Nonsymmetric variants of the prekernel and the prenucleolus

Guni Orshan & Peter Sudhölter

International Journal of Game Theory

ISSN 0020-7276
Volume 41
Number 4

Int J Game Theory (2012) 41:809-828
DOI 10.1007/s00182-011-0294-6

Springer

Int J Game Theory (2012) 41:809–828
DOI 10.1007/s00182-011-0294-6

Nonsymmetric variants of the prekernel and the prenucleolus

Guni Orshan · Peter Sudhölter

Accepted: 27 June 2011 / Published online: 10 July 2011

Abstract A solution on a class of TU games that satisfies the axioms of the pre-nucleolus or -kernel except the equal treatment property and is single valued for two-person games, is a nonsymmetric pre-nucleolus (NSPN) or -kernel (NSPK). We investigate the NSPKs and NSPNs and their relations to the positive prekernel and to the positive core. It turns out that any NSPK is a subsolution of the positive prekernel. Moreover, we show that an arbitrary NSPK, when applied to a TU game, intersects the set of preimputations whose dissatisfactions coincide with the dissatisfactions of an arbitrary element of any other NSPK applied to this game. This result also provides a new proof of sufficiency of the characterizing conditions for NSPKs introduced by Orshan (Non-symmetric prekernels, discussion paper 60. Center for Rationality, The Hebrew University of Jerusalem, 1994). Any NSPN belongs to "its" NSPK. Several classes of NSPNs are presented, all of them being subsolutions of the positive core. We show that any NSPN is a subsolution of the positive core provided that it satisfies the equal treatment property on an infinite subset of the universe of potential players. Moreover, we prove that, for any game whose prenucleolus is in its anticore, any NSPN coincides with the prenucleolus.

G. Orshan
Department of Agricultural Economics and Management, The Hebrew University,
P.O. Box 12, Rehovot 76100, Israel
e-mail: orshan@agri.huji.ac.il

G. Orshan
Department of Mathematics, The Open University of Israel, 1 University Road, P.O. Box 808,
Raanana 43107, Israel

P. Sudhölter (✉)
Department of Business and Economics, University of Southern Denmark, Campusvej 55,
5230 Odense M, Denmark
e-mail: psu@sam.sdu.dk

Keywords TU game · Solution concept · Kernel · Nucleolus · Core · Equal treatment

JEL Classification C71

0 Dedication

This study is dedicated to our teacher and friend Bezalel Peleg, who initiated the joint research of the authors in 1993. Indeed, at that time the second author showed that the anonymity axiom cannot be removed in Sobolev's (1975) famous characterization of the prenucleolus, and when asking whether this logical independence of anonymity was known, Bezalel Peleg indicated that the first author, completing his PhD thesis supervised by Michael Maschler, was aware of a solution to this problem by his investigations on the nonsymmetric prekernels. In fact, he supervised the first author for one year when Maschler was abroad and, hence, motivated this research, as is also documented by Sudhölter and Peleg (2000).

1 Introduction

The prenucleolus and the prekernel are widely acceptable solutions for cooperative transferable utility games. Introduced as auxiliary solutions of the prebargaining set, they became important solutions in their own rights, heavily supported by the fact that they can be justified by simple and intuitive axioms. Both are closely related because they share many properties and because one, the prenucleolus, is a subsolution of the other. Two of these properties, *anonymity* (AN), requiring that the solution is independent of the names of the players, and the *equal treatment property* (ETP), requiring that any element of the solution of a game assigns the same payoffs to players that are substitutes, may be used, together with further axioms, to characterize these solutions (see Theorems 2.1, 2.2).

This paper investigates the roles of AN and ETP in the aforementioned axiomatizations. Indeed, it may be desirable to apply a solution that has the properties of the pre-nucleolus or -kernel with the exception of AN or ETP. In order to mention one example of this kind, note that bankruptcy problems may be modeled as cooperative TU games (see, e.g., Aumann and Maschler 1985). However, in a bankruptcy problem, some of the creditors may be ranked so that ETP or AN are not possible.

A solution that may violate AN is called *nonsymmetric prenucleolus* (NSPN) if it satisfies the remaining axioms that characterize the prenucleolus. Similarly, a *nonsymmetric prekernel* (NSPK) may violate ETP, but it must assign a single proposal to any two-person game (this axiom is abbreviated by 2-SIVA) and it must satisfy the remaining characterizing axioms of the prekernel.

Thus, an NSPK is a solution that satisfies nonemptiness (NE), Pareto optimality (PO), covariance (COV), the reduced game property (RGP) and its converse (CRGP). Even without requiring NE, a solution satisfying the remaining axioms is determined as soon as it is defined for all 0-1 and all 0-(-1) normalized two-person games (see Sect. 3). Orshan (1994) describes how an NSPK must behave on these two-person games. Hence, he presents conditions that are necessary and sufficient for nonemptiness. The conditions are used to demonstrate that an NSPK is contained in the *positive prekernel*, a solution introduced and characterized by Sudhölter and Peleg (2000). A preimputation of a game belongs to the positive prekernel if it differs from some preimputation in the prekernel only inasmuch as it may assign different amounts to *satisfied* coalitions; that is, to coalitions that have non-positive excesses. Moreover, a new proof of sufficiency of these conditions is provided that yields, as one important byproduct, that any NSPK of any game intersects the *positive core* of the game. The positive core that arises from the prenucleolus in the same manner as the positive prekernel arises from the prekernel (i.e., an element of the positive core assigns the same dissatisfaction to the coalitions as the prenucleolus does) was used as an auxiliary solution (Sudhölter 1993), but Orshan and Sudhölter (2010) show that this solution is interesting in its own right.

Any NSPN is contained in a unique NSPK, but an NSPK may contain distinct NSPNs. Though all NSPNs constructed in this paper are subsolutions of the positive core, we do not know whether this inclusion is valid for an arbitrary NSPN. However, it is shown that any NSPN that satisfies a mild additional condition is a subsolution of the positive core. Moreover, we show that if the prenucleolus of a game is an element of the anticore of the game, then it coincides with any of the NSPNs of the game.

It should be noted that the proofs of several results in Sect. 3 use the characterization result of NSPKs, i.e., Theorem 3.5, whose proof was only published by Orshan (1994), in a version of his PhD thesis. However, we recall or give some sketches or hints about parts of the proof[1] that are relevant for the present paper. Thus, Sect. 3 is, at least basically, self-contained.

The paper is organized as follows. In Sect. 2 the necessary notation and definitions, Peleg's (1986) axiomatization of the prekernel, and Sobolev's (1975) axiomatization of the prenucleolus together with two variants (see Orshan 1993; Orshan and Sudhölter 2003) are presented.

In Sect. 3 we use Orshan's (1994) characterization of NSPKs (Theorem 3.5) to show that any NSPK is a subsolution of the positive prekernel. However, it is demonstrated that this result may be easily obtained independently without applying Theorem 3.5 for any NSPK that satisfies one further plausible axiom, called *reasonableness*. Moreover, the main result of this section (Theorem 3.8), also deduced without applying Theorem 3.5, states that if σ is an arbitrary solution that satisfies the condition, then for any element x of σ applied to a TU game, any other solution that satisfies the condition when applied to the same game intersects the set of preimputations that treat all dissatisfied coalitions in the same way as x. Inserting the prekernel for σ, Theorem 3.8 yields (a) a new proof of the sufficiency of Orshan's (1994) conditions and (b) that any NSPK of a game intersects the positive core of the game.

In Sect. 4, examples of NSPNs are provided that assign to a game the nucleolus with respect to an arbitrary face of the positive core of the game. It is shown that a broad variety of NSPKs contains an NSPN, and it is demonstrated that each NSPK of a considerably large class contains distinct NSPNs.

[1] We do not offer the complete proof because it is not new (in fact more than 16 years old) and quite technical. The authors should be happy to send an electronic copy of Orshan's (1994) paper on demand to those readers who are interested in the details nevertheless.

In Sect. 5 we show that an NSPN that satisfies ETP on an infinite subset U' of the universe U of players must be contained in the positive core. Though any NSPN that satisfies ETP on U' must select the prenucleolus for any game whose players are elements of U', the examples of Sect. 4 demonstrate that there exist many nontrivial distinct NSPNs that satisfy ETP on U' and do not coincide with the prenucleolus in general, provided that U contains at least one player that does not belong to U'. Moreover, it is shown that any NSPN satisfies AN when restricted to the class of games that have a nonempty anticore (the *anticore* of a game is the *core* of the dual game). This result may be used to show that an NSPN selects the prenucleolus if the prenucleolus is in the anticore, but it may not be expanded to show that the NSPN is contained in the positive core for an arbitrary game that possesses a nonempty anticore. The proofs of both theorems (Theorems 5.1, 5.3) are in particular independent of the results of Orshan (1993), Orshan (1994), Sudhölter (1993), and Orshan and Sudhölter (2003, 2010) who employ some versions of AN or ETP in almost all characterizations of solutions.

Finally, Sect. 6 is devoted to the proof of a technical lemma (Lemma 5.2) that is used in the proof of Theorem 5.1.

2 Notation, solutions, and properties

Let U, $|U| \geq 4$, be the universe of players containing, without loss of generality, $1, \ldots, k$ whenever $|U| \geq k$. A (cooperative TU) *game* is a pair (N, v) such that $\emptyset \neq N \subseteq U$ is finite and $v : 2^N \to \mathbb{R}$, $v(\emptyset) = 0$. For any game (N, v) let

$$X^*(N, v) = \{x \in \mathbb{R}^N \mid x(N) \leq v(N)\} \quad \text{and} \quad X(N, v) = \{x \in \mathbb{R}^N \mid x(N) = v(N)\}$$

denote the set of *feasible* and *Pareto optimal* feasible payoffs (*preimputations*), respectively. We use $x(S) = \sum_{i \in S} x_i$ $(x(\emptyset) = 0)$ for every $S \in 2^N$ and every $x \in \mathbb{R}^N$ as a convention. Additionally, x_S denotes the restriction of x to S, i.e. $x_S = (x_i)_{i \in S}$, and we write $x = (x_S, x_{N \setminus S})$. For $x \in \mathbb{R}^N$, $S \subseteq N$, and distinct players $k, \ell \in N$ let

$$e(S, x, v) = v(S) - x(S) \text{ and } s_{k\ell}(x, v) = \max_{S \subseteq N \setminus \{\ell\} : k \in S} e(S, x, v)$$

denote the *excess* of S and the *maximum surplus*[2] of k over ℓ, respectively, at x with respect to (N, v). The *prekernel* (see Davis and Maschler 1965) of (N, v) is given by

$$\mathcal{PK}(N, v) = \{x \in X(N, v) \,|\, s_{k\ell}(x, v) = s_{\ell k}(x, v) \quad \text{for all k } \in N,\ \ell \in N \setminus \{k\}\}.$$

For $X \subseteq \mathbb{R}^N$ let $\mathcal{N}((N, v); X)$ denote the *nucleolus* of (N, v) with respect to X, i.e., the set of members of X that lexicographically minimize the nonincreasingly ordered vector of excesses of the coalitions (see Schmeidler 1969). It is well-known that the nucleolus with respect to $X^*(N, v)$ is a singleton, the unique element of which is called the *prenucleolus* of (N, v) and is denoted by $\nu(N, v)$.

