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Axiomatizations of symmetrically weighted solutions

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Abstract If the excesses of the coalitions in a transferable utility game are weighted, then we show that the arising weighted modifications of the well-known (pre)nucleolus and (pre)kernel satisfy the equal treatment property if and only if the weight system is symmetric in the sense that the weight of a subcoalition of a grand coalition may only depend on the grand coalition and the size of the subcoalition. Hence, the symmetrically weighted versions of the (pre)nucleolus and the (pre)kernel are symmetric, i.e., invariant under symmetries of a game. They may, however, violate anonymity, i.e., they may depend on the names of the players. E.g., a symmetrically weighted nucleolus may assign the classical nucleolus to one game and the per capita nucleolus to another game.

We generalize Sobolev's axiomatization of the prenucleolus and its modification for the nucleolus as well as Peleg's axiomatization of the prekernel to the symmetrically weighted versions. Only the reduced games have to be replaced by suitably modified reduced games whose definitions may depend on the weight system. Moreover, it is shown that a solution may only satisfy the mentioned sets of modified axioms if the weight system is symmetric.

Keywords TU game · Nucleolus · Kernel

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1 Introduction

The (pre)nucleolus (Schmeidler 1969; Sobolev 1975) and the prekernel (Davis and Maschler 1965; Maschler et al. 1972) are among the most well-known and important solution concepts for (cooperative transferable utility) games. Their status is heavily supported by the fact that they can be justified by simple and intuitive axioms. The definition of each of these solution concepts is based on the excesses of the coalitions that may be regarded as their dissatisfactions. The excess of a coalition S in a game (N, v) at some payoff vector x is the difference between the worth of S, $v(S)$, and the amount $\sum_{i\in S} x_i = x(S)$ that is distributed to S. Classically, the excesses of any two coalitions are treated as equally important, regardless of coalition sizes and composition.

Variants of the nucleolus and the kernel result from modifying these excesses. Wallmeier (1983), e.g., considers weighted excesses $\frac{v(S)-x(S)}{f(|S|)}$ where f is a nondecreasing function. The special case that $f(s) = s$ results in the so-called "per capita" nucleolus and kernel (Grotte 1970; Young et al. 1982; Albers 1977). The resulting weighted versions of the solution concepts share many properties with the classical solution concepts.

Derks and Haller (1999) consider arbitrarily weighted excesses, where the weight of the excess of a subcoalition S of N may depend on N and S. They show that two weighted nucleoli coincide if and only if the two weight systems coincide up to a positive multiplication factor. We adopt their setting and first of all extend this result to weighted prenucleoli and weighted (pre)kernels. Moreover, we prove that given a weight system the arising weighted modifications of the (pre)nucleolus and the (pre)kernel satisfy *anonymity* (AN) or the *equal treatment property* (ETP) if and only if the weight system is anonymous or symmetric, respectively. Here a weight system is anonymous if up to a multiple the weight of a subcoalition S of a grand coalition N may only depend on the sizes of S and N. Further, a weight system is symmetric if the weight of any subcoalition S of N may depend on the size of S and also on the grand coalition. Hence, anonymity implies symmetry.

The foregoing results motivate the study of the symmetrically weighted (pre)nucleoli and prekernels from an axiomatic point of view. According to Sobolev (1975) the classical prenucleolus is axiomatized[1] by *single-valuedness* (SIVA), *covariance under strategic equivalence* (COV), *symmetry* (SYM), and the *reduced game property* (RGP), provided that there are infinitely many potential players. Snijders (1995) showed that the nucleolus is axiomatized similarly; only RGP has to be replaced by a suitable modification that ensures that the reduction of an imputation is an imputation of the reduced game. For interesting variants of Sobolev's famous result see Orshan (1993) and Orshan and Sudhölter (2003). Moreover, Peleg (1986) shows that the prekernel is axiomatized by *non-emptiness*, *Pareto optimality*, ETP, COV, RGP, and the *converse reduced game property* (CRGP). We show that after adjusting the definition of the reduced game for the given weight system, the corresponding weighted variants of the (pre)nucleolus and prekernel are axiomatized by the same sets of axioms as in the classical context provided that the weight system is symmetric; only RGP and CRGP refer to the adjusted variants of the reduced game now. Moreover, we show that symmetry of the weight system is necessary (and sufficient) for the existence of a solution that satisfies the mentioned sets of modified axioms.

It should be emphasized that, unlike in the classical case, symmetry of the weight system does not imply that the arising weighted solution concepts satisfy AN. E.g., a symmetric

[1] An axiomatization is a characterization by axioms that are logically independent of each other. The logical independence, in particular of the anonymity axiom, was proved by Sudhölter (1993).

weight system may result in the classical prenucleolus for one part of the player society and in the per capita prenucleolus for the other part.

