Determinacy of equilibrium in outcome game forms[☆]

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ABSTRACT

We show the generic finiteness of the number of probability distributions on outcomes induced by Nash equilibria for two-person game forms such that either (i) one of the players has no more than two strategies or (ii) both of the players have three strategies, and (iii) for outcome game forms with three players, each with at most two strategies. Finally, we exhibit an example of a game form with three outcomes and three players for which the Nash equilibria of the associated game induce a continuum of payoffs for an open non-empty set of utility profiles.

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1. Introduction

In normal form games with an arbitrary number of players the payoffs of which may be perturbed independently Rosenmüller (1971) and Wilson (1971) simultaneously proved (see also Harsanyi, 1973) that generically there is a finite number of equilibria. This result was extended to extensive form games by Kreps and Wilson (1982). On the other hand, Govindan and McLennan (2001) and Kukushkin et al. (2008) show that the situation for outcome game forms is entirely different. These authors construct some game forms for which there is a continuum of equilibrium distributions on outcomes (i.e., probability distributions on outcomes induced by Nash equilibria) of the associated games for an open non-empty set of utility profiles.

A natural question is to determine which outcome game forms admit a finite number of equilibrium distributions on outcomes. For example, Mas-Colell (2010) proved that for two player game forms the number of equilibrium payoffs are generically finite,

and Govindan and McLennan (2001) proved that for game forms with two outcomes and any number of players the number of equilibrium distributions is generically finite. Similar results were obtained for game forms with two players and three outcomes (González-Pimienta, 2010) and sender–receiver cheap-talk games (Park, 1997). Using semi-algebraic geometry techniques Govindan and McLennan (1998) showed in an unpublished manuscript the generic finiteness of the number of equilibrium distributions on outcomes, when the associated game is either a two player zero sum or a common interest game. This result is also proved by Litan and Marhuenda (2012) using elementary linear algebra.

In this paper we prove the generic finiteness of the number of equilibrium distributions on outcomes for two-person outcome game forms in which one of the players has no more than two strategies or both of the players have three strategies. The results of González-Pimienta (2010) and Govindan and McLennan (2001) imply that for two-person outcome game forms with at most three outcomes the number of equilibrium distributions is generically finite. We provide an example of an outcome game form with three outcomes and three players for which the Nash equilibria of the associated games induce a continuum of probability distributions on outcomes for an open non-empty set of utility profiles. Finally, we show that for outcome game forms with three players, each with at most two strategies, generic finiteness of the number of equilibrium distributions on outcomes is obtained.

2. Outcome game forms

Let Ω be a finite nonempty set. Denote by $\Delta(\Omega)$ the set of probability measures on Ω, and let $\Delta_+(\Omega)$ denote its subset of

strictly positive elements. A *(finite, pure) L-person outcome game form* (on Ω) is defined by Govindan and McLennan (2001) as a tuple $(S^1, \ldots, S^L, \phi)$ such that, for all $\ell \in \{1, \ldots, L\}$, S^ℓ is a finite nonempty set, and $\phi = (\phi_s)_{s \in S} \in \Omega^S$, where Ω is called the set of *outcomes* and $S = S^1 \times \cdots \times S^L$. Note that we do not consider *general* outcome game forms the definition of which only requires that $\phi \in \Delta(\Omega)^S$. A profile $u = (u^1, \ldots, u^L) \in \mathbb{R}^\Omega \times \cdots \times \mathbb{R}^\Omega$ defines the *associated* finite L-person game $g_u^\phi = (S^1, \ldots, S^L, u^1 \circ \phi, \ldots, u^L \circ \phi)$, where "$\circ$" denotes "composition". Recall that a Nash equilibrium (NE) of g_u^ϕ is a tuple $x = (x^1, \ldots, x^L)$ such that, for all $\ell \in \{1, \ldots, L\}$, $x^\ell \in \Delta(S^\ell)$, and for all $s^\ell \in S^\ell$,

$$x^\ell(s^\ell) > 0 \Rightarrow \sum_{s^{-\ell} \in S^{-\ell}} \left(\prod_{\ell' \in \{1, \ldots, L\} \setminus \{\ell\}} x^{\ell'}(s^{\ell'}) \right) \left(u^\ell \left(\phi_{(s^\ell, s^{-\ell})} \right) \right.$$
$$\left. - u^\ell \left(\phi_{(\widetilde{s}^\ell, s^{-\ell})} \right) \right) \geqslant 0 \quad \text{for all } \widetilde{s}^\ell \in S^\ell,$$

where $S^{-\ell} = \times_{\ell' \in \{1, \ldots, L\} \setminus \{\ell\}} S^{\ell'}$. Moreover, recall that an NE $(x^1, \ldots, x^L)$ is a *completely mixed* NE (CMNE) if $x^\ell \in \Delta_+(S^\ell)$ for all $\ell = 1, \ldots, L$.