[2] Sometimes, if (N, v) is fixed, we omit v and simply write $e(S, x)$ and $s_{k\ell}(x)$.

In general, a *solution* σ associates with each game (N, v) a subset of $X^*(N, v)$. Let σ and σ' be solutions. We say that σ' is a *subsolution* of σ if $\sigma'(N, v) \subseteq \sigma(N, v)$ for all games (N, v). Moreover, σ

(1) is *covariant under strategic equivalence* (COV) if, for all games (N, v), (N, w) satisfying $w = \beta v + z$ for some $\beta > 0$ and $z \in \mathbb{R}^N$, the equation $\sigma(N, w) = \beta\sigma(N, v) + z$ holds. (Here we use the convention that identifies $z \in \mathbb{R}^N$ with the *additive* coalitional function, again denoted by z, on the player set N defined by $z(S) = \sum_{i \in S} z_i$ for all $S \in 2^N$. Also note that the games v and w are called *strategically equivalent*.);
(2) is *nonempty* (NE) if $\sigma(N, v) \neq \emptyset$ for every game (N, v);
(3) is *Pareto optimal* (PO) if $\sigma(N, v) \subseteq X(N, v)$ for every game (N, v);
(4) is *single-valued* (SIVA) if $|\sigma(N, v)| = 1$ for every game (N, v);
(5) is *anonymous* (AN) if the following condition is satisfied for all games (N, v): If $\pi : N \to U$ is an injection, then $\sigma(\pi(N), \pi v) = \pi(\sigma(N, v))$, where $\pi v(\pi(S)) = v(S)$ for all $S \subseteq N$ and, for any $x \in \mathbb{R}^N$, $y = \pi(x) \in \mathbb{R}^{\pi(N)}$ is given by $y_{\pi(i)} = x_i$ for all $i \in N$ (in this case the games (N, v) and $(\pi(N), \pi v)$ are *isomorphic*);
(6) satisfies the *equal treatment property* (ETP) if for every game (N, v), for every $x \in \sigma(N, v)$, $x_k = x_\ell$ for all *substitutes* $k, \ell \in N$ (k and ℓ are *substitutes* if $v(S \cup \{k\}) = v(S \cup \{\ell\})$ for all $S \subseteq N \setminus \{k, \ell\}$);
(7) is *reasonable* (REAS) if, for every game (N, v), for every $x \in \sigma(N, v)$, and for every $i \in N$,

$$\min_{S \subseteq N \setminus \{i\}} (v(S \cup \{i\}) - v(S)) \leq x_i \leq \max_{S \subseteq N \setminus \{i\}} (v(S \cup \{i\}) - v(S));$$

(8) satisfies the *reduced game property* (RGP) if for any game (N, v), for every $\emptyset \neq S \subseteq N$, and any $x \in \sigma(N, v)$, $x_S \in \sigma(S, v^{S,x})$ (the *reduced game* $(S, v^{S,x})$ with respect to S and x is defined by $v^{S,x}(\emptyset) = 0$, $v^{S,x}(S) = v(N) - x(N \setminus S)$, and $v^{S,x}(T) = \max_{Q \subseteq N \setminus S}(v(T \cup Q) - x(Q))$ for $\emptyset \neq T \subsetneq S$);
(9) satisfies the *converse reduced game property* (CRGP) if for every game (N, v) with $|N| \geq 2$ the following condition is satisfied for every $x \in X(N, v)$: If, for every $S \subseteq N$ with $|S| = 2$, $x_S \in \sigma(S, v^{S,x})$, then $x \in \sigma(N, v)$;
(10) satisfies the *reconfirmation property* (RCP) if, for any game (N, v), for every $\emptyset \neq S \subseteq N$, for any $x \in \sigma(N, v)$ and $y \in \sigma(S, v^{S,x})$, $(y, x_{N \setminus S}) \in \sigma(N, v)$.

For interpretations and discussions, in particular of the variants (8), (9), and (10) of the reduced game property, see Peleg (1986) and Hwang and Sudhölter (2001).

We now recall the classical characterizations of the prenucleolus and the prekernel.

Theorem 2.1 (Sobolev 1975) *If* $|U| = \infty$*, then the prenucleolus is the unique solution that satisfies* SIVA*,* COV*,* AN*, and* RGP.

Theorem 2.2 (Peleg 1986) *The prekernel is the unique solution that satisfies* NE*,* PO*,* COV*,* ETP*,* RGP*, and* CRGP.

Note that in Theorem 2.2 no assumption on the cardinality of U is needed.

Remark 2.3 (1) Orshan (1993) shows that AN may be replaced by ETP in Theorem 2.1. As RGP and RCP are equivalent for solutions that satisfy SIVA, the prenucleolus is, hence, characterized by SIVA, COV, ETP, and RCP. Surprisingly, if ETP and RCP are used, then SIVA may be weakened. In fact, Orshan and Sudhölter (2003) show that the prenucleolus is characterized by NE, COV, ETP, and RCP, provided that $|U| = \infty$.

(2) By means of an example, Peleg and Sudhölter (2003, Remark 6.3.3) show that Theorem 2.1 and the modifications of Remark 2.3 (1) are no longer valid, if the condition $|U| = \infty$ is deleted.

In view of Remark 2.3 (2), a solution σ is called a *nonsymmetric prenucleolus* (NSPN) if it satisfies SIVA, COV, and RGP, provided that $|U| = \infty$.

In order to define nonsymmetric prekernels we do not simply delete ETP in Theorem 2.2, because there are many "pathological" examples (e.g., the solution $X(\cdot, \cdot)$ assigning the set of all preimputations to a game) that satisfy the remaining axioms. One basic property of the prekernel is kept. Indeed, the prekernel of any two-person game coincides with its prenucleolus. Hence, the prekernel of any two-person game is single valued. A solution σ is 2-SIVA if $|\sigma(N, v)| = 1$ for any game (N, v) with $|N| = 2$. We say that σ is a *nonsymmetric prekernel* if σ satisfies 2-SIVA, NE, PO, COV, RGP, and CRGP (see Orshan 1994).

Remark 2.4 (1) A solution that satisfies PO, RGP, and CRGP, is uniquely determined by the 2-person games in the following sense. If σ is a solution that satisfies PO and RGP, then there exists a unique solution $\widetilde{\sigma}$ that satisfies PO, RGP, and CRGP, and coincides with σ for any 2-person game. So CRGP may be replaced by "maximality" (see Remark 3.12 of Orshan 1994 or Remark 3.7 of Hwang and Sudhölter 2001).

(2) As SIVA, COV, and RGP imply PO (see Sobolev 1975), we conclude that every NSPN σ is a subsolution of a unique NSPK $\widetilde{\sigma}$ defined by $\widetilde{\sigma}(N, v) = \sigma(N, v)$ for any game (N, v) with $|N| \leq 2$.

3 Nonsymmetric prekernels and the positive prekernel

In this section we show that for any element of an NSPK of a game, any other NSPK of the same game contains an element with coinciding dissatisfactions. See Corollary 3.9 for the precise formulation of this statement. Moreover, we show that any NSPK is a subsolution of the positive prekernel. The main result of this section, Theorem 3.8, does not only imply Corollary 3.9 and is used for motivation in the introduction of Sect. 5, but it also provides a new proof of sufficiency of Orshan's (1994) characterizing properties of NSPKs (see Theorem 3.5). We do not re-prove this theorem completely, but we present several ideas and parts of the proof so that our results or minor modifications of them are independent of the cited thesis.

Let σ be a solution that satisfies 2-SIVA, PO, COV, RGP, and CRGP. In view of Remark 2.4 and of COV, σ is determined as soon as it is defined for all 0-1 and 0-(-1) normalized 2-person games. (Indeed, up to strategic equivalence a 2-person game $(\{k, \ell\}, v)$ satisfies $v(\{k\}) = v(\{\ell\}) = 0$ and $v(\{k, \ell\}) \in \{1, -1, 0\}$.) However, if $v(\{k,\ell\}) = 0$, then, by 2-SIVA and COV, $\sigma(\{k,\ell\}, tv) = t\sigma(\{k,\ell\}, v)$ for $t > 0$, so $\sigma(\{k,\ell\}, v) = \{0\}$ by NE. For all distinct $k, \ell \in U$, let $a^{\sigma}_{k\ell} = a_{k\ell}$ denote the k-coordinate of the unique element of σ applied to the 0-1 normalized game on $\{k,\ell\}$ and let $b^{\sigma}_{k\ell} = b_{k\ell}$ be the k-coordinate of the 0-(−1) normalized game on $\{k,\ell\}$. By PO,

$$a_{k\ell} + a_{\ell k} = 1; \tag{3.1}$$

$$b_{k\ell} + b_{\ell k} = -1. \tag{3.2}$$

Let (N, v) be a game, and let $x \in \sigma(N, v)$. Moreover, let $k, \ell \in N$, $k \neq \ell$, and let $S = \{k,\ell\}$. If $s_{k\ell}(x) + s_{\ell k}(x) < 0$, then the reduced game $(S, v^{S,x})$ is strategically equivalent to the 0-1 normalized game on S so that[3] $a_{k\ell}s_{\ell k}(x) = a_{\ell k}s_{k\ell}(x)$. If $s_{k\ell}(x)+s_{\ell k}(x) > 0$, then $(S, v^{S,x})$ is strategically equivalent to the 0-(−1) normalized game so that $b_{k\ell}s_{\ell k}(x) = b_{\ell k}s_{k\ell}(x)$. Finally, if $s_{k\ell}(x) + s_{\ell k}(x) = 0$, then $(S, v^{S,x})$ is strategically equivalent to the "flat" (0-0 normalized) game so that $0 = s_{\ell k}(x) = s_{k\ell}(x)$. Hence, by RGP and CRGP,

$$\sigma(N, v) = \left\{ x \in X(N, v) \,\middle|\, \begin{array}{ll} a_{k\ell}s_{\ell k}(x) = a_{\ell k}s_{k\ell}(x) \text{ if } s_{k\ell}(x) + s_{\ell k}(x) < 0 \\ b_{k\ell}s_{\ell k}(x) = b_{\ell k}s_{k\ell}(x) \text{ if } s_{k\ell}(x) + s_{\ell k}(x) \geq 0 \end{array} \text{ for all } k, \ell \in N, k \neq \ell \right\}. \tag{3.3}$$

The converse is also valid: If σ is defined by (3.3) and $a_{k\ell}$, $b_{k\ell}$ satisfy (3.1) and (3.2) for all distinct $k, \ell \in U$, then σ satisfies 2-SIVA, PO, COV, RGP, and CRGP.