This paper is organized as follows. In Sect. 2 the relevant notation is introduced, the definition of and some known facts about the weighted (pre)nucleolus are recalled, and the weighted prekernel is introduced. In Sect. 3 it is shown that two weighted solutions coincide if and only if their underlying weight systems coincide up to a multiplication factor. Moreover, we show that a weighted solution satisfies AN or ETP if and only if the weight system is anonymous or symmetric, respectively. Finally, we show that a symmetrically weighted prekernel is compact-valued, and we present an example of a non-symmetrically weighted prekernel that is not compact-valued. In Sect. 4 the weighted variants of the reduced games are introduced and it is shown that the weighted prenucleolus and prekernel satisfy the reduced game property. Section 5 is devoted to the generalization of Sobolev's axiomatization to symmetrically weighted prenucleoli. We also show that if there exists a solution that satisfies this set of modified axioms, then the weight system must be symmetric. Moreover, it is shown that Snijders' axiomatization may be suitably modified for weighted nucleoli. Finally, in Sect. 6 the axiomatization of symmetrically weighted prekernels is presented that resembles Peleg's axiomatization of the classical prekernel. Moreover, we show that symmetry of the weight system is necessary for the existence of a solution that satisfies the axioms of this axiomatization.

2 Preliminaries

Let U be a set, the *universe of players*, containing, without loss of generality, $1,\dots,k$ whenever $|U|\geq k$. Here and in the sequel, if D is a set, then $|D|$ denotes the cardinality of D. A *coalition* is a finite nonempty subset of U. Let $\mathcal{F}$ denote the set of coalitions. A (cooperative transferable utility) *game* is a pair (N,v) such that $N\in\mathcal{F}$ and $v:2^N\to\mathbb{R}$, $v(\emptyset)=0$. Let (N,v) be a game. We call N its grand coalition and denote the set of all proper nonempty subcoalitions of N by $\mathcal{F}^N$, i.e., $\mathcal{F}^N=2^N\setminus\{\emptyset,N\}$. Let

$$X^*(N,v)=\big\{x\in\mathbb{R}^N\mid x(N)\leq v(N)\big\},$$

$$X(N,v)=\big\{x\in\mathbb{R}^N\mid x(N)=v(N)\big\},\quad\text{and}$$

$$I(N,v)=\big\{x\in X(N,v)\mid x_i\geq v\big(\{i\}\big)\text{ for all }i\in N\big\}$$

denote the set of *feasible payoffs*, the set of *Pareto optimal* feasible payoffs (*preimputations*), and the set of *individually rational* preimputations (*imputations*) of (N,v), respectively, where $x(S)=\sum_{i\in S}x_i$ $(x(\emptyset)=0)$ for $S\subseteq N$ and $x\in\mathbb{R}^N$. For $S\subseteq N$ and $x\in\mathbb{R}^N$, x_S denotes the restriction of x to S, i.e., $x_S=(x_i)_{i\in S}$, and $e(S,x,v)=v(S)-x(S)$ denotes the *excess* of S at x.

A *solution* σ assigns a subset $\sigma(N,v)$ of $X^*(N,v)$ to any game (N,v). Its restriction to a set Γ of games is again denoted by σ. A solution on Γ is the restriction to Γ of a solution.

In order to recall the definition of the "weighted (pre)nucleolus", we employ and recall Justman's (1977) notion of the "generalized nucleolus".

Let D be a finite nonempty set, X be a set, let $h:X\to\mathbb{R}^D$. Define $\theta:X\to\mathbb{R}^{|D|}$ by

$$\theta_t(x)=\max_{T\subseteq D,|T|=t}\min_{i\in T}h_i(x)\quad\text{for all }x\in X\text{ and all }t=1,\dots,|D|,$$

that is, for any $x\in X$, $\theta(x)$ is the vector, whose components are the numbers $h_i(x)$, $i\in D$, arranged in nonincreasing order. Let $\geq_{lex}$ denote the lexicographical order of $\mathbb{R}^{|D|}$. The *nucleolus* of h with respect to (w.r.t.) X, $\mathcal{NUC}(h, X)$, is defined by

$$\mathcal{NUC}(h, X) = \{x \in X \mid \theta(y) \geq_{lex} \theta(x) \text{ for all } y \in X\}.$$

Remark 2.1 Justman (1977) proved the following statements.

(1) If X is nonempty and compact and if all $h_i, i \in D$, are continuous, then $\mathcal{NUC}(h, X) \neq \emptyset$.
(2) If X is convex and all $h_i, i \in D$, are convex, then $\mathcal{NUC}(h, X)$ is convex and $h_i(x) = h_i(y)$ for all $i \in D$ and all $x, y \in \mathcal{NUC}(h, X)$.