We say that a subset of a Euclidean space is *generic* if it contains an open and dense subset of this Euclidean space. Let $\mathbb{R}[\Omega]$ denote the ring of real polynomials in the $|\Omega|$ variables $\omega \in \Omega$. Note that a polynomial f of $\mathbb{R}[\Omega]$ defines a continuous polynomial function $\mathbb{R}^\Omega \to \mathbb{R}$ that we again denote by f. We frequently use the fact that, if $0 \neq f \in \mathbb{R}(\Omega)$, then $\{u \in \mathbb{R}^\Omega : f(u) \neq 0\}$ is an open and dense, hence generic, subset of $\mathbb{R}^\Omega$. With $f = \prod_{\omega \in \Omega} \prod_{\omega' \in \Omega \setminus \{\omega\}} (\omega - \omega')$ we obtain that

$$\mathcal{V} = \left\{ u \in \mathbb{R}^\Omega : u(\omega) \neq u(\omega') \text{ for all } \omega, \omega' \in \Omega \text{ with } \omega \neq \omega' \right\}$$

is a generic subset of $\mathbb{R}^\Omega$.

3. Distributions on outcomes and minimality for two-person outcome game forms

Let (S^1, S^2, ϕ) be an outcome game form with two players, where $S^1 = \{1, 2, \ldots, m\}$, $S^2 = \{1, 2, \ldots, n\}$ and $S = S^1 \times S^2$.

Kukushkin et al. (2008) provide an example of an outcome game form with two players in which, for a non-empty open set of utility profiles, there is a continuum of outcome distributions induced by the Nash equilibria. In that example the first player has three strategies and the second player has four strategies. The next theorem shows that the example is minimal in terms of strategies.

Theorem 3.1. *If (S^1, S^2, ϕ) is a 2-person outcome game form such that*

$$\min\{|S^1|, |S^2|\} \leqslant 2 \quad \text{or} \quad |S^1| = |S^2| = 3,$$

then there is a generic set $\mathcal{W} \subseteq \mathbb{R}^\Omega$ such that for any $u^1, u^2 \in \mathcal{W}$ the set of CMNEs of the game g_u^ϕ induce finitely many probability distributions on outcomes.

Proof. We may assume that the rows of ϕ (that is, $\phi_{i\cdot} = (\phi_{ij})_{j \in S^2}$, $i \in S^1$) are pairwise distinct. Indeed, for any utility profile in $\mathbb{R}^\Omega \times \mathbb{R}^\Omega$ the set of distributions on outcomes induced by (completely mixed) Nash equilibria is not changed if multiple rows are eliminated. A similar assumption refers to the columns of ϕ (that is, $\phi_{\cdot j} = (\phi_{ij})_{i \in S^1}$, $j \in S^2$). Hence, we may assume without loss of generality that

$$|\{\phi_{i\cdot} : i \in S^1\}| = m, \qquad |\{\phi_{\cdot j} : j \in S^2\}| = n, \quad \text{and} \quad m \leqslant n. \tag{1}$$

Let $(u^1, u^2) \in \mathcal{V} \times \mathcal{V}$. For $(i, j) \in S^1 \times S^2$ denote $u_{ij} = u^2(\phi_{ij})$. We distinguish three cases.

Case 1: $m = 1$.

As $u^2 \in \mathcal{V}$, any Nash equilibrium selects the unique $\arg\max_{j \in S_2} u_{1j}$, and, hence a CMNE can only exist if $n = 1$.

Case 2: $m = 2$.

If $n = 2$ and there exists a CMNE (x, y), then $x_1 u_{11} + x_2 u_{21} = x_1 u_{12} + x_2 u_{22}$ because player 2 is indifferent between her strategies. As $x_2 = 1 - x_1$, we conclude that

$$x_1(u_{11} + u_{22} - u_{21} - u_{12}) = u_{22} - u_{21} \quad \text{and}$$
$$x_2(u_{11} + u_{22} - u_{21} - u_{12}) = u_{11} - u_{12}.$$

By (1) and as $u^2 \in \mathcal{V}$, $u_{22} \neq u_{21}$ or $u_{11} \neq u_{12}$ so that x is uniquely determined. By exchanging the roles of the players, it follows that y is also unique.