An NSPK satisfies 2-SIVA, PO, COV, RGP, CRGP, and NE. In order to describe the impact of NE on the $a_{k\ell}$ and $b_{k\ell}$, we shall now briefly review Sect. 6 of Orshan (1994) and start with the following lemma.

Lemma 3.1 *Let σ be an* NSPK. *For all distinct $k, \ell \in U$, $a^{\sigma}_{k\ell} \geq 0$ and $b^{\sigma}_{k\ell} = -\frac{1}{2}$.*

Note that the $b_{k\ell}$ have no longer to be specified and (3.3) simplifies to

$$\sigma(N, v) = \left\{ x \in X(N, v) \,\middle|\, \begin{array}{ll} a_{k\ell}s_{\ell k}(x) = a_{\ell k}s_{k\ell}(x) & \text{if } s_{k\ell}(x) + s_{\ell k}(x) < 0 \\ s_{\ell k}(x) = s_{k\ell}(x) & \text{if } s_{k\ell}(x) + s_{\ell k}(x) \geq 0 \end{array} \text{ for all } k, \ell \in N, k \neq \ell \right\}. \tag{3.4}$$

Note that Lemma 3.1 implies that an NSPK satisfies REAS. Indeed, let (N, v) be a game and $k \in N$. If $x \in X(N, v)$ satisfies $x_k < \min_{S \subseteq N \setminus \{k\}}(v(S \cup \{k\}) - v(S))$, then $e(S \cup \{k\}, x) > e(S, x)$ for all $S \subseteq N \setminus \{k\}$ and $e(\{k\}, x) > 0$. Hence, if $R \subseteq N$ satisfies $e(R, x) \geq e(T, x)$ for all $T \subseteq N$, then $e(R, x) > 0$ and $k \in R \neq N$. We conclude that, for any $\ell \in N \setminus R$, $s_{k\ell}(x) > s_{\ell k}(x)$ and $s_{k\ell}(x) > 0$ so that x is not a member of any NSPK. Similarly, if $x_k > \max_{S \subseteq N \setminus \{k\}}(v(S \cup \{k\}) - v(S))$, and $R \subseteq N$ satisfies, again, $e(R, x) \geq e(T, x)$ for all $T \subseteq N$, then $k \notin R$ and $e(R, x) \geq e(N \setminus \{k\}, x) > 0$ so that $R \neq \emptyset$. We conclude that for any $\ell \in R$, $s_{\ell k}(x) > s_{k\ell}(x)$ and $s_{\ell k}(x) > 0$ so that x cannot be a member of any NSPK.

[3] Note that $s_{ij}(y) := s_{ij}(y, v) = s_{ij}\left(y_T, v^{T,y}\right)$ for all distinct $i, j \in T$, $\emptyset \neq T \subseteq N$, and $y \in \mathbb{R}^N$.

Lemma 3.1 is one of Orshan's (1994) main results. Instead of repeating the details of its quite sophisticated proof, we now present a simpler and shorter proof of the following weaker version of the lemma. In fact, instead of deducing it, we assume REAS.

Proposition 3.2 *If σ is an* NSPK *that satisfies* REAS, *then $a^\sigma_{k\ell} \geq 0$ and $b^\sigma_{k\ell} = -\frac{1}{2}$ for all distinct $k, \ell \in U$.*

Proof Let $k, \ell \in U, k \neq \ell$. In order to show that $b^\sigma_{k\ell} = -\frac{1}{2}$, choose $i, j \in U \setminus \{k, \ell\}, i \neq j$, let $N = \{i, j, k, \ell\}$, and define (N, v) by $v(S) = -1$ for $\{i, j\} \subseteq S \subseteq N$ and $v(T) = 0$ for all other subsets T of N. By NE there exists $x \in \sigma(N, v)$. As k and ℓ are *null-players* of (N, v) (i.e., $v(S \cup \{k\}) = v(S \cup \{\ell\}) = v(S)$ for all $S \subseteq N$), REAS implies that $x_k = x_\ell = 0$. By PO, $x_i + x_j = -1$ so that $s_{k\ell}(x, v) = s_{\ell k}(x, v) = \max\{0, -x_i, -x_j, 1\} > 0$ and, hence, $b^\sigma_{k\ell} = b^\sigma_{\ell k}$.

In order to show the remaining statement, we assume, on the contrary, that there exist distinct players $k, \ell \in U$ with $a^\sigma_{k\ell} < 0$. Let $i \in U \setminus \{k, \ell\}$ and $M = \{k, \ell, i\}$. Define (M, w) by $w(S) = 1$ for $\{k, \ell\} \subseteq S \subseteq M$ and $w(T) = 0$ for all other subsets T of M. By NE there exists $y \in \sigma(M, w)$. As i is a null-player of (M, w), REAS and RGP imply $y_i = 0$, $y_k = a^\sigma_{k\ell}$, and $y_\ell = a^\sigma_{\ell k}$. Hence, $s_{\ell i}(y, w) = e(\{k, \ell\}, y, w) = 0$ and $s_{i\ell}(y, w) = e(\{i, k\}, y, w) = -y_k = -a^\sigma_{k\ell} > 0$ so that a contradiction to (3.3) and the first part has been obtained. □

Lemma 3.1 implies that any NSPK is a subsolution of the positive prekernel whose definition is now recalled. Let $t_+ = \max\{t, 0\}$ denote the positive part of a real number t. Let (N, v) be a game and $x \in \mathbb{R}^N$. Note that the excess $e(S, x, v)$, if positive, is interpreted as dissatisfaction of S and may be used in a bargaining process by players of S to object against x (see, e.g., Davis and Maschler 1967). These considerations suggest to define the *positive prekernel* of a game (N, v) to be the set

$$\mathcal{PK}_+(N, v) = \{x \in X(N, v) \mid s_{k\ell}(x)_+ = s_{\ell k}(x)_+ \quad \text{for all} k, \ell \in N, k \neq \ell\}.$$

Remark 3.3 For an analysis of the positive prekernel see Sudhölter and Peleg (2000), who present also the following axiomatization: The positive prekernel is the maximum solution that satisfies NE, AN, REAS, RGP, and CRGP; i.e., $\mathcal{PK}_+$ satisfies NE, AN, REAS, RGP, and CRGP, and any solution σ that also satisfies these axioms[4] is a subsolution of $\mathcal{PK}_+$. Note that Remark 2.4 (1) does not guarantee that CRGP implies "maximality", because the axioms may not determine the solution for 2-person games. In fact, "maximality" is needed, because the (symmetric) prekernel satisfies NE, AN, REAS, RGP, and CRGP as well and it does not coincide with the positive prekernel.

Corollary 3.4 *Any* NSPK *is a subsolution of the positive prekernel.*

Proof Let σ be an NSPK, and let (N, v) be a game. Let $x \in \sigma(N, v)$, and let $k, \ell \in N, k \neq \ell$. If $s_{k\ell}(x) + s_{\ell k}(x) < 0$, then (3.3) implies $s_{k\ell}(x), s_{\ell k}(x) \leq 0$, because $a_{k\ell}, a_{\ell k} \geq 0$ by Lemma 3.1. If $s_{k\ell}(x) + s_{\ell k}(x) \geq 0$, then (3.3) implies $s_{k\ell}(x) = s_{\ell k}(x)$, because $b_{k\ell} = b_{\ell k} \neq 0$ by Lemma 3.1. Hence, $x \in \mathcal{PK}_+(N, v)$. □

For all distinct $k, \ell \in U$, let $a_{k\ell} \in \mathbb{R}$ and $\boldsymbol{a} = (a_{k\ell})_{k,\ell \in U, k \neq \ell}$. Then the mapping $\boldsymbol{a}$ is said to *generate legal chains* if for all distinct players $j, k, \ell \in U$ the following conditions are fulfilled:

[4] In an earlier version from 1998, available as Preprint No. 10 of 1997/98 of *The Edmund Landau Center for Research in Mathematical Analysis* (see http://www.ma.huji.ac.il/~landau/preprint98/preprint10.pdf), they use an argument that is similar to the proof of Proposition 3.2 to show that AN and REAS may be replaced by COV and the *strong null-player property* that requires that the solution assigns (a) 0 to any null-player of a game (N, v) and (b) the restriction to N of the solution of $(N \cup \{i\}, v)$ if i is a null-player.

$$a_{k\ell} \geq 0; \tag{3.5}$$
$$a_{k\ell} + a_{\ell k} = 1; \tag{3.6}$$
$$a_{jk} = 1 = a_{k\ell} \Rightarrow a_{j\ell} = 1; \tag{3.7}$$
$$a_{jk} = \frac{1}{2} = a_{k\ell} \Rightarrow a_{j\ell} = \frac{1}{2}; \tag{3.8}$$
$$a_{jk} \notin \left\{0, \frac{1}{2}, 1\right\} \Rightarrow a_{j\ell} \in \{0, 1\}. \tag{3.9}$$

Now we are ready for presenting the characterization.

Theorem 3.5 (Orshan 1994) *The solution σ is an* NSPK if and only if there exists a mapping $\boldsymbol{a} = (a_{k\ell})_{k,\ell \in U, k \neq \ell}$ that generates legal chains such that, for any game (N, v), $\sigma(N, v)$ is given by (3.4).