Let us recall the definition of a "weighted (pre)nucleolus" (Derks and Haller 1999). A *weight system* is a system $\boldsymbol{p} = (p^N)_{N\in\mathcal{F}}$ such that, for every $N \in \mathcal{F}$, $p^N = (p^N_S)_{S\in\mathcal{F}^N}$, the *weight system* for N, satisfies $p^N_S > 0$ for all $S \in \mathcal{F}^N$. Let $\boldsymbol{p}$ be a weight system and (N, v) be a game. The *weighted prenucleolus* and the *weighted nucleolus* of (N, v) according to $\boldsymbol{p}$, $\mathcal{PN}^{\boldsymbol{p}}(N, v)$ and $\mathcal{N}^{\boldsymbol{p}}(N, v)$, are defined by

$$\mathcal{PN}^{\boldsymbol{p}}(N, v) = \mathcal{NUC}\big((p^N_S e(S, \cdot, v))_{S\in\mathcal{F}^N}, X(N, v)\big) \quad \text{and}$$
$$\mathcal{N}^{\boldsymbol{p}}(N, v) = \mathcal{NUC}\big((p^N_S e(S, \cdot, v))_{S\in\mathcal{F}^N}, I(N, v)\big).$$

As each of the excess functions $e(S, \cdot, v) : \mathbb{R}^N \to \mathbb{R}$ is affine linear, it is convex so that, by Remark 2.1, $\mathcal{N}^{\boldsymbol{p}}(N, v)$ is a singleton whose unique element is denoted by $\nu^{\boldsymbol{p}}_I(N, v)$ provided that $I(N, v) \neq \emptyset$. For any $x \in X(N, v)$, we may replace $X(N, v)$ by the compact, nonempty, and convex polyhedral set

$$\Big\{y \in X(N, v) \,\Big|\, \max_{S\in\mathcal{F}^N} p^N_S e(S, y, v) \leq \max_{T\in\mathcal{F}^N} p^N_T e(T, x, v)\Big\}$$

in the definition of $\mathcal{PN}^{\boldsymbol{p}}(N, v)$ so that this weighted prenucleolus is also a singleton whose unique element is denoted by $\nu^{\boldsymbol{p}}(N, v)$. Note that if the core of a game (N, v),

$$C(N, v) = \{x \in X(N, v) \mid e(S, x, v) \leq 0 \text{ for all } S \subseteq N\},$$

is nonempty, then $\nu^{\boldsymbol{p}}(N, v) = \nu^{\boldsymbol{p}}_I(N, v) \in C(N, v)$ for any weight system $\boldsymbol{p}$.

A solution σ is called a *weighted (pre)nucleolus* if there exists a weight system $\boldsymbol{p}$ such that $\sigma = (\mathcal{P})\mathcal{N}^{\boldsymbol{p}}$. If all weights are identical, then the arising weighted (pre)nucleolus is the classical (pre)nucleolus (Schmeidler 1969; Sobolev 1975). In this case we frequently omit the superscript $\boldsymbol{p}$, i.e., $\nu(N, v)$ and, for $I(N, v) \neq \emptyset$, $\nu_I(N, v)$ denote the classical prenucleolus point and the classical nucleolus point of (N, v), respectively.

Other weight systems were considered by Wallmeier (1983) who investigated weighted nucleoli where the weights may only depend on and are weakly decreasing in coalition size. This property is satisfied by, e.g., the classical nucleolus and by the *per capita nucleolus*, i.e., the weighted nucleolus according to the inverse cardinalities of the coalitions as weights, i.e., $p^N_S = \frac{1}{|S|}$ for each $N \in \mathcal{F}$ and $S \in \mathcal{F}^N$ (Grotte 1970).

Let $\boldsymbol{p}$ be a weight system, (N, v) be a game, and $x \in \mathbb{R}^N$. The ***p**-weighted excess game* of (N, v) at x, denoted by $(N, v^{\boldsymbol{p}}_x)$, is defined by

$$v^{\boldsymbol{p}}_x(S) = \begin{cases} e(N, x, v), & \text{if } S = N, \\ p^N_S e(S, x, v), & \text{if } S \in \mathcal{F}^N. \end{cases} \tag{2.1}$$

We conclude that $e(S, 0, v^{\boldsymbol{p}}_x) = p^N_S e(S, x, v)$ for all $S \subseteq N$. Moreover, $0 \in X(N, v^{\boldsymbol{p}}_x)$ if and only if $x \in X(N, v)$. Thus, we have deduced the following result.