We now assume that $n \geqslant 3$.

If there are $j, j' \in S^2$, $j \neq j'$, such that $\phi_{1j} = \phi_{1j'}$, then by (1) we may assume that $u_{2j} < u_{2j'}$ so that column j is not a best response to any completely mixed strategy of player 1. Therefore, there does not exist any CMNE in this subcase. Similarly, we may proceed if $\phi_{2j} = \phi_{2j'}$. Therefore, we shall now assume that

$$n = |\{\phi_{ij} : j \in S^2\}| \quad \text{for } i = 1, 2. \tag{2}$$

By (1) there exists $j \in S^2$ such that $\phi_{1j} \neq \phi_{2j}$. By (2) there exists $j' \in S^2 \setminus \{j\}$ such that $\phi_{2j'} \neq \phi_{1j}$ so that we may assume without loss of generality that

$$\phi_{11} \neq \phi_{21} \quad \text{and} \quad \phi_{11} \neq \phi_{22}. \tag{3}$$

By (2) and (3),

$$\phi_{11} \notin \{\phi_{12}, \phi_{13}, \phi_{21}, \phi_{22}\}. \tag{4}$$

Define $f \in \mathbb{R}[\Omega]$ by

$$f = (\phi_{11} - \phi_{12})(\phi_{21} - \phi_{23}) - (\phi_{11} - \phi_{13})(\phi_{21} - \phi_{22}). \tag{5}$$

Claim. *If $f(u^2) \neq 0$ then the game g_u^ϕ has no CMNE.*

If x is a mixed strategy of player 1 such that player 2 is indifferent between the payoff columns $u_{\cdot 1}$, $u_{\cdot 2}$, and $u_{\cdot 3}$, then

$$x \cdot \begin{pmatrix} u_{11} - u_{12} & u_{11} - u_{13} & 1 \\ u_{21} - u_{22} & u_{21} - u_{23} & 1 \end{pmatrix} = (0, 0, 1). \tag{6}$$

We conclude that

$$\det \begin{pmatrix} u_{11} - u_{12} & u_{11} - u_{13} & 1 \\ u_{21} - u_{22} & u_{21} - u_{23} & 1 \\ 0 & 0 & 1 \end{pmatrix} = 0.$$

Therefore, $f(u^2) = 0$ so that our claim has been proved.

Now the proof in this case can be completed. Indeed, it suffices to show that $f \neq 0$ because then $\mathcal{V} \times \{u^2 \in \mathcal{V} : f(u^2) \neq 0\}$ is a generic set of utility profiles for which no CMNE exist. Define $\omega := \phi_{11}$. We may assume that there exists $u \in \mathcal{V}$ such that $f(u) = 0$. For $\varepsilon > 0$ let $u^\varepsilon \in \mathbb{R}^\Omega$ differ from u only inasmuch as $u^\varepsilon(\omega) = u(\omega) + \varepsilon$. If $\phi_{23} \neq \omega$, then $f(u^\varepsilon) = f(u) + \varepsilon(u^{\varepsilon 22} - u^{\varepsilon 23}) = \varepsilon(u_{22} - u_{23}) \neq 0$. If $\phi_{23} = \omega$, then

$$f(u^\varepsilon) = \varepsilon(u_{22} + u_{12} - 2u_{11}) - \varepsilon^2.$$

Hence, we conclude that $f \neq 0$.

Case 3: $m = n = 3$.

It suffices to describe a suitable generic set of utilities for player 2. We distinguish two cases.

(1) There exists $i \in S_1$ such that $\phi_{i1} = \phi_{i2} = \phi_{i3}$. We may assume that $i = 3$. Let $u \in \mathcal{V}$. If x' is a completely mixed strategy of player 1 such that player 2 is indifferent between the columns, then let $x = \frac{1}{x'_1 + x'_2}(x'_1, x'_2) \in \Delta_+(\{1, 2\})$ and observe that (6) must hold. Also, we may assume that (2) holds because otherwise there exist two payoff columns that differ only in one coordinate so that a completely mixed Nash equilibrium is ruled out. By (1) there exists $\ell \in S_2$ such that $\phi_{1\ell} \neq \phi_{2\ell}$ so that (3) may be assumed and the proof may be finished by literally copying the corresponding part of the case $m = 2$.