It should be noted that the proof of the remaining part of the "only if" direction, i.e., the verification of (3.7)–(3.9), is also quite technical and not presented in this paper. Regarding the "if" direction we shall show a stronger result that is interesting in its own right. Let $\boldsymbol{a} = (a_{k\ell})_{k,\ell \in U, k \neq \ell}$ be a mapping that satisfies (3.5) and (3.6). Define, for any game (N, v), all distinct players $k, \ell \in N$, and any $x \in X(N, v)$,

$$f^{\boldsymbol{a}}_{k\ell}(x, v) = \begin{cases} \min\left\{s_{k\ell}(x) - s_{\ell k}(x), s_{k\ell}(x) - \frac{a_{k\ell}}{a_{\ell k}} s_{\ell k}(x)\right\}, & \text{if } a_{k\ell} \leq a_{\ell k}, \\ \max\left\{s_{k\ell}(x) - s_{\ell k}(x), \frac{a_{\ell k}}{a_{k\ell}} s_{k\ell}(x) - s_{\ell k}(x)\right\}, & \text{if } a_{k\ell} \geq a_{\ell k}. \end{cases} \tag{3.10}$$

The following lemma of Orshan (1994) is useful.

Lemma 3.6 *Let $\boldsymbol{a} = (a_{k\ell})_{k,\ell \in U, k \neq \ell}$ be a mapping that satisfies (3.5) and (3.6), and let σ be defined by (3.4). Let (N, v) be a game, and let $f_{k\ell}(x) = f^{\boldsymbol{a}}_{k,\ell}(x, v)$ for all $x \in X(N, v)$ and all distinct $k, \ell \in N$.*

(1) *For all distinct $k, \ell \in N$, the mapping $f_{k\ell} : X(N, v) \to \mathbb{R}$ is continuous.*
(2) *For all distinct $k, \ell \in N$, $f_{k\ell} = -f_{\ell k}$.*
(3) *$\sigma(N, v) = \{x \in X(N, v) \mid f_{k\ell}(x) = 0$ for all distinct $k, \ell \in N\}$.*

Proof Any excess function $e(S, \cdot) : X(N, v) \to \mathbb{R}$ is continuous for any $S \subseteq N$ so that, as a maximum of such functions, $s_{k\ell}$ is also continuous and the continuity of $f_{k\ell}$ follows. Statements (2) and (3) are straightforward consequences of (3.4) and (3.10), respectively. □

In order to derive the desired nonemptiness of the solution defined by (3.4), the following preparatory lemma is needed.

Lemma 3.7 *Let $\boldsymbol{a} = (a_{k\ell})_{k,\ell \in U, k \neq \ell}$ be a mapping that generates legal chains, and let (N, v) be a game. For any $x \in X(N, v)$, the binary relation $\succ_x$ on N defined by $k \succ_x \ell$ if $k \neq \ell$ and $f^{\boldsymbol{a}}_{k,\ell}(x, v) > 0$ is asymmetric and transitive.*

We just present a brief sketch of the proof. By (2) of Lemma 3.6, $\succ_x$ is asymmetric. Moreover, if $k, \ell \in N$, $k \neq \ell$, then by (3.5) and (3.6),

$$k \succ_x \ell \Leftrightarrow (s_{k\ell}(x) > s_{\ell k}(x) \text{ and } s_{k\ell}(x) > 0) \text{ or } (a_{\ell k} s_{k\ell}(x) > a_{k\ell} s_{\ell k}(x) \text{ and } s_{k\ell}(x) \leq 0).$$

The proof that $\succ_x$ is transitive is technical, requires to distinguish cases (see (3.7)–(3.9)), and uses, e.g., the transitivity of the "outweigh relation" (k *outweighs* ℓ if $s_{k\ell}(x) > s_{\ell k}(x)$, see Lemma 5.1 of Davis and Maschler 1965).

Orshan (1994) used Lemma 3.7 to show with the help of the KKM lemma that $\sigma(N, v)$ defined by (3.4) is nonempty, that is, the sufficiency ("if") part of Theorem 3.5. We shall now use an equivalent criterion, namely Brouwer's fixed point theorem, to prove a stronger result. To this end, let $x \in X(N, v)$ and denote

$$Z = Z(N, v, x) = \{z \in X(N, v) \mid e(S, z)_+ = e(S, x)_+ \text{ for all } S \subseteq N\}.$$

Thus, Z is the set of all preimputations such that a coalition keeps its dissatisfaction at x if it has some, and all coalitions satisfied at x remain satisfied.

Theorem 3.8 *Let $\boldsymbol{a}^\sigma$ and $\boldsymbol{a}^{\sigma'}$ be two mappings that generate legal chains, and let σ and σ' be the corresponding solutions defined by (3.4), where $a_{k\ell} = a^\sigma_{k\ell}$ and $a_{k\ell} = a^{\sigma'}_{k\ell}$ for all distinct $k, \ell \in U$, respectively. Let (N, v) be a game, and let $x \in \sigma'(N, v)$. Then*

$$\sigma(N, v) \cap Z(N, v, x) \neq \emptyset.$$

Proof Let $n = |N|$ and $Z = Z(N, v, x)$. Then Z is nonempty, compact, and convex. Let $f_{k\ell}(x) = f^{\boldsymbol{a}}_{k\ell}(x, v)$ for all $x \in X(N, v)$ and all distinct $k, \ell \in N$ (see (3.10)). Define, for any $z \in Z$, $g(z) = y \in \mathbb{R}^N$ by

$$y_k = z_k + \frac{1}{n^2} \sum_{\ell \in N \setminus \{k\}} f_{k\ell}(z) \quad \text{for all } k \in N.$$

By (2) of Lemma 3.6, $y(N) = z(N)$ so that $y \in X(N, v)$. In order to show that $y \in Z$ it is remarked that, by Corollary 3.4, for distinct $k, \ell \in N$,

$$s_{k\ell}(z) \leq 0 \Rightarrow s_{k\ell}(z) \leq f_{k\ell}(z); \tag{3.11}$$

$$s_{k\ell}(z) > 0 \Rightarrow f_{k\ell}(z) = 0. \tag{3.12}$$

Now, let $S \subseteq N$, $S \neq \emptyset$. Note that, again by (2) of Lemma 3.6,

$$y(S) = z(S) + \frac{1}{n^2} \sum_{k \in S, \ell \in N \setminus S} f_{k\ell}(z). \tag{3.13}$$

Let $k \in S$ and $\ell \in N \setminus S$. We distinguish two cases. If $e(S, z) > 0$, then $s_{k\ell}(z) \geq e(S, z) > 0$ and, by (3.12), $y(S) = z(S)$ and $e(S, y) = e(S, x)$. If $e(S, z) \leq 0$, then, by (3.11) or (3.12) respectively, $f_{k\ell}(z) \geq e(S, z)$. As $|S| \cdot |N \setminus S| < n^2$, $e(S, y) \leq 0$. We conclude that $g(z) \in Z$.

By (1) of Lemma 3.6, $g : Z \to Z$ is continuous and, by Brouwer's fixed point theorem, there exists $\widehat{z} \in Z$ such that $g(\widehat{z}) = \widehat{z}$. So we have $\sum_{\ell \in N \setminus \{k\}} f_{k\ell}(\widehat{z}) = 0$ for all $k \in N$. Now, Lemma 3.7 implies that $f_{k\ell}(\widehat{z}) = 0$ for all $k \in N$ and $\ell \in N \setminus \{k\}$. The proof is complete by (3) of Lemma 3.6. □

Note that Theorem 3.8 applied to $\sigma' = \mathcal{PK}$ (i.e., $a^\sigma_{k\ell} = \frac{1}{2}$ for all distinct $k, \ell \in U$) yields nonemptiness of any σ satisfying the assumptions of this theorem. Hence, σ is an NSPK.

Corollary 3.9 *Let (N, v) be a game and σ, σ' be* NSPK*s. If $x' \in \sigma'(N, v)$, then there exists $x \in \sigma(N, v)$ such that $e(S, x, v)_+ = e(S, x', v)_+$ for all $S \subseteq N$.*

Remark 3.10 In addition to 2-SIVA, NE, PO, COV, RGP, and CRGP, nonsymmetric prekernels have many further properties in common with the prekernel. We mention only two of them (see Sects. 7 and 9 of Orshan 1994). (1) Any NSPK applied to any 3-person game is a singleton. (2) Any NSPK applied to any convex game is a singleton in the core of the game. The proofs are generalizations of the proofs for the prekernel due to Davis and Maschler (1965) and Maschler et al. (1972).

4 Examples of nonsymmetric prenucleoli

In order to present examples of NSPNs, we first recall the definition of the positive core. Let (N, v) be a game. The *positive core* of (N, v) is the set $\mathcal{C}_+(N, v) = Z(N, v, \nu(N, v))$, that is,

$$\mathcal{C}_+(N, v) = \{x \in X^*(N, v) \mid (e(S, x, v))_+ = e(S, \nu(N, v), v)_+ \text{ for all } S \subseteq N\}. \quad (4.1)$$

By the definition of the prenucleolus (see Sect. 2), (4.1) may be written as

$$\mathcal{C}_+(N, v) = \{x \in X^*(N, v) \mid x(S) \geq v(S) - e(S, \nu(N, v), v)_+ \text{ for all } S \subseteq N\}. \quad (4.2)$$

Hence, $\mathcal{C}_+(N, v)$ is a *polytope*, i.e., a compact nonempty polyhedral convex set, that contains the prenucleolus $\nu(N, v)$ and coincides with the core of (N, v), denoted by $\mathcal{C}(N, v)$, if this core is nonempty.

Remark 4.1 For interpretations and illustrations of (4.1) see Orshan and Sudhölter (2010) who also present several characterizations of the positive core by simple properties, thereby providing a theoretical justification of this nonempty core extension. E.g., they prove the following result: Assume that $|U| = \infty$, and let σ be a solution that contains the prenucleolus.[5] Then σ satisfies REAS, COV, AN, RGP, and RCP, if and only if σ coincides with one of the following solutions: (a) The prenucleolus; (b) The positive core; (c) The relative interior of the positive core. Moreover, the property that the prenucleolus is a subsolution of σ may be replaced by several sets of axioms. We mention only one. We call a solution σ *convex valued* (CON), if $\sigma(N, v)$ is a convex set for any game (N, v). Then the foregoing result remains valid if "contains the prenucleolus" is replaced by "satisfies NE and CON".