Proposition 2.2 *Let $\boldsymbol{p}$ be a weight system, (N, v) be a game, and $x \in X(N, v)$. Then*

(1) $x = \nu^{\boldsymbol{p}}(N, v)$ *if and only if* $0 = \nu(N, v_x^{\boldsymbol{p}})$;
(2) $x = \nu_I^{\boldsymbol{p}}(N, v)$ *if and only if* $0 = \nu_I(N, v_x^{\boldsymbol{p}})$.

Whether or not a (pre)imputation of a game coincides with the weighted (pre)nucleolus can be checked with the help of suitable modifications of Kohlberg's (1971) "Property I" or "Property II", the characterization of the (pre)nucleolus by balanced collections of coalitions—see also Potters and Tijs (1992).

Let Γ_I denote the set of all games (N, v) with $I(N, v) \neq \emptyset$, i.e., $(N, v) \in \Gamma_I$ if and only if $v(N) \geq \sum_{i \in N} v(\{i\})$. Some more notation is useful. For any game (N, v), $X \subseteq \mathbb{R}^N$, $x \in X$, and any bijective mapping $\pi : U \to U$, denote $\pi(N) = \{\pi(i) \mid i \in N\}$, $\pi x = (x_{\pi(i)})_{i \in N} \in \mathbb{R}^{\pi(N)}$, $\pi X = \{\pi x \mid x \in X\}$, and let $(\pi(N), \pi v)$ be the game defined by $\pi v(\pi(S)) = v(S)$ for all $S \subseteq N$. A permutation π of N is called a *symmetry* of (N, v), abbreviated by $\pi \in \mathcal{SYM}(N, v)$, if $(\pi(N), \pi v) = (N, v)$.

Let σ be a solution on a set Γ of games. We recall some intuitive and desirable properties. The solution σ satisfies

- *non-emptiness* (NE) if $\sigma(N, v) \neq \emptyset$ for all $(N, v) \in \Gamma$;
- *single-valuedness* (SIVA) if $|\sigma(N, v)| = 1$ for all $(N, v) \in \Gamma$;
- *Pareto optimality* (PO) if $\sigma(N, v) \in X(N, v)$ for all $(N, v) \in \Gamma$;
- *anonymity* (AN) if, for all $(N, v) \in \Gamma$ and all bijective mappings $\pi : U \to U$ we have $\sigma(\pi(N), \pi v) = \pi\sigma(N, v)$;
- *symmetry* (SYM) if $\pi\sigma(N, v) = \sigma(N, v)$ for all $(N, v) \in \Gamma$ and all symmetries π of (N, v);
- *covariance under strategic equivalence* (COV) if for any $(N, v), (N, w) \in \Gamma$, $\alpha > 0$, and $\beta \in \mathbb{R}^N$ the following condition is valid: If $w(S) = \alpha v(S) + \beta(S)$ for all $S \subseteq N$, then $\sigma(N, w) = \alpha\sigma(N, v) + \beta$;
- the *equal treatment property* (ETP) if for all $(N, v) \in \Gamma$, all $x \in \sigma(N, v)$, and all $k, \ell \in N$ the following condition is satisfied: If k and ℓ are *substitutes*, i.e., $v(S \cup \{k\}) = v(S \cup \{\ell\})$ for all $S \subseteq N \setminus \{k, \ell\}$, then $x_k = x_\ell$.

Note that AN implies SYM. Furthermore, each weighted prenucleolus clearly satisfies NE, SIVA, PO, and COV on any set of games. Moreover, a weighted nucleolus satisfies PO and COV on any set of games, and it satisfies NE and SIVA if and only if Γ is contained in Γ_I. We now introduce the "weighted (pre)kernel".

Let $\boldsymbol{p}$ be a weight system, (N, v) be a game, $x \in \mathbb{R}^N$, and $k, \ell \in N$, $k \neq \ell$. The *maximum* $\boldsymbol{p}$*-weighted surplus* of k over ℓ at x (w.r.t. (N, v)) is defined by

$$s_{k\ell}^{\boldsymbol{p}}(x, v) = \max\left\{p_S^N e(S, x, v) \mid k \in S \subseteq N \setminus \{\ell\}\right\}.$$