(2) For any $i \in S_1$, $|\{\phi_{ij} : j \in S_2\}| \geqslant 2$.

Consider again the polynomial f defined in (5). If x is a mixed strategy that makes player 2 indifferent between all columns, then

$$x \cdot \begin{pmatrix} u_{11} - u_{12} & u_{11} - u_{13} & 1 \\ u_{21} - u_{22} & u_{21} - u_{23} & 1 \\ u_{31} - u_{32} & u_{31} - u_{33} & 1 \end{pmatrix} = (0, 0, 1). \tag{7}$$

Claim. *If the system of equations* (7) *has multiple solutions, then* $f(u^2) = 0$ *and* $|\{\phi_{ij} : j \in S_2\}| = 3$, *for every* $i \in S_1$.

As (7) has multiple solutions, the columns of the matrix are not linearly independent. Thus, there exists $z \in \mathbb{R}^3$, $z \neq 0$, such that

$$\begin{pmatrix} u_{11} - u_{12} & u_{11} - u_{13} & 1 \\ u_{21} - u_{22} & u_{21} - u_{23} & 1 \\ u_{31} - u_{32} & u_{31} - u_{33} & 1 \end{pmatrix} \cdot z = (0, 0, 0).$$

Multiplying both sides of (7) by z yields $z_3 = 0$. Moreover, as $u \in \mathcal{V}$, by (1), $\phi_{\cdot 1} \neq \phi_{\cdot 2}$ so that $z_1 \neq 0$, and similarly $z_2 \neq 0$. Hence, we may assume (after replacing z by z/z_1 if necessary) that $z_1 = 1$. Hence, with $\lambda = -z_2$, we have

$$u_{11} - u_{12} = \lambda(u_{11} - u_{13})$$
$$u_{21} - u_{22} = \lambda(u_{21} - u_{23}) \quad \text{and}$$
$$u_{31} - u_{32} = \lambda(u_{31} - u_{33}).$$

Since $u \in \mathcal{V}$ and every row has at least two outcomes, all of the above differences are nonzero. We conclude that $|\{\phi_{ij} : j \in S_2\}| = 3$ for all $i \in S_1$ and the claim follows.

Now the proof can be completed. If, for any $u \in \mathcal{V}$, (7) has one or no solution, then $\mathcal{W} = \mathcal{V}$ has the desired properties. In the other case, let $u \in \mathcal{V}$ such that (7) has multiple solutions. Hence, $|\{\phi_{ij} : j \in S_2\}| = 3$ for all $i \in S_1$ so that (2) holds. Now the proof may be finished by literally copying the relevant part of Case 2: By (1) there exists $j \in S_2$ such that $\phi_{1j} \neq \phi_{2j}$, and by (2) there exists $j' \in S^2 \setminus \{j\}$ such that $\phi_{1j} \neq \phi_{2j'}$. We may assume without loss of generality that $j = 1$ and $j' = 2$ so that $f \neq 0$. $\square$

4. Outcome game forms with three players

The first example of an outcome game form in which there is a continuum of distributions on outcomes induced by the Nash equilibria of the associated games for an open non-empty set of utility profiles was provided by Govindan and McLennan (2001). Their example had three players and six outcomes. On the other hand, for any outcome game form with two outcomes they prove generic finiteness of the number of Nash equilibrium outcome distributions, and González-Pimienta (2010) shows this generic finiteness for two-person game forms with three outcomes. Finally, Kukushkin et al. (2008) provide an example of an outcome game form with two players and four outcomes in which there is a continuum of outcome distributions induced by the Nash equilibria of the associated games for an open non-empty set of utility profiles. Thus, it is natural to ask if four is the minimum number of outcomes needed to construct outcome game forms

which do not generically have finitely many distributions on Ω induced by Nash equilibria. However, the next example shows that the results in González-Pimienta (2010) cannot be extended to three players. That is, a three-person game form with three outcomes a, b, and c may allow a continuum of outcome distributions induced by Nash equilibria for an open non-empty set of utility profiles.

In Section 4.2 we show the generic finiteness for outcome game forms with three players, each with at most two strategies.