The following notation will be used to construct the desired examples of NSPNs. A *configuration* of U is a pair $(\mathcal{U}, \succ)$ such that $\mathcal{U} \subseteq 2^U \setminus \{\emptyset\}$, $U = \bigcup\{S \mid S \in \mathcal{U}\}$, and $\succ$ is a total order on $\mathcal{U}$. Let $(\mathcal{U}, \succ)$ be a configuration. For any $N \subseteq U$ denote $\mathcal{N}^{\succ} = \{S \cap N \mid S \in \mathcal{U}\} \setminus \{N, \emptyset\}$. If $k, \ell \in U$, then $k \sim \ell$ if, for all $S \in \mathcal{U}, k \in S$ if and only if $\ell \in S$. The relation $\sim$ is an equivalence relation and the set of equivalence classes, denoted by $\mathcal{U}^{\succ}$, is a partition of U. Say that $(\mathcal{U}, \succ)$ is *feasible* if for any finite nonempty $N \subseteq U$ and any $T \in \mathcal{N}^{\succ}$ there exists a maximal element $S(T, N)$ in $\{S \in \mathcal{U} \mid S \cap N = T\}$, i.e., $S(T, N) \in \mathcal{U}$, $S(T, N) \cap N = T$, and if $Q \in \mathcal{U}$, $Q \cap N = T$, then $Q \not\succ S(T, N)$.

Note that a configuration $(\mathcal{U}, \succ)$ is automatically feasible, if one of the following conditions is satisfied:

[5] That is, $\nu(N, v) \in \sigma(N, v)$ for any game (N, v).

$$\text{The inverse of } \succ \text{ is a well-ordering;} \tag{4.3}$$
$$|\{S \in \mathcal{U} \mid R \subseteq S\}| < \infty \text{ for all } R \in \mathcal{U}^{\succ}; \tag{4.4}$$
$$|\{S \in \mathcal{U} \mid R \cap S = \emptyset\}| < \infty \text{ for all } R \in \mathcal{U}^{\succ}. \tag{4.5}$$

If $(\mathcal{U}, \succ)$ is feasible and N is a finite nonempty subset of U, then let $\succ^N$ be the total order on $\mathcal{N}^{\succ}$ that is *induced* by $\succ$, that is, for $P, Q \in \mathcal{N}^{\succ}$, $P \succ^N Q$ if and only if $S(P, N) \succ S(Q, N)$.

Now we are ready to define the NSPN $\sigma^{\succ}$ generated by the feasible configuration $(\mathcal{U}, \succ)$. Let N be a finite nonempty subset of U, and let $S^1, \ldots, S^t$ be determined by

$$\mathcal{N}^{\succ} = \{S^1, \ldots, S^t\} \quad \text{and} \quad S^1 \succ^N \cdots \succ^N S^t. \tag{4.6}$$

For any game (N, v) define

$$\sigma_0^{\succ}(N, v) = \{x \in \mathcal{C}_+(N, v) \mid (x(S^1), \ldots, x(S^t)) \geq_{lex} (y(S^1), \ldots, y(S^t)) \text{ for all } y \in \mathcal{C}_+(N, v)\}, \tag{4.7}$$

where $\geq_{lex}$ denotes the lexicographical order, i.e., for $x, y \in \mathbb{R}^t$, $x \geq_{lex} y$ is defined by $x = y$ or there is $i = 1, \ldots, t$ with $x_j = y_j$ for $j < i$ and $x_i > y_i$, and note that $\sigma_0^{\succ}(N, v)$ is recursively determined by

$$X^0 = \mathcal{C}_+(N, v) \text{ and } X^i = \{x \in X^{i-1} \mid x(S^i) \geq y(S^i) \text{ for all } y \in X^{i-1}\} \text{ for all } i = 1, \ldots, t, \tag{4.8}$$

so that $\sigma_0^{\succ}(N, v) = X^t$. Hence, $\sigma_0^{\succ}(N, v)$ is a polytope.

Remark 4.2 Let (N, v) be a game. If $(\mathcal{U}, \succ)$ is a feasible configuration, then, by (4.7) or (4.8), $\sigma_0^{\succ}(N, v)$ is a nonempty *face*[6] of $\mathcal{C}_+(N, v)$. We remark that the opposite is also true: If C is a nonempty face of $\mathcal{C}_+(N, v)$, then there exists a feasible configuration $(\mathcal{U}, \succ)$ such that $C = \sigma_0^{\succ}(N, v)$. In order to show this statement, note that, by (4.2) there exists $\mathcal{S} \subseteq 2^N \setminus \{\emptyset, N\}$ such that

$$C = \{x \in \mathcal{C}_+(N, v) \mid x(S) = v(S) - e(S, \nu(N, v), v)_+ \quad \text{for all } S \in \mathcal{S}\}.$$

Now, $C = \sigma_0^{\succ}(N, v)$ for any feasible configuration $(\mathcal{U}, \succ)$ that satisfies $\mathcal{N}^{\succ} = \{N \setminus S \mid S \in \mathcal{S}\}$ (see (4.6) for the definition of $\mathcal{N}^{\succ}$). Moreover, e.g., with $\mathcal{U} := \{U \setminus S \mid S \in \mathcal{S}\}$ and any total order $\succ$ on $\mathcal{U}$, $(\mathcal{U}, \succ)$ is feasible (because $|\mathcal{U}| < \infty$) and, hence satisfies the required property.

Define (see Sect. 2 for the definition of $\mathcal{N}((N, v); X)$, the nucleolus of (N, v) with respect to a subset X of $\mathbb{R}^N$)

$$\sigma^{\succ}(N, v) = \mathcal{N}((N, v); \sigma_0^{\succ}(N, v)). \tag{4.9}$$

Lemma 4.3 *For a feasible configuration* $(\mathcal{U}, \succ)$, $\sigma^{\succ}$ *defined by (4.9) is an* NSPN.

---

[6] A *face* of a convex set C is a convex subset C' of C such that every line segment in C with a relative interior point in C' has both endpoints in C' (see Rockafellar 1970, Sect. 18). E.g., the extreme points of C are its 0-dimensional faces.

Proof Schmeidler (1969) shows that the nucleolus of a game with respect to a nonempty compact convex set is a singleton and the proof of COV is straightforward. In order to show RGP, let (N, v) be a game and $\emptyset \neq S \subseteq N$. Let $\mathcal{R}^N = \{R \cap N \mid R \in \mathcal{U}^{\succ}\} \setminus \{\emptyset\}$, that is, $\mathcal{R}^N$ is the *coalition structure* of N generated by the partition $\mathcal{U}^{\succ}$ of U. Note that, for any $x \in \sigma_0^{\succ}(N, v)$,

$$\sigma_0^{\succ}(N, v) = \{y \in \mathcal{C}_+(N, v) \mid y(R) = x(R) \text{ for all } R \in \mathcal{R}^N\}. \tag{4.10}$$

Moreover, note that $\mathcal{R}^S = \{R \cap S \mid R \in \mathcal{R}^N\} \setminus \{\emptyset\}$, that is, $\mathcal{R}^S$ is the coalition structure of N *reduced* to S. This fact together with RGP and RCP of $\mathcal{C}_+$ (see Peleg and Sudhölter 2003, Theorem 6.3.14) implies that $\sigma_0^{\succ}$ satisfies RGP and RCP as well.

Let the *derived game* $(N, v_{\succ})$ be the game that differs from (N, v) only inasmuch as $v_{\succ}(R) = x(R)$ for any $R \in \mathcal{R}^N$ and any $x \in \sigma_0^{\succ}(N, v)$. By RGP and RCP of $\sigma_0^{\succ}$, $(v^{S,x})_{\succ} = (v_{\succ})^{S,x}$ for any $x \in \sigma_0^{\succ}(N, v)$. Now, $\sigma^{\succ}(N, v)$ coincides with the prenucleolus of the game with coalition structure $(N, v_{\succ}, \mathcal{R}^N)$. As the prenucleolus on games with coalition structures satisfies RGP (see, e.g., Peleg and Sudhölter 2003, Theorem 5.2.7), our proof is complete. □

Remark 4.4 Let (N, v) be a game. By Remark 4.2, for any face of $\mathcal{C}_+(N, v)$ there exists a feasible configuration $(\mathcal{U}, \succ)$ such that the $\sigma^{\succ}$ assigns the nucleolus with respect to this face to (N, v). It should be remarked that, in order to show that AN is logically independent of the remaining axioms in Sobolev's axiomatization of the prenucleolus (Theorem 2.1), Sudhölter (1993) used an example that is a special case of a solution $\sigma^{\succ}$ defined by (4.9), namely the case that all elements of $\mathcal{U}$ are singletons so that $\succ$ may be regarded as a total order relation on U. Hence, under this restriction, just the extreme points of the positive core that are lexicographically maximal according to some order of the players may be selected. The following example shows that this restriction is crucial even if also lexicographically minimal core elements may be selected: Let (N, v) be the symmetric 5-person game ($N = \{1, \ldots, 5\}$) defined by

$$v(S) = \begin{cases} 0, & \text{if} |S| \leq 1, \\ 1, & \text{if} |S| = 2, \\ 8, & \text{if } |S| = 3, \\ 11, & \text{if} |S| = 4, \\ 23, & \text{if } S = N. \end{cases}$$

Let $x = (11, 4, 4, 4, 0)$, $y = (12, 3, 3, 3, 2)$, and $z = (8, 7, 7, 1, 0)$. Derks and Kuipers (2002) show that x, y, and z are extreme points of $\mathcal{C}(N, v)$ and that $\mathcal{C}(N, v)$ is the convex hull of the vectors x, y, and z and of their permuted variants so that the core has 100 extreme points; 20 corresponding to the permutations of the vectors x and y and 60 corresponding to the permutations of z. As the core is nonempty, it coincides with the positive core. Clearly, the permutations of y are the lexicographical maximal elements and the permutations of z are the lexicographical minimal elements of the (positive) core with respect to the orderings of N.