Definition 2.3 The *weighted prekernel* and *weighted kernel* according to $\boldsymbol{p}$, $\mathcal{PK}^{\boldsymbol{p}}$ and $\mathcal{K}^{\boldsymbol{p}}$, respectively, of a game (N, v) are defined by

$$\mathcal{PK}^{\boldsymbol{p}}(N, v) = \left\{x \in X(N, v) \mid s_{k\ell}^{\boldsymbol{p}}(x, v) = s_{\ell k}^{\boldsymbol{p}}(x, v) \text{ for all } k, \ell \in N, k \neq \ell\right\} \quad \text{and}$$
$$\mathcal{K}^{\boldsymbol{p}}(N, v) = \left\{x \in I(N, v) \mid s_{k\ell}^{\boldsymbol{p}}(x, v) \geq s_{\ell k}^{\boldsymbol{p}}(x, v) \text{ or } x_k = v(\{k\}) \text{ for all } k, \ell \in N, k \neq \ell\right\}.$$

The kernel, i.e., the weighted kernel according to the weight system that assigns to all coalitions identical weights, was introduced by Davis and Maschler (1965), whereas the prekernel was first considered by Maschler et al. (1972). If $\boldsymbol{p}$ is omitted as a superscript at $(\mathcal{P})\mathcal{K}$, then the classical (pre)kernel is meant.

Remark 2.4

(1) Similarly as in the classical case it is easily verified that the weighted prenucleolus of any game belongs to its weighted prekernel. Moreover, if the game has imputations, then its weighted nucleolus belongs to its weighted kernel.
(2) For any weight system $\boldsymbol{p}$, any game (N, v), and any $x \in X(N, v)$:

$$x \in \mathcal{PK}^{\boldsymbol{p}}(N, v) \iff 0 \in \mathcal{PK}\big(N, v_x^{\boldsymbol{p}}\big).$$
$$x \in \mathcal{K}^{\boldsymbol{p}}(N, v) \iff 0 \in \mathcal{K}\big(N, v_x^{\boldsymbol{p}}\big).$$

(3) Weighted (pre)kernels satisfy COV.

3 Anonymity, symmetry, and the equal treatment property

We first extend a result for weighted nucleoli to the three other aforementioned weighted solutions (Derks and Haller 1999, Theorem 1).

Theorem 3.1 *Let $N \in \mathcal{F}$. Two weighted nucleoli, prenucleoli, kernels, or prekernels coincide for all $(N, v) \in \Gamma_I$ if and only if the two weight systems coincide for N up to a positive multiplication factor.*

Proof The "if-part" is obvious. In order to show the "only if-part" let $\boldsymbol{p}$ and $\boldsymbol{p}'$ be weight systems and σ and σ' be the respective arising weighted solutions. Let $p = p^N$ and $p' = p'^N$ be the corresponding weight systems for N and let $S \in \mathcal{F}^N$. Let

$$\mathcal{T} = \big\{T \in F^N \setminus \{S\} \,\big|\, T \neq \{i\} \text{ for all } i \in N \setminus S\big\}$$

and $t > \max_{Q,R \in \mathcal{F}^N} \frac{p_Q p'_R}{p'_Q p_R}$. Note that $t > 1$. Define (N, v) by $v(S) = -\frac{1}{p_S}$, $v(\{i\}) = -\frac{1}{p_{\{i\}}}$ for all $i \in N \setminus S$, $v(N) = 0$, and $v(T) = -\frac{t}{p_T}$ for all $T \in \mathcal{T}$. Further, let $x = 0 \in \mathbb{R}^N$ and note that $x \in I(N, v)$. Then $p_S e(S, x, v) = p_{\{i\}} e(\{i\}, x, v) = -1$ for all $i \in N \setminus S$ and $p_T e(T, x, v) = -t$ for all $T \in \mathcal{T}$. Let $y = \nu^{\boldsymbol{p}}(N, v)$. Then $y(S) \geq x(S)$, $y_i \geq x_i$ for all $i \in N \setminus S$, and $y(N) = 0$ so that $y(S) = x(S)$ and $y_i = x_i$ for all $i \in N \setminus S$. Thus, $y(T) \geq x(T)$ for all $T \in \mathcal{T}$ so that $y = x$. By Remark 2.4(1), $x \in \sigma(N, v) = \sigma'(N, v)$ so that $x \in \mathcal{PK}^{\boldsymbol{p}}(N, v)$. By the definition of t,

$$s_{k\ell}^{\boldsymbol{p}'}(x, v) = p'_S v(S) = -\frac{p'_S}{p_S} \quad \text{and} \quad s_{\ell k}^{\boldsymbol{p}'}(x, v) = p'_{\{\ell\}} v\big(\{\ell\}\big) = -\frac{p'_{\{\ell\}}}{p_{\{\ell\}}}$$
$$\text{for all } k \in S, \text{ and } \ell \in N \setminus S.$$

Since S has been chosen in $\mathcal{F}^N$ arbitrarily, we have

$$\frac{p'_S}{p_S} = \frac{p'_{\{\ell\}}}{p_{\{\ell\}}} \quad \text{for all } S \in \mathcal{F}^N \text{ and all } \ell \in N \setminus S. \tag{3.1}$$