4.1. An example with three players and three outcomes

Let $\Omega = \{a, b, c\}$, $S_1 = \{N, E, S, W\}$, $S_2 = \{L, R\}$ and $S_3 = \{U, D\}$. We use the notation of Section 3 of Govindan and McLennan (2001) and consider the game form

$$\begin{array}{cc}
\begin{array}{c} U \\ \begin{array}{cc} L & R \end{array} \end{array} &
\begin{array}{c} D \\ \begin{array}{cc} L & R \end{array} \end{array}
\end{array}$$

$$\begin{array}{c} N \\ E \\ S \\ W \end{array}
\begin{pmatrix} a & a \\ b & b \\ a & c \\ b & a \end{pmatrix}
\begin{array}{c} N \\ E \\ S \\ W \end{array}
\begin{pmatrix} c & c \\ a & a \\ a & c \\ b & a \end{pmatrix}$$

so that player 1 selects the row, player 2 the column, and player 3 the matrix. Moreover, for $i \in \{1, 2, 3\}$, $a_i = u_i(a)$, $b_i = u_i(b)$, and $c_i = u_i(c)$. If $a_i > \max\{b_i, c_i\}$ for all $i \in \{1, 2, 3\}$, then we may define, for any p with $0 \leqslant p \leqslant t := \frac{a_3 - b_3}{2a_3 - b_3 - c_3}$, the strategy profile $X(p) = ((p, q, r, s), (y, 1 - y), (z, 1 - z))$ by

$$z = \frac{a_1 - c_1}{2a_1 - b_1 - c_1} = y; \tag{8}$$

$$q = \frac{a_3 - c_3}{a_3 - b_3}p; \tag{9}$$

$$r = \left(1 - \left(\frac{2a_3 - b_3 - c_3}{a_3 - b_3}\right)p\right)\frac{a_2 - b_2}{2a_2 - b_2 - c_2}; \tag{10}$$

$$s = \frac{a_2 - c_2}{a_2 - b_2}r. \tag{11}$$

It is straightforward to show that $X(p)$ is a Nash equilibrium that induces the payoff $\frac{a_1^2 - b_1 c_1}{2a_1 - b_1 - c_1}$ for the row player. Let $\pi_2(p)$ denote the payoff of the column player. We may easily compute

$$\pi_2(0) = \frac{a_2^2 - b_2 c_2}{2a_2 - b_2 - c_2}$$

and

$$\pi_2(t) = \frac{(a_3 - b_3)((a_1 - c_1)a_2 + (a_1 - b_1)c_2) + (a_3 - c_3)((a_1 - c_1)b_2 + (a_1 - b_1)a_2)}{(2a_3 - b_3 - c_3)(2a_1 - b_1 - c_1)}$$

so that $\pi_2(0) \neq \pi_2(t)$ for a nonempty open subset of utility profiles. Moreover, if $\pi_2(0) \neq \pi_2(t)$, then by continuity of $\pi_2 : [0, t] \to \mathbb{R}$, there is a continuum of payoffs of the column player induced by the completely mixed Nash equilibria $X(p)$, $0 < p < t$. A similar statement is valid for the matrix player.

On the other hand, we show in Section 4.2 that if each of the players has at most two strategies, generically the number of probability distributions on outcomes induced by the Nash equilibria of the game is finite.

4.2. Outcome game forms with three players and two strategies each

Let Ω be the finite non-empty set of outcomes. We consider an outcome game form (S^1, S^2, S^3, ϕ) with three players and two strategies each, i.e., we assume that $S^1 = S^2 = S^3 = \{1, 2\}$. We denote the $3|\Omega|$ variables of a polynomial in $\mathbb{R}[\Omega \times \{1, 2, 3\}]$

by ω^ℓ, where $\omega \in \Omega$ and $\ell \in \{1, 2, 3\}$. For any finite $\mathcal{F} \subseteq \mathbb{R}[\Omega \times \{1, 2, 3\}]$, let $U_{\mathcal{F}}$ be defined by

$$U_{\mathcal{F}} = \left\{ u \in \mathbb{R}^{\Omega} \times \mathbb{R}^{\Omega} \times \mathbb{R}^{\Omega} : \prod_{f \in \mathcal{F} \setminus \{0\}} f(u) \neq 0 \right\}.$$

As remarked in Section 2, $U_{\mathcal{F}}$ is an open and dense subset of $\left(\mathbb{R}^{\Omega}\right)^3$.

Now we are able to prove the following main result of this section.

Theorem 4.1. *For any three-person pure outcome game form* (S^1, S^2, S^3, ϕ) *with* $S^1 = S^2 = S^3 = \{1, 2\}$ *there is a generic set* U *of utility profiles such that, for any* $u = (u^1, u^2, u^3) \in U$, *the set of CMNEs of the game* g_u^ϕ *induce finitely many probability distributions on outcomes.*

Proof. By Theorem 3.1, we may assume that none of the players $i = 1, 2, 3$ is a dummy, where i is *dummy* if ϕ is invariant under any permutation of S^i.