By Remark 2.4 (2), any NSPN is a subsolution of a unique NSPK. We now determine the NSPK σ that contains $\sigma^{\succ}$ as a subsolution. Let $k, \ell \in U, k \neq \ell$. As $\sigma^{\succ}$ is a subsolution of the positive core and the prenucleolus of a game is an element of the prekernel of the game, $b_{k\ell}^{\sigma} = -\frac{1}{2}$ (see (3.2)). Now, we determine $a_{k\ell}^{\sigma}$ (see (3.1)). As $\mathcal{U}^{\succ}$ is a partition of U, there are unique $P, Q \in \mathcal{U}^{\succ}$ such that $k \in P$ and $\ell \in Q$. With $N = \{k, \ell\}$ let (N, v) be the 0-1 normalized game, and let x be the unique element of $\sigma^{\succ}(N, v)$, i.e., $x_k = a_{k\ell}^{\sigma}, x_\ell = a_{\ell k}^{\sigma}$. The following 2 cases may occur:

(1) $P = Q$: Then $x = \nu(N, v)$ so that $a^\sigma_{k\ell} = \frac{1}{2}$.
(2) $P \neq Q$: Then $\mathcal{N}^\succ = \{\{k\}, \{\ell\}\}$ and there are two possible subcases: If $\{k\} \succ^N \{\ell\}$, then $a^\sigma_{kl} = 1$. If $\{\ell\} \succ^N \{k\}$, then $a^\sigma_{kl} = 0$.

Note that these considerations just depend on the equivalence classes P and Q and not on their representatives k and ℓ. Hence, for any $P, Q \in \mathcal{U}^\succ$, $P \neq Q$, there exists a maximal element, denoted by $S(P, Q)$, in $\mathcal{U}$ that contains P and does not contain Q. By a slight abuse of notation we write $P \succ Q$ if $S(P, Q) \succ S(Q, P)$. These observations show that σ is determined as follows:

$$a^\sigma_{k\ell} = 1 \text{ for all } k \in P, \ell \in Q, P, Q \in \mathcal{U}^\succ \text{ with } P \succ Q; \tag{4.11}$$
$$a^\sigma_{k\ell} = 0 \text{ for all } k \in P, \ell \in Q, P, Q \in \mathcal{U}^\succ \text{ with } Q \succ P; \tag{4.12}$$
$$a^\sigma_{k\ell} = \frac{1}{2} \text{ for all } k, \ell \in P, P \in \mathcal{U}^\succ. \tag{4.13}$$

Thus, $a^\sigma_{k\ell} \in \left\{0, \frac{1}{2}, 1\right\}$ for all distinct $k, \ell \in U$.

Conversely, let σ be an NSPK such that $a_{k\ell} = a^\sigma_{k\ell} \in \left\{0, \frac{1}{2}, 1\right\}$ for all distinct $k, \ell \in U$ such that (3.8) and (3.9) are satisfied. We shall present two special examples of feasible configurations that generate NSPNs that are subsolutions of σ. For any distinct players $k, \ell \in U$ say that $k \sim \ell$ if $a_{k\ell} = \frac{1}{2}$. Let $\mathcal{U}$ denote the set of equivalence classes with respect to $\sim$ and define $\mathcal{U}_c = \{U \setminus R \mid R \in \mathcal{U}\}$. Moreover, define $\succ$ on $\mathcal{U}$ by $P \succ R$ iff $a_{k\ell} = 1$ for any $k \in P$ and $\ell \in R$ for all $P, R \in \mathcal{U}$, and define $\succ_c$ of $\mathcal{U}_c$ by $U \setminus R \succ_c U \setminus P$ if $P \succ R$ for all $P, Q \in \mathcal{U}$. By (4.4) and (4.5), respectively, and by (3.8) and (3.9), $(\mathcal{U}, \succ)$ and $(\mathcal{U}_c, \succ_c)$ are feasible configurations. Moreover, $\mathcal{U}^\succ = (\mathcal{U}_c)^{\succ_c} = \mathcal{U}$ and by the above construction, $\sigma^\succ$ and $\sigma^{\succ_c}$ are subsolutions of σ. The next example shows that $\sigma^\succ \neq \sigma^{\succ_c}$ if $|\mathcal{U}| \geq 4$.

Example 4.5 Let σ be an NSPK defined by $a_{k\ell} \in \left\{0, \frac{1}{2}, 1\right\}$ for all distinct $k, \ell \in U$. We assume that σ generates at least 4 equivalence classes. Hence, we may assume without loss of generality that $a_{12} = a_{23} = a_{34} = 1$. Let $N = \{1, 2, 3, 4\}$ and define (N, v) by

$$v(\emptyset) = v(N) = v(\{1, 2\}) = v(\{1, 3\}) = v(\{2, 4\}) = v(\{3, 4\}) = 0 \text{ and } v(S) = -2, \text{ otherwise.}$$

With $x = (1, -1, -1, 1)$ it is straightforward to verify that $\mathcal{C}(N, v)$ is the convex hull of x and $-x$. Moreover, if $S^1, \ldots, S^t$ are defined by (4.6), then $S^i = \{i\}$ and $t = 4$ so that the set X^1 of (4.8) is defined by $X^1 = \{z \in \mathcal{C}(N, v) \mid z_1 \geq y_1 \text{ for all } y \in \mathcal{C}(N, v)\}$. We conclude that $X^1 = \{x\}$ and, hence, $\sigma^\succ(N, v) = \{x\}$. In the other case, if the S^i are defined as in (4.6), but now for $\succ_c$ rather than $\succ$, then $S^i = N \setminus \{5 - i\}$, $t = 4$, and $X^1 = \{z \in \mathcal{C}(N, v) \mid z(\{1, 2, 3\}) \geq y(\{1, 2, 3\}$ for all $y \in \mathcal{C}(N, v)\}$. We conclude that $\sigma^{\succ_c}(N, v) = \{-x\}$.

The next example shows that there exist NSPKs with $a_{k\ell} \notin \left\{0, \frac{1}{2}, 1\right\}$ for some $k, \ell \in U$.

Example 4.6 Let σ be an NSPK such that $a_{12} \notin \left\{0, \frac{1}{2}, 1\right\}$ and $a_{k\ell} \in \{0, 1\}$ for all $k, \ell \in U$, $k \neq \ell$, with $\{k, \ell\} \neq \{1, 2\}$. Let σ' be the NSPK that differs from σ only inasmuch as $a_{12} = a_{21} = \frac{1}{2}$, and let $(\mathcal{U}, \succ)$ be any feasible configuration such that $\sigma^\succ$ is a subsolution of σ'. The NSPN in σ, $\widetilde{\sigma}$, is defined as follows. Let (N, v) be a game. If $\{1, 2\} \not\subseteq N$, then $\widetilde{\sigma}(N, v) = \sigma^\succ(N, v)$. If $\{1, 2\} \subseteq N$, then let x be the unique element of $\sigma^\succ(N, v)$. If $s_{12}(x, v) \geq 0$, then define $\sigma(N, v) = \{x\}$. If $\alpha := s_{12}(x, v) < 0$, then let $y \in \mathbb{R}^N$ differ from x only inasmuch as $y_1 = x_1 + \varepsilon$ and $y_2 = x_2 - \varepsilon$, where $\varepsilon = \alpha(a_{21} - a_{12})$. As $s_{21}(x, v) = \alpha$, we may conclude that $a_{12}s_{21}(y, v) = a_{21}s_{12}(y, v)$ so that $y \in \sigma(N, v)$. Define $\widetilde{\sigma}(N, v) = \{y\}$. The proof that $\widetilde{\sigma}$ satisfies RGP is straightforward.

5 Nonsymmetric prenucleoli and the positive core

According to Sect. 3, any NSPK is a subsolution of the positive prekernel, and all NSPNs constructed in Sect. 4 are subsolutions of the positive core. Moreover, by Remark 2.4 (2), any NSPN is a subsolution of its corresponding NSPK. Finally, as the prenucleolus is a subsolution of the prekernel, by Theorem 3.8 (applied to $\sigma' = \mathcal{PK}$ and to $x = \nu(N, v)$), the positive core of a game intersects any NSPK.

Thus, in view of the preceding paragraph, the question arises, whether there is the following parallel between the concepts of NSPKs and the positive prekernel on the one hand, and the concepts of NSPNs and the positive core on the other hand: Is an NSPN a subsolution of the positive core?

Unfortunately, unlike in the case of NSPKs, we do not have a complete characterization of NSPNs. This indicates that we also do not know an answer to this question in general.

However, this section provides two "partial" answers to this question (see Theorem 5.1 and Theorem 5.3).

First, an affirmative answer may be deduced under some "innocent" further condition. This condition "only" requires that infinitely many potential players have to be treated equally. As, in view of Remark 2.3 (1), $|U| = \infty$ has anyway to be assumed, this condition may be satisfied and may be regarded as not very demanding.

Theorem 5.1 *Let $U' \subseteq U$ be an infinite set. If σ is an* NSPN *such that σ satisfies* ETP *on the set of all games (N, v) with $N \subseteq U'$, then σ is a subsolution of the positive core.*

Let $U' \subseteq U$. It should be remarked that for any feasible configuration $(\mathcal{U}, \succ)$ as defined in Sect. 4 such that $U' \subseteq P$ for some $P \in \mathcal{U}^{\succ}$, the NSPN $\sigma^{\succ}$ defined by (4.6) satisfies ETP on the set of all games with players in U', because, by (4.13), equivalent players are treated equally. Thus, under the assumptions of Theorem 5.1, if $U \setminus U' \neq \emptyset$, then there is a nontrivial NSPN, i.e., an NSPN that does not coincide with the prenucleolus. Indeed, by (4.11)–(4.13) and in view of Remark 2.3 (1), $\sigma^{\succ}$ coincides with the prenucleolus if and only if $U \in \mathcal{U}^{\succ}$, i.e., $\mathcal{U} = \{U\}$.

In order to show Theorem 5.1, the following lemma, whose proof is postponed to Sect. 6, is useful.

Lemma 5.2 *Assume that $|U| = \infty$. Let σ be an* NSPN *and (N, v) be a game. Let $\{x\} = \sigma(N, v)$ and $\pi : N \to U$ be an injection. Let $(\pi(N), u)$ be given by*

$$u(\pi(S)) = \max\{v(S), x(S)\} \quad \text{forall} \quad S \subseteq N. \tag{5.1}$$

Then $\sigma(\pi(N), u) = \{\pi x\}$.

Proof of Theorem 5.1 By Theorem 2.1 and Remark 2.3 (1), for any game (N', v') with $N' \subseteq U'$, $\sigma(N', v')$ coincides with the prenucleolus. Now, let (N, v) be a game, $\{x\} = \sigma(N, v)$, and let $(\pi(N), u)$ be defined as in Lemma 5.2 such that, moreover, $\pi(N) \subseteq U'$. The requirement $\pi(N) \subseteq U'$ can be matched, because $|U'| = \infty$. By Lemma 5.2, $\{\pi x\} = \sigma(\pi(N), u)$, and, hence, $\pi x = \nu(\pi(N), u)$.