Applying (3.1) to singletons S yields $\frac{p'_{\{k\}}}{p_{\{k\}}} = \frac{p'_{\{\ell\}}}{p_{\{\ell\}}} = c$ for all $k, \ell \in N$. Hence, (3.1) applied to an arbitrary proper nonempty coalition S of N yields $p'_S = c p_S$. □

Theorem 3.1 enables us to characterize those weight systems $\boldsymbol{p}$ that result in weighted solutions that satisfy anonymity. We call a weight system $\boldsymbol{p}$ *anonymous* if, for all $N, N' \in \mathcal{F}$ with $|N| = |N'|$, there exists $c = c(N, N') > 0$ such that $p_{S'}^{N'} = cp_S^N$ for all $S \in \mathcal{F}^N$ and $S' \in \mathcal{F}^{N'}$ with $|S| = |S'|$. In this case $p_S^N = p^N(|S|)$ and $p^{N'}(s) = c(N, N')p^N(s)$ for all $s = 1, \ldots, |N| - 1$.

Theorem 3.2 *Let* $\boldsymbol{p}$ *be a weight system,* $\Gamma \supseteq \Gamma_I$*, and* $\sigma^{\boldsymbol{p}}$ *be one of the following solutions on* Γ: $\mathcal{N}^{\boldsymbol{p}}, \mathcal{PN}^{\boldsymbol{p}}, \mathcal{K}^{\boldsymbol{p}}$*, or* $\mathcal{PK}^{\boldsymbol{p}}$*. Then* $\sigma^{\boldsymbol{p}}$ *satisfies* AN *if and only if* $\boldsymbol{p}$ *is anonymous.*

Proof The "if-part" is straightforward and left to the reader. In order to show the "only-if-part" let N, N', S, S' be coalitions with $|N| = |N'|$, $|S| = |S'|$, $S \in \mathcal{F}^N$, and $S' \in \mathcal{F}^{N'}$. Let $\pi : U \to U$ be a bijection such that $\pi(S) = S'$, and $\pi(N) = N'$. Let $(N, v) \in \Gamma$. Let the weight system $\boldsymbol{p}'$ be defined by $p'^M_R = p_{\pi(R)}^{\pi(M)}$ for all $M \in \mathcal{F}$ and $R \in \mathcal{F}^M$.

For all $T \in \mathcal{F}^N$, $x \in X(N, v)$ we have

$$p'^N_T e(T, x, v) = p'^N_T e\big(\pi(T), \pi x, \pi v\big) = p_{\pi(T)}^{\pi(N)} e\big(\pi(T), \pi x, \pi v\big). \tag{3.2}$$

Therefore, e.g., $s_{k\ell}^{\boldsymbol{p}'}(x, v) = s_{\pi(k)\pi(\ell)}^{\boldsymbol{p}}(\pi x, \pi v)$ for all $x \in \mathbb{R}^N$ and $k, \ell \in N$, $k \neq \ell$. Now, let $\sigma^{\boldsymbol{p}'}$ be the weighted solution according to $\boldsymbol{p}'$ defined in an analogous way as $\sigma^{\boldsymbol{p}}$ (e.g., $\sigma^{\boldsymbol{p}'}$ is the $\boldsymbol{p}'$-weighted nucleolus if and only if $\sigma^{\boldsymbol{p}}$ is the $\boldsymbol{p}$-weighted nucleolus). By (3.2), $\pi\sigma^{\boldsymbol{p}'}(N, v) = \sigma^{\boldsymbol{p}}(\pi(N), \pi v)$. On the other hand, by AN, $\pi\sigma^{\boldsymbol{p}}(N, v) = \sigma^{\boldsymbol{p}}(\pi(N), \pi v)$ as well. As $\Gamma \supseteq \Gamma_I$, $\sigma^{\boldsymbol{p}}$ and $\sigma^{\boldsymbol{p}'}$ coincide on the set of games with player set N whose imputation sets are nonempty. By Theorem 3.1 there exists $c > 0$ with $p^N = c \cdot p'^N$. Hence, $p_S^N = c \cdot p'^N_S = c \cdot p_{\pi(S)}^{\pi(N)} = c \cdot p_{S'}^{N'}$. □

It should be noted that the weights of an arbitrary weight system $\boldsymbol{p}$ can be *normalized* so that $\sum_{S \in \mathcal{F}^N} p_S^N$ is a constant that may only depend on $|N|$ (e.g., the constant $2^{|N|} - 2$) for all $N \in \mathcal{F}$ without changing the mentioned weighted solutions. If $\boldsymbol{p}$ is normalized, then it is anonymous if and only if $c(N, N') = 1$ for all $N, N' \in \mathcal{F}$ with $|N| = |N'|$, i.e., with $|N| = n$ and $|S| = s$, $p_S^N = p^N(s) = p(s, n)$ for $s = 1, \ldots, n - 1$ and $n \in \mathbb{N}$, $2 \leq n \leq |U|$.