Note that a generic set U of utility profiles has the requested property if and only if, for each $u \in U$ such that g_u^ϕ has infinitely many CMNEs, these CMNEs only induce finitely many probability distributions on Ω. As finite intersections of generic sets are generic, it suffices to find, for each player $i = 1, 2, 3$, a generic set U' of utility profiles with the following property: For each $u \in U'$ such that infinitely many values of $x^i \in \Delta_+(S^i)$ can be extended to CMNEs of g_u^ϕ, i.e., $|\{x^i : x \text{ is a CMNE of } g_u^\phi\}| = \infty$, all those CMNEs induce only finitely many probability distributions on Ω. In what follows, we construct $\mathcal{F} \subseteq \mathbb{R}[\Omega \times \{1, 2, 3\}]$ such that $U' = U_{\mathcal{F}}$ has the aforementioned property for an arbitrary player, say player $i = 1$. In fact we show that the probability distribution on Ω induced by the mentioned infinitely many CMNEs is unique.

The proof is organized as follows. Given a CMNE, an inspection of the polynomial (denoted by H in the sequel) the coefficients of which are the probabilities of the outcomes enables us to construct the aforementioned set $\mathcal{F}$ of polynomials that do not depend on the CMNE. By distinguishing cases the proof is finished.

Let $u = (u^1, u^2, u^3) \in \left(\mathbb{R}^{\Omega}\right)^3$ and let $((x_1, x_2), (y_1, y_2), (z_1, z_2))$ be a CMNE of g_u^ϕ. Define the polynomial $H \in \mathbb{R}[\Omega]$ by

$$H = \sum_{i,j,k=1}^{2} \phi_{ijk} x_i y_j z_k.$$

Hence, the probability of each outcome ω is its coefficient in the polynomial H. Let $x = x_1, x_2 = 1 - x, y = y_1, y_2 = 1 - y, z = z_1, z_2 = 1 - z$. The polynomial H may be written as

$$H = Axyz + Bxy + Cxz + Dyz - Ex - Fy - Gz + \phi_{222},$$

where $A, \dots, G \in \mathbb{R}[\Omega]$ are defined by

$$A = \phi_{111} - \phi_{121} - \phi_{112} + \phi_{122} - \phi_{211} + \phi_{221} + \phi_{212} - \phi_{222},$$
$$B = \phi_{112} - \phi_{122} - \phi_{212} + \phi_{222},$$
$$C = \phi_{121} - \phi_{122} - \phi_{221} + \phi_{222},$$
$$D = \phi_{211} - \phi_{212} - \phi_{221} + \phi_{222}, \quad (12)$$
$$E = \phi_{222} - \phi_{122},$$
$$F = \phi_{222} - \phi_{212},$$
$$G = \phi_{222} - \phi_{221}.$$

For $i \in \{1, 2, 3\}$, each of the polynomials $A, \dots, G$ may be identified with a polynomial A_i in $\mathbb{R}[\Omega \times \{1, 2, 3\}]$ by formally replacing the variable ω with the variable ω^i. Note that with this identification, $A_i \neq A_j$ for $i \neq j$, since A_i and A_j are defined on a different sets of variables. Let $a_i = A(u^i), \dots, g_i = G(u^i)$. As $((x, 1-x), (y, 1-$

$y), (z, 1-z))$ is a CMNE, each of the players is indifferent between her pure strategies. Therefore, we have the equations

$$a_1 yz + b_1 y + c_1 z = e_1; \quad (13)$$
$$a_2 xz + b_2 x + d_2 z = f_2; \quad (14)$$
$$a_3 xy + c_3 x + d_3 y = g_3. \quad (15)$$

By (13) and (15),

$$y(a_1 z + b_1) = e_1 - c_1 z \quad \text{and} \quad y(a_3 x + d_3) = g_3 - c_3 x$$

so that

$$(e_1 - c_1 z)(a_3 x + d_3) = (a_1 z + b_1)(g_3 - c_3 x). \quad (16)$$

By (14), $z(a_2 x + d_2) = f_2 - b_2 x$ and, by (16),

$$z(x(a_3 c_1 - a_1 c_3) + a_1 g_3 + c_1 d_3) = x(a_3 e_1 + b_1 c_3) + d_3 e_1 - b_1 g_3.$$