In order to recall a characterization of the prenucleolus due to Kohlberg (1971), let (M, w) be any game and $z \in X(M, w)$. For any $\alpha \in \mathbb{R}$ denote $\mathcal{D}(\alpha, z, w) = \{S \subseteq M \mid e(S, z, w) \geq \alpha\}$. Then $z = \nu(M, w)$ if and only if, for any $\alpha \in \mathbb{R}$, the following condition is satisfied for all $y \in \mathbb{R}^M$: If $y(S) \geq 0$ for all $S \in \mathcal{D}(\alpha, z, w)$, then $y(S) = 0$ for all $S \in \mathcal{D}(\alpha, z, w)$. Applied to $(\pi(N), u)$, by the definition of u, this criterion yields the following condition for any $y \in \mathbb{R}^N$ and any $\alpha > 0$: If $y(S) \geq 0$ for all $S \in \mathcal{D}(\alpha, x, v)$, then $y(S) = 0$ for all $S \in \mathcal{D}(\alpha, x, v)$. By a simple modification (see Peleg and Sudhölter 2003, (6.3.8)) of Kohlberg's criterion, $x \in \mathcal{C}_+(N, v)$. □

The second "partial" answer is provided by the following theorem.

Theorem 5.3 *Let $|U| = \infty$ and σ be an* NSPN. *Then σ satisfies* AN *when restricted to the set of games that have a nonempty anticore.*[7]

The following remark will be used in the proof of Theorem 5.3.

Remark 5.4 Let (N, v) be a game, $x \in \mathcal{PK}_+(N, v)$, and $k \in N$. Then there exist $S, T \subseteq N$ such that $k \in S$, $k \notin T$, and $e(S, x, v) = e(T, x, v) = \max_{R \subseteq N} e(R, x, v)$. (We don't exclude the possibilities $S = N$ and $T = \emptyset$.)

Proof of Theorem 5.3 Let (N, v) be a game that has a nonempty anticore, and let $\pi : N \to U$ be an injection. Let $N' = \pi(N)$, $v' = \pi v$, and $\sigma(N, v) = \{x\}$. It has to be shown that $\sigma(N', v') = \{\pi(x)\}$. By COV of σ and of the anticore, we may assume that $v(N) = 0$ and $v(S) \geq 0$ for all $S \subseteq N$. Let $k \in N$. By the infinity assumption on $|U|$, we may assume that $N' = (N \setminus \{k\}) \cup \{k^*\}$ for some $k \in N$, $k^* \in U \setminus N$, and that $\pi(k) = k^*$ and $\pi(\ell) = \ell$ for all $\ell \in N \setminus \{k\}$. Let $\alpha \leq -\max_{S \subseteq N} v(S)$, $\widetilde{N} = N \cup \{k^*\}$, and define the game $(\widetilde{N}, w)$ by

$$w(S) = \begin{cases} v(S), & \text{if} S \subseteq N \setminus \{k\}, \\ v(S \setminus \{k^*\}), & \text{if} \{k, k^*\} \subseteq S, \\ \alpha, & \text{otherwise}, \end{cases}$$

for any $S \subseteq \widetilde{N}$. Let $\{z\} = \sigma(\widetilde{N}, w)$. It remains to show that, for all $S \subseteq N'$,

$$w^{N', z}(S) = \begin{cases} v(S), & \text{if} k^* \notin S, \\ v((S \setminus \{k^*\}) \cup \{k\}) - z_k, & \text{if} k^* \in S. \end{cases} \tag{5.2}$$

Indeed, if (5.2) is valid, then, by COV and RGP, $\{\pi(x)\} = \sigma(N', v')$.

Assume, on the contrary, that (5.2) is not valid. Then two cases may occur:

(1) There exists $S \subseteq N \setminus \{k\}$ such that $v(S) = w(S) < w(S \cup \{k\}) - z_k = \alpha - z_k$. As $v(S) \geq 0$, $w(\{k\}) - z_k = \alpha - z_k > 0$. We conclude that $s_{kk^*}(z, w) > 0$. As $z \in \mathcal{PK}_+(\widetilde{N}, w)$, $s_{k^*k}(z, w) = s_{kk^*}(z, w)$. As k and k^* are substitutes, $z_{k^*} = z_k$. Now, let $T \subseteq \widetilde{N}$ such that $k \notin T$. If $k^* \notin T$, then

$$e(T, z, w) = v(T) - z(T) < v(T \cup \{k\}) - z(T) - z_k - z_{k^*} = e(T \cup \{k, k^*\}, z, w),$$

because $-z_k - z_{k^*} = -2z_k > -2\alpha \geq \max_{S,T \subseteq N} v(S) - v(T)$. If $k^* \in T$, then

$$e(T, z, w) = \alpha - z(T) < v((T \setminus \{k^*\}) \cup \{k\}) - z(T) - z_k = e(T \cup \{k\}, z).$$

In view of Remark 5.4 the desired contradiction has been obtained.

[7] The *anticore* of a game (N, v) is the set of all $x \in X(N, v)$ such that $x(S) \leq v(S)$ for all $S \subseteq N$.

(2) There exists $S \subseteq N \setminus \{k\}$ such that $v(S \cup \{k\}) - z_k = w(S \cup \{k, k^*\}) - z_k < w(S \cup \{k\}) = \alpha$. As $v(S \cup \{k\}) \geq 0$, $z_k > -\alpha$ so that $z(\widetilde{N} \setminus \{k\}) = -z_k < \alpha$. We conclude that $s_{k^*k}(w, z) \geq e(\widetilde{N} \setminus \{k\}, z, w) = \alpha + z_k > 0$ and, as in the first case, $z_k = z_{k^*}$, because k and k^* are substitutes. Let $S \subseteq \widetilde{N}$ with $k \in S$. If $k^* \in S$, then

$$e(S, z, w) = v(S \setminus \{^*\}) - z(S) < v(S \setminus \{k, k^*\}) - z(S \setminus \{k, k^*\}) = e(S \setminus \{k, k^*\}, z, w),$$

because $z_k + z_{k^*} > -2\alpha \geq \max_{S,T \subseteq N} v(S) - v(T)$. If $k^* \notin S$, then $e(T, z, w) = \alpha - z(T) < e(T \setminus \{k\}, z, w)$ and, hence, the Remark 5.4 again yields the desired contradiction. □

Corollary 5.5 *Assume that* $|U| = \infty$*, and let* σ *be an* NSPN. *If* (N, v) *is a game such that* $\nu(N, v)$ *is an element of the anticore of* (N, v)*, then* $\sigma(N, v) = \{\nu(N, v)\}$.

Proof By COV, we may assume that $\nu(N, v) = 0 \in \mathbb{R}^N$. As $\nu(N, v)$ is an element of the anticore of (N, v), $v(S) \geq 0$ for all $S \subseteq N$ and $v(N) = 0$. According to Sobolev (1975), there exists a transitive game[8] (M, w) that satisfies the following properties:

(1) $N \subseteq M$.
(2) $w(T) \in \{v(S) \mid S \subseteq N\}$ for all $T \subseteq M$ and $w(M) = v(N) = 0$.
(3) With $z = 0 \in \mathbb{R}^M$, $w^{N,z} = v$.

By (2), $w(S) \geq 0$ for all $S \subseteq M$ and $w(M) = 0$ so that z is in the anticore of (M, w). By Theorem 5.3, SIVA, and PO, $\{z\} = \sigma(M, w)$. By SIVA, RGP, and (3), $\{\nu(N, v)\} = \sigma(N, v)$. □

Remark 5.6 Let σ be an NSPN and (N, v) be a game. Note that the game $(\pi(N), u)$ defined in Lemma 5.2 has a nonempty anticore. Thus, if σ coincides with the prenucleolus for any game that has a nonempty anticore, then σ must be a subsolution of the positive core. However, the proof of Corollary 5.5 does not show that $\sigma(N, v) = \{\nu(N, v)\}$ for any game whose anticore is nonempty, because the transitive game (M, w) defined in this proof may not inherit from (N, v) the property of having a nonempty anticore.

6 The Proof of Lemma 5.2

As $|U| = \infty$, we may assume that $\pi(N) \cap N = \emptyset$. Denote $\pi(i) = i^*$ for all $i \in N$, $S^* = \{i^* \mid i \in S\}$ for every $S \subseteq N$, and $M = N \cup N^*$. Let $\beta = \max_{S,T \subseteq N} v(S) - v(T)$ and $\alpha < -2|N|\beta$. Let (M, w) be defined be the following formula, where $S, T \subseteq N$:

$$w(S \cup T^*) = \begin{cases} v(S), & \text{if } S = T \\ 0, & \text{if } S = \emptyset \\ v(N), & \text{if } S = N \text{ and } T = \emptyset \\ \alpha, & \text{otherwise} \end{cases} \tag{6.1}$$

Let $\{y\} = \sigma(M, w)$, $\mu = \max_{S,T \subseteq N} e(S \cup T^*, y, w)$, and $\rho = \max_{k \in N} (s_{kk^*}(y, w))_+$. Note that $\rho = \max_{k \in N} (s_{k^*k}(y, w))_+$ by Corollary 3.4.

It suffices to show that $\rho = 0$. Indeed, if $\rho = 0$, then $y(N) \geq v(N)$ and $y_{N^*} \geq 0$. Hence, by PO of σ, $y(N) = v(N)$ and $y_{N^*} = 0$, thus $w^{N,y} = v$ by (6.1) and $y_N = x$ by RGP. Moreover, by REAS, $x_i \geq -\beta$ for all $i \in N$, thus $w^{N^*,y}(S^*) = (v(S) - x(S))_+$ for every $S \subseteq N$. Hence, COV completes the proof.

[8] A game is *transitive* if its symmetry group is transitive.

In order to show that $\rho = 0$ we assume, on the contrary, $\rho > 0$ and proceed by showing the following 7 claims. This procedure finally leads to the desired contradiction. Let

$$N_0 = \{i \in N \mid y_i < 0\},\ \overline{N_0} = \{i \in N \mid y_i \leq 0\},\ N'_0 = \{i \in N \mid y_{i^*} < 0\}, \text{ and}$$
$$\overline{N'_0} = \{i \in N \mid y_{i^*} \leq 0\}.$$

Claim 1 $e(N, y, w) < \mu$: Assume the contrary. Then $s_{k^*k}(y, w) = s_{kk^*}(y, w) = \mu > 0$ for all $k \in N$ by Corollary 3.4. By PO of σ there exists $i \in N$ with $y_{i^*} > 0$. Let $s_{i^*i}(y, w)$ $(= \mu)$ be attained by $S \cup T^*$, that is,

$$S, T \subseteq N,\ i \notin S,\ i \in T, \quad \text{and} \quad e(S \cup T^*, y, w) = \mu.$$

By (6.1), $w(S \cup T^*) \leq w(S \cup (T^* \setminus \{i^*\}))$, thus

$$\mu = e(S \cup T^*, y, w) \leq w(S \cup (T^* \setminus \{i^*\})) - y(S) - y(T^*) < e(S \cup (T^* \setminus \{i^*\}), y, w),$$

which is impossible.

Claim 2 $\rho < \mu$: Assume the contrary and let $k \in N$ satisfy $s_{kk^*}(y, w) = \mu$. Let $s_{kk^*}(y, w)$ be attained by $S \cup T^*$, that is, $S, T \subseteq N,\ k \in S,\ k \notin T$, and $e(S \cup T^*, y, w) = \mu$. By Claim 1, $w(S \cup T^*) = \alpha$. For every pair (Q, Q') satisfying $N_0 \subseteq Q \subseteq \overline{N_0}$ and $N'_0 \subseteq Q' \subseteq \overline{N'_0}$ we have

$$\mu \geq e(Q \cup Q'^*, y, w) \geq \alpha - y(Q) - y(Q'^*) \geq \alpha - y(S) - y(T^*) = \mu, \tag{6.2}$$

thus all inequalities of (6.2) are, in fact, equalities. Hence, $N_0 \subseteq S \subseteq \overline{N_0},\ N'_0 \subseteq T \subseteq \overline{N'_0}$, and

$$(N_0 \setminus \overline{N'_0}) \cup (N'_0 \setminus \overline{N_0}) \neq \emptyset. \tag{6.3}$$

Indeed, (6.3) follows from the inequality $\alpha - y(R) - y(R^*) < v(R) - y(R) - y(R^*)$ which is true for every $R \subseteq N$.

Two cases may occur:

(1) $N_0 \setminus \overline{N'_0} \neq \emptyset$: Then there exists $i \in N_0 \setminus \widetilde{N}_0$. If additionally $|N_0| \geq 2$, then

$$\begin{aligned} s_{ii^*}(y, w) &\geq e(N_0 \cup N'^*_0, y, w) = \alpha - y(N_0) - y(N'^*_0) \\ &= \mu > e((N_0 \setminus \{i\}) \cup N'^*_0 \cup \{i^*\}, y, w). \end{aligned}$$

Thus $s_{i^*i}(y, w)$ is attained by $N'^*_0 \cup \{i^*\}$ in any case. As $y_{i^*} > 0$, $e(N'^*_0 \cup \{i^*\}, y, w) < e(N'^*_0, y, w)$. Hence $N'_0 \neq \emptyset$ and $s_{\ell^*\ell}(y, w) > \mu$ for all $\ell \in N'_0$ which is impossible.

(2) $N'_0 \setminus \overline{N_0} \neq \emptyset$: Then there exists $j \in N'_0 \setminus \overline{N_0}$. By Claim 1,

$$\begin{aligned} s_{j^*j}(y, w) &= e(N_0 \cup N'^*_0, y, w) = \mu \\ &> \max\left\{e(N_0 \cup \{j\} \cup (N'^*_0 \setminus \{j^*\}), y, w), e(N, y, w)\right\} = s_{jj^*}(y, w) \end{aligned}$$

so that the contradiction is obtained.

Claim 3 $y_k + y_{k^*} \geq -\beta$ for all $k \in N$: Assume, on the contrary, that there exists $k \in N$ with $y(\{k, k^*\}) < -\beta$. By Remark 5.4 there exists $S \cup T^* \subseteq M$ with $e(S \cup T^*, y, w) = \mu$ and $k \notin S$. By Claim 2, $k \notin T$. However,

$$w(S \cup \{k\} \cup T^* \cup \{k^*\}) \geq w(S \cup T^*) - \beta,$$

thus $e(S \cup \{k\} \cup T^* \cup \{k^*\}, y, w) > e(S \cup T^*, y, w) = \mu$ which is impossible.

Claim 4 $e(N, y, w) < \rho$: Assume the contrary. As in the proof of Claim 1, by PO, there exists $i \in N$ with $y_{i^*} > 0$. Let $s_{i^*i}(y, w)$ $(= s_{ii^*}(y, w) = \rho)$ be attained by $S \cup T^*$. Then

$$e(S \cup (T^* \setminus \{i^*\}), y, w) > e(S \cup T^*, y, w) = \rho > 0 \tag{6.4}$$

implies $S = T \setminus \{i\} \neq \emptyset$. As $v(S \cup T^*) = \alpha$, (6.4) implies

$$\alpha > y(S) + y(T^*) > y(S) + y(S^*). \tag{6.5}$$

Now, (6.5) is impossible by Claim 3.

Claim 5 $\rho \leq \alpha - y(N_0) - y({N_0'}^*)$: Let $k \in N$ satisfy $s_{kk^*}(y, w) = \rho$. By Claim 4 $s_{kk^*}(y, w)$ is attained by $S \cup T^* \subseteq M$ satisfying $w(S \cup T^*) = \alpha$. Hence our claim follows immediately.

Claim 6 $N_0' \subseteq \overline{N_0}$: Assume, on the contrary, there exists $k \in N_0' \setminus \overline{N_0}$. By Claim 5,

$$s_{k^*k}(y, w) \geq e(N_0 \cup {N_0'}^*, y, w) \geq \alpha - y(N_0) - y({N_0'}^*) \geq \rho,$$

thus $s_{k^*k}(y, w) = \rho$. By Claim 4,

$$s_{kk^*}(y, w) = e(N_0 \cup \{k\} \cup ({N_0'}^* \setminus \{k^*\}), y, w) < \alpha - y(N_0) - y({N_0'}^*),$$

which is impossible.

Claim 7 $N_0 \subseteq \overline{N_0'}$: Assume, on the contrary, there exists $k \in N_0 \setminus \overline{N_0'}$. By Claim 5,

$$s_{kk^*}(y, w) \geq e(N_0 \cup {N_0'}^*, y, w) \geq \alpha - y(N_0) - y({N_0'}^*) \geq \rho,$$

thus $s_{kk^*}(y, w) = \rho$. As $e((N_0 \setminus \{k\}) \cup {N_0'}^* \cup \{k^*\}, y, w) < \alpha - y(N_0) - y({N_0'}^*)$, $s_{k^*k}(y, w)$ must be attained by ${N_0'}^* \cup \{k^*\}$. Hence, $N_0' \neq \emptyset$. The observation

$$\rho \geq e({N_0'}^*, y, w) > -y({N_0'}^*) - y_{k^*} = e({N_0'}^* \cup \{k^*\}, y, w)$$

yields a contradiction.

By Claims 6 and 7

$$y(N_0 \cup N_0') = y(N_0) \quad \text{and} \quad y(N_0^* \cup {N_0'}^*) = y({N_0'}^*)$$

and, by Claim 5, $y(N_0) + y({N_0'}^*) < \alpha - \rho < \alpha$. On the other hand, by Claim 3,

$$y(N_0 \cup N_0') + y(N_0^* \cup {N_0'}^*) \geq -|N|\beta > \alpha,$$

which is impossible.

Acknowledgements We are grateful to the associate editor and two anonymous referees of this journal for their comments that helped to improve the writing of this paper. This research is supported by the Spanish Ministerio de Ciencia e Innovación under project ECO2009-11213, co-funded by the ERDF, and the second author was supported by the Edmund Landau Center for Research in Mathematical Analysis and Related Areas and by the Center for the Study of Rationality at the Hebrew University of Jerusalem.

References

Aumann RJ, Maschler M (1985) Game theoretic analysis of a bankruptcy problem from the Talmud. J Econ Theory 36:195–213

Davis M, Maschler M (1965) The kernel of a cooperative game. Nav Res Logist Q 12:223–259

Davis M, Maschler M (1967) Existence of stable payoff configurations for cooperative games. In: Shubik M (ed) Essays in mathematical economics in honor of Oskar Morgenstern. Princeton University Press, Princeton, pp 39–52

Derks J, Kuipers J (2002) On the number of extreme points of the core of a transferable utility game. In: Borm P, Peters H (eds) Chapters in game theory in Honor of Stef Tijs, theory and decision library, series c: game theory, mathematical programming and operations research. Kluwer Academic Publishers, Dordrecht, pp 83–97

Hwang Y-A, Sudhölter P (2001) Axiomatizations of the core on the universal domain and other natural domains. Int J Game Theory 29:597–623

Kohlberg E (1971) On the nucleolus of a characteristic function game. SIAM J Appl Math 20:62–66

Maschler M, Peleg B, Shapley LS (1972) The kernel and bargaining set for convex games. Int J Game Theory 1:73–93

Orshan G (1993) The prenucleolus and the reduced game property: Equal treatment replaces anonymity. Int J Game Theory 22:241–248

Orshan G (1994) Non-symmetric prekernels. Discussion paper 60. Center for Rationality, The Hebrew University of Jerusalem

Orshan G, Sudhölter P (2003) Reconfirming the prenucleolus. Math Oper Res 28:283–293

Orshan G, Sudhölter P (2010) The positive core of a cooperative game. Int J Game Theory 39:113–136

Peleg B (1986) On the reduced game property and its converse. Int J Game Theory 15:187–200

Peleg B, Sudhölter P (2003) Introduction to the theory of cooperative games. Kluwer Academic Publishers, Dordrecht

Rockafellar RT (1970) Convex analysis. Princeton University Press, Princeton

Schmeidler D (1969) The nucleolus of a characteristic function game. SIAM J Appl Math 17:1163–1170

Sobolev AI (1975) The characterization of optimality principles in cooperative games by functional equations. In: Vorobiev NN (ed) Mathematical methods in the social sciences, vol 6. Academy of Sciences of the Lithuanian SSR, Vilnius, pp 95–151 (in Russian)

Sudhölter P (1993) Independence for characterizing axioms of the pre-nucleolus. Working paper 220. Institute of Mathematical Economics, University of Bielefeld

Sudhölter P, Peleg B (2000) The positive prekernel of a cooperative game. Int Game Theory Rev 2:287–305