We now characterize weight systems that result in weighted solutions satisfying the equal treatment property. We call a weight system $\boldsymbol{p}$ *symmetric* if, for all $N \in \mathcal{F}$, p_S^N may only depend on the size of the subcoalition, i.e., $p_S^N = p^N(s)$ for all $S \in \mathcal{F}^N$ where $s = |S|$.

Theorem 3.3 *Let* $\boldsymbol{p}$ *be a weight system,* $\Gamma \supseteq \Gamma_I$*, and* σ *be one of the following solutions on* Γ: $\mathcal{N}^{\boldsymbol{p}}, \mathcal{PN}^{\boldsymbol{p}}, \mathcal{K}^{\boldsymbol{p}}$*, or* $\mathcal{PK}^{\boldsymbol{p}}$*. Then* σ *satisfies* ETP *if and only if* $\boldsymbol{p}$ *is symmetric.*

Proof The "if-part" is an obvious consequence of the definitions of the considered weighted solutions. In order to show the "only-if-part" let σ be one of the considered solutions and let it satisfy ETP. Assume, on the contrary, that $\boldsymbol{p}$ does not satisfy the desired property. Hence, there exists a coalition N and some $S, S' \in \mathcal{F}^N$ with $|S| = |S'|$ such that $p_S^N \neq p_{S'}^N$. It remains to show that σ violates ETP. As S' arises from S by a sequence of replacements of one player by one other player, we may assume that $|S \setminus S'| = 1$. Let T, k, ℓ be determined by $S = T \cup \{k\}$ and $S' = T \cup \{\ell\}$. Let (N, v) be the game defined by $v(N) = v(T) = v(N \setminus T) = 0$, $v(T \cup \{i\}) = -1$ for all $i \in N \setminus T$, and $v(R) = \frac{-p_S^N - p_{S'}^N}{\min\{p_Q^N | Q \in \mathcal{F}^N\}}$ for all other $R \in \mathcal{F}^N$. Then $(N, v) \in \Gamma_I$. Let $y = \nu^{\boldsymbol{p}}(N, v)$. By Remark 2.4(1), $y \in \sigma(N, v)$. As σ satisfies ETP and as all players inside T are substitutes and all players in $N \setminus T$ are substitutes as well, there exist $\alpha, \beta \in \mathbb{R}$ such that $y_i = \alpha$ for all $i \in T$ and $y_j = \beta$ for all $j \in N \setminus T$. As $y(N) =$

$v(N) = 0$, $|T|\alpha + |N \setminus T|\beta = 0$. Let $x = 0 \in \mathbb{R}^N$. Then $e(T, x, v) = e(N \setminus T, x, v) = 0$ and $e(R, x, v) < 0$ for all $R \in \mathcal{F}^N \setminus \{T, N \setminus T\}$. By the definition of the weighted prenucleolus, $e(T, y, v) = e(N \setminus T, y, v) = 0$. Hence, $y(T) = y(N \setminus T) = 0$, implying $|T|\alpha = \beta = 0$, i.e., $y = x$.

For any $R \in \mathcal{F}^N \setminus \{S\}$ with $k \in R \not\ni \ell$, the definition of v gives

$$p_R^N e(R, y, v) \leq p_R^N \frac{-p_S^N - p_{S'}^N}{p_R^N} < -p_S^N = p_S^N e(S, y, v).$$

A similar statement is valid when switching the roles of k and ℓ, so $s_{k\ell}^{\boldsymbol{p}}(y, v) = -p_S^N \neq -p_{S'}^N = s_{\ell k}^{\boldsymbol{p}}(y, v)$. Hence, $y \notin \mathcal{PK}^{\boldsymbol{p}}(N, v)$, $y \notin \mathcal{K}^{\boldsymbol{p}}(N, v)$ and the desired contradiction is obtained by Remark 2.4(1). □

Remark 3.4 A symmetric weight system generates weighted solutions that do not only satisfy ETP, but also satisfy SYM.