We conclude that

$$(x(a_3 c_1 - a_1 c_3) + a_1 g_3 + c_1 d_3)(f_2 - b_2 x)$$
$$\quad = (x(a_3 e_1 + b_1 c_3) + d_3 e_1 - b_1 g_3)(a_2 x + d_2)$$

so that, with

$$r = a_1 b_2 c_3 - a_3 b_2 c_1 - a_2 a_3 e_1 - a_2 b_1 c_3,$$
$$p = a_3 c_1 f_2 + a_2 b_1 g_3 - a_1 c_3 f_2 - a_1 b_2 g_3$$
$$\quad - b_2 c_1 d_3 - a_3 d_2 e_1 - b_1 c_3 d_2 - a_2 d_3 e_1, \quad (17)$$
$$q = a_1 f_2 g_3 + c_1 d_3 f_2 + b_1 d_2 g_3 - d_2 d_3 e_1,$$

we have

$$rx^2 + px + q = 0. \quad (18)$$

Let $P, Q, R \in \mathbb{R}[\Omega \times \{1, 2, 3\}]$ be the polynomials that correspond to p, q, r, that is,

$$R = A_1 B_2 C_3 - A_3 B_2 C_1 - A_2 A_3 E_1 - A_2 B_1 C_3,$$
$$P = A_3 C_1 F_2 + A_2 B_1 G_3 - A_1 C_3 F_2 - A_1 B_2 G_3$$
$$\quad - B_2 C_1 D_3 - A_3 D_2 E_1 - B_1 C_3 D_2 - A_2 D_3 E_1, \quad (19)$$
$$Q = A_1 F_2 G_3 + C_1 D_3 F_2 + B_1 D_2 G_3 - D_2 D_3 E_1.$$

We now define the set $\mathcal{F} \subseteq \mathbb{R}[\Omega \times \{1, 2, 3\}]$ as follows: $\mathcal{F}$ consists of all polynomials of the form

$$P, Q, R, \alpha_i, \alpha_i \beta_j \pm \alpha_j \beta_i, \alpha_1 \beta_2 \gamma_3 + \alpha_2 \beta_3 \gamma_1 \quad \text{with}$$
$$\alpha, \beta, \gamma \in \{A, \dots, G\} \text{ and } i, j \in \{1, 2, 3\}, \ i \neq j.$$

Note that $\mathcal{F}$ is finite and it contains all polynomials that are explicitly used in the present proof.

We now assume that $u = (u^1, u^2, u^3) \in U_{\mathcal{F}}$ and that there are infinitely many, hence at least three, values of x that can be extended to CMNEs of g_u^ϕ. By (18), $p = q = r = 0$. Therefore, $P, Q, R \in \mathcal{F}$ implies that $P = Q = R = 0$. In order to show that all the CMNEs generate a unique probability distribution on Ω, we distinguish the following cases:

Case 1: $A = 0$. The system of equations (13)–(15) is linear. The determinant of the associated matrix is $b_1 c_3 d_2 + b_2 c_1 d_3$. There is more than one solution if and only if this determinant vanishes. As $B_1 C_3 D_2 + B_2 C_1 D_3 \in \mathcal{F}$, we conclude that $B_1 C_3 D_2 + B_2 C_1 D_3 = 0$ so that, in particular, $BCD + BCD = 2BCD = 0$. As the cases $C = 0$ and $D = 0$ can be treated similarly, we only consider the case $B = 0$. Then the system (13)–(15) becomes

$$c_1 z = e_1; \quad (20)$$
$$d_2 z = f_2; \quad (21)$$
$$c_3 x + d_3 y = g_3. \quad (22)$$

If, in addition, $C = 0$, then $E_1 \in \mathcal{F}$ implies that $E_1 = 0$, hence $E = 0$ so that $\phi_{122} = \phi_{222}$. From $C = B = 0$ we conclude that $\phi_{121} = \phi_{221}$ and $\phi_{112} = \phi_{212}$. Finally, $A = 0$ now implies $\phi_{111} = \phi_{211}$ so

that player 1 would be a dummy which was excluded. Similarly, if $D = 0$, $F_2 \in \mathcal{F}$ implies that $F_2 = 0$, i.e., player 2 would be a dummy which was also excluded. Hence, $C \neq 0 \neq D$. As $z > 0$, (20) and (21) together with $C_1, E_1, D_2, F_2, E_1D_2 - C_1F_2 \in \mathcal{F}$ imply that $E_1, F_2 \neq 0$ and $E_1D_2 - C_1F_2 = 0$. Consequently, there is a constant μ such that

$$\frac{f_2}{d_2} = \frac{e_1}{c_1} = \mu.$$

Therefore, $F = \mu D$, $E = \mu C$, and, by (20), $z = \mu$ so that z is uniquely determined. The polynomial H becomes $H = Cxz + Dyz - Ex - Fy - Gz + \phi_{222} = \phi_{222} - G\mu$, and the distribution induced on outcomes is unique.

