij}((u^1, v^1) + \beta \cdot (s, t))] = \begin{bmatrix} 0 & (0+\beta)^* & 1+2\beta & 2+3\beta \\ 6-\beta & 0 & (0+\beta)^* & 1+2\beta \\ 4-2\beta & 4-\beta & 0 & (0+\beta)^* \\ 6-3\beta & 4-2\beta & (2-\beta)^\diamond & 0 \end{bmatrix},$$

where $*$ refers to the worst satisfied coalition at the current imputation, and $\diamond$ refers to the penultimate coalition. Compared to the worst satisfied mixed coalition consisting of the dummy seller 0 with buyer 1 with satisfaction $0 + \beta$, the next worst hit coalition is the one with seller 3 and buyer 2 and with satisfaction $f_{32} = 2 - \beta$ which is the first one to reach the same level as the worst hit one when improved. To reach this common level we equate $2 - \beta = 0 + \beta$ and we get $\beta = 1$ and $(u^2, v^2) = (0, 5, 2, 3; 0, 1, 3, 5)$. It can be shown that (u^2, v^2) is the u-worst corner (v-best corner) in D^1.

$$
\begin{array}{c|ccc|l}
0 & 1^* & 3 & 5 & \\
\hline
5 & \boxed{0} & 1^* & 3 & -1 \\
2 & 3 & \boxed{0} & 1^* & -2 \quad \beta = 1. \\
3 & 2 & 1^* & \boxed{0} & -2 \\
\hline
 & +1 & +2 & +2 &
\end{array}
$$

The updated distance matrix is compactly represented by

Here the left-side border frame is the column vector u^2 and the top border frame is the row vector v^2 for all the non-dummy players. We also have the right-side frame with the column vector $s = (-1, -2, -2)^T$ and the bottom frame with the row vector $t(1, 2, 2)$.

They are derived from the following considerations: To improve further from $(u^2, v^2) = (0, 5, 2, 3; 0, 1, 3, 5)$ we need to find a direction to move inside D^1. Observe that the starred entries represent the worst hit coalitions at the current point. If the rows and columns are numbered 0,1,2,3 as before, the satisfactions of sellers with dummy buyers are given by the left-side frame. The satisfaction of buyers with dummy sellers are given by the entries of the top frame. Thus we have a starred value 1 at entries (0,1), (1,2), (2, 3) and (3, 2). This means that $f_{01}(u, v) \geq 1, f_{12}(u, v) \geq 1, f_{23}(u, v) \geq 1, f_{32}(u, v) \geq 1$ for all $(u, v) \in D^1$. Since on D^1 we have $f_{22}(u, v) = f_{33}(u, v) \equiv 0$, we have $f_{23}(u, v) = f_{32}(u, v) \equiv 1$ $\forall\ (u, v) \in D^1$. Thus the new direction (s, t) must satisfy $s_0 + t_1 \geq 1, s_1 + t_2 \geq 1, s_2 + t_3 = 0, s_3 + t_2 = 0$. Thus the direction is $(s, t)(0, -1, -2, -2 : 0, 1, 2, 2)$.

Thus we notice that on the new set $D^2 \subseteq D^1$ not only the coalitions $(1, 1), (2, 2), (3, 3)$ of buyer-seller pairs have constant value for the satisfaction at the imputations but also have constant satisfaction for the coalitions (2,3), (3,2). Thus what were originally boxed coalitions for D^1 are also boxed for D^2 and so are $(2, 3), (3, 2)$ coalitions. Now to determine the new step size β for the new direction we proceed as follows. Consider the matrix

$$
\begin{array}{c|ccc|l}
 & (1+\beta^*) & 3+2\beta & 5+2\beta & \\
\hline
5-\beta & \boxed{0} & (1+\beta)^\diamond & 2+\beta & -1 \\
(2-2\beta)^\diamond & 3-\beta & \boxed{0} & \boxed{1} & -2 \quad \beta = 1 \\
(3-2\beta) & (2-\beta) & \boxed{1} & \boxed{0} & -2 \\
\hline
 & +1 & +2 & +2 &
\end{array}
$$

The decreasing distance $f_{20} = 2 - 2\beta$ is the first to reach the increasing second smallest distance $1 + \beta$. It happens when $\beta = 1/3$. So the maximal distance in this direction is $\beta = 1/3$, and the u-worst corner of the set D^2 of points with the second smallest distance $4/3$ is $(u^3, v^3) = (0, 14/3, 4/3, 7/3; 0, 4/3, 11/3, 17/3)$. The updated distance matrix is

$$[f_{ij}(u^3, v^3)] = \begin{array}{c|c|c|c|} 0 & 4/3^* & 11/3 & 17/3 \\ \cline{2-4} 14/3 & \boxed{0} & 4/3^* & 7/3 \\ \cline{2-4} 4/3^* & 8/3 & \boxed{0} & \boxed{1} \\ \cline{2-4} 7/3 & 5/3 & \boxed{1} & \boxed{0} \\ \cline{2-4} \end{array} .$$

Again (u^3, v^3) is the u-worst corner (v-best corner) in D^2. To move inside D^2, we look for direction (s, t). Using the starred entries, (s, t) must satisfy $s_0 + t_1 \geq 4/3$, $s_1 + t_2 \geq 4/3$, $s_2 + t_0 \geq 4/3$. Also since $f_{23}(u, v) = f_{32}(u, v) \equiv 1$ on D^2, we easily find the above system of inequalities inconsistent. Thus no more movement inside is possible. We have reached the lexicographic center.

Remark 5.1. Starting with the worst set of commissions for all sellers and using Kuhn's Hungarian method [12], the algorithm locates the unique set of commissions that again favors all the buyers in the restricted new domain D of commissions. The next step is to locate the unique direction (s, t) and the unique step size β in finding the new set of commissions. We have not used in our example any efficient procedure to find the direction (s, t). Solymosi and Raghavan [26] develop an explicit graph-theoretic algorithm to find this direction. The decomposition of the payoff space and the lattice structure of the feasible set at each iteration are utilized in associating a directed graph. If the graph is acyclic, the problem of finding the new direction (s, t) can be transformed to determine the longest path to each vertex of the graph. Cycles are used to collapse vertices so that the graph has fewer vertices. The algorithm stops when the graph is reduced to just one vertex. The assignment game is the simplest of cooperative games which are balanced and hence have a nonempty core. The Real Estate Game was first considered by Shapley and Shubik [23]. The same problem was viewed in the context of competitive pricing of indivisible goods by Gale [6]. Pooling peoples' utility functions amounts to interpersonal comparisons and hence has remained alien to mainstream economists. For a version of the Real Estate Game without side payments see [22].

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