Let $\boldsymbol{p}$ be an arbitrary weight system and (N, v) be a game. A system $\mathcal{S} = (S^{k\ell})_{(k,\ell)\in N\times N, k\neq\ell}$ is a *constellation* if $k \in S^{k\ell} \subseteq N \setminus \{\ell\}$ for any $k, \ell \in N$, $k \neq \ell$. Hence, $\mathcal{PK}^{\boldsymbol{p}}(N, v)$ is the union taken over all constellations $\mathcal{S}$ of the sets $X_{\mathcal{S}}$ given by

$$X_{\mathcal{S}} = \big\{x \in X(N, v) \mid p_S^N e(S, x, v) \leq p_{S^{k\ell}}^N e\big(S^{k\ell}, x, v\big) = p_{S^{\ell k}}^N e\big(S^{\ell k}, x, v\big) \geq p_T^N e(T, x, v)$$
$$\text{for all } k, \ell \in N, \text{ for all } k \in S \subseteq N \setminus \{\ell\} \text{ and all } \ell \in T \subseteq N \setminus \{k\}\big\}.$$

Hence, the weighted prekernel of (N, v) is, similarly to the classical case, a finite union of polyhedral sets.

Proposition 3.5 *If $\boldsymbol{p}$ is a symmetric weight system, then for any game (N, v), $\mathcal{PK}^{\boldsymbol{p}}(N, v)$ is compact.*

Proof Assume, on the contrary, that $\mathcal{PK}^{\boldsymbol{p}}(N, v)$ is not compact. Let $\mathcal{S} = (S^{k\ell})_{(k,\ell)\in N\times N, k\neq\ell}$ be a constellation such that $X_{\mathcal{S}}$ is unbounded. Let $(x^r)_{r\in\mathbb{N}}$ be an unbounded sequence of elements of $X_{\mathcal{S}}$. Then, after replacing our sequence by a suitable subsequence, if necessary, there exist $S, T \in \mathcal{F}^N$ with $S \cap T = \emptyset$ such that $x_i^r \underset{r\to\infty}{\longrightarrow} -\infty$, $x_j^r \underset{r\to\infty}{\longrightarrow} \infty$ for all $i \in S$, $j \in T$, and $\{x_i^r \mid r \in \mathbb{N}, i \in N \setminus (S \cup T)\}$ is bounded. Let, for all $r \in \mathbb{N}$, $\mu^r = \max_{R\in\mathcal{F}^N} p_R^N e(R, x^r, v)$.

Let $r \in \mathbb{N}$ and $k \in S$. Let i, j be such that $s_{ij}^{\boldsymbol{p}}(x^r, v) = \mu^r$. If $k \in S^{ij}$, then $s_{kj}^{\boldsymbol{p}}(x^r, v) = \mu^r$. If $k \notin S^{ij}$, then $\mu^r = s_{ik}^{\boldsymbol{p}}(x^r, v) = s_{ki}^{\boldsymbol{p}}(x^r, v)$, where the second equality follows from the fact that $x^r \in \mathcal{PK}^{\boldsymbol{p}}(N, v)$. Hence, there exists a player m with $s_{km}^{\boldsymbol{p}}(x^r, v) = \mu^r$.

As $e(\{i\}, x^r, v) \underset{r\to\infty}{\longrightarrow} \infty$ for any $i \in S$, we have $\lim_{r\to\infty} \mu^r = \infty$. Since $p_{S^{km}}^N e(S^{km}, x^r, v) \underset{r\to\infty}{\longrightarrow} \infty$, there exists a player $\ell \in T \setminus S^{km}$. Then $\mu^r \geq p_{S^{k\ell}}^N e(S^{k\ell}, x^r, v) \geq p_{S^{km}}^N e(S^{km}, x^r, v) = \mu^r$ for all $r \in \mathbb{N}$ as well, and hence, $\mu^r = p_{S^{k\ell}}^N e(S^{k\ell}, x^r, v)$. Denote $R = S^{\ell k} \cup \{k\} \setminus \{\ell\}$. Since $\boldsymbol{p}$ is symmetric, $p_R^N e(R, x^r, v) > p_{S^{\ell k}}^N e(S^{\ell k}, x^r, v) = p_{S^{k\ell}}^N e(S^{k\ell}, x^r, v) = \mu^r$ for r taken sufficiently large, so the desired contradiction has been obtained. □

We now show that the condition that the weight system $\boldsymbol{p}$ is symmetric cannot be omitted in Proposition 3.5. Indeed, we present an example of a non-symmetric weight system that results in a weighted prekernel that is not bounded provided that $|U| \geq 5$.

Example 3.6 Let $N = \{1, \dots, 5\}$ and p^N be defined by

$$p^N_S = 7 \quad \text{if } \big|S \cap \{1,2,3\}\big| = 2 \text{ and } \big|S \cap \{4,5\}\big| = 1 \text{ and } p^N_S = 1 \text{ otherwise},$$

for all $S \in \mathcal{