Case 2: $A \neq 0$ and $B = 0$. As $0 = R = -A_2A_3E_1$, $E = 0$ because $A_2, A_3, E_1 \in \mathcal{F}$. As $z \neq 0$, (13) implies that $a_1y + c_1 = 0$ so that $C \neq 0$ and y is uniquely determined. As $P = 0$, $(A_3C_1 - A_1C_3)F_2 = 0$, and, as $Q = 0$, $(A_1G_3 + C_1D_3)F_2 = 0$. If $F = 0$, then, by (14), $a_2x + d_2 = 0$ (because $z \neq 0$) so that x is uniquely determined which was excluded. Hence, $A_3C_1 = A_1C_3$ and $A_1G_3 + C_1D_3 = 0$. As $C \neq 0$, we may define $\mu = a_1/c_1$. Then, $A = \mu C$ and $D = -\mu G$.

Now, from (13), we have that $0 = a_1y + c_1 = (\mu y + 1)c_1$. Since $c_1 \neq 0$, we obtain $\mu y + 1 = 0$. Therefore, y is uniquely determined and $Ay = -C$. Substituting in (15), we obtain $G = yD$. The polynomial that determines the probabilities on outcomes becomes $H = Axyz + Cxz + Dyz - Fy - Gz + \phi_{222} = -Cxz + Cxz + Gz - Fy - Gz + \phi_{222} = \phi_{222} - Fy$ and the distribution induced on outcomes is unique.

Case 3: $A \neq 0$ and $C = 0$. This case may be treated analogously to Case 2.

Case 4: $ABC \neq 0$. We first claim that either $A = \mu B$ or $A = \mu C$, for some non null $\mu \in \mathbb{R}$. Since $R = 0$, we have that $B_2(A_1C_3 - A_3C_1) = A_2(A_3E_1 + B_1C_3)$. If $A_1C_3 - A_3C_1 = 0$ we conclude that $A_1 = \frac{a_3}{c_3}C_1$, and the claim follows. Otherwise, we have that

$$A_2 = B_2 \frac{a_1c_3 - a_3c_1}{a_3e_1 + b_1c_3},$$

and again the claim follows. The following subcases might occur:

(1) $A = \mu B$ for some $\mu \neq 0$. Hence, $A_1B_2 = A_2B_1$ and therefore $R = -\mu B_2B_3(C_1 + \mu E_1)$ so that $R = 0$ and $\mu, B \neq 0$ imply that

$C_1 = -\mu E_1$. It follows that $C = -\mu E$. Substituting $A = \mu B$ and $C = -\mu E$ into (13), we obtain $b_1y(\mu z + 1) = e_1(\mu z + 1)$. We consider the following two subcases.

(1.1) $\mu z + 1 \neq 0$. Then, $y = e_1/b_1$ and substituting this value into (15), we obtain that

$$x\mu\left(\frac{b_3e_1}{b_1} - e_3\right) = g_3 - \frac{d_3e_1}{b_1}.$$

Since x is not uniquely determined and $\mu \neq 0$, $b_3e_1 = e_3b_1$ and, hence, $g_3b_1 = d_3e_1$. With $\nu = e_1/b_1$ we obtain $E = \nu B$ and $G = \nu D$ so that $y = \nu$. Therefore, $H = \mu\nu Bxz + \nu Bx - \mu\nu Bxz + \nu Dz - \nu Bx$

$$- \nu F - \nu Dz + \phi_{222} = \phi_{222} - \nu F,$$

and the distribution induced on outcomes is unique.

(1.2) $z = -1/\mu$. From (14) we obtain that $d_2z = f_2$. Hence, $D = -\mu F$. Therefore, $Az = \mu Bz = -B$, $Cz = -\mu zE = E$ and $Dz = -\mu zF = F$. Hence, $H = \phi_{222} - Gz = \phi_{222} + \mu G$. Thus, the distribution induced on outcomes is unique.

(2) The subcase $A = \mu C$ for some $\mu \neq 0$ can be treated similarly to (1). $\square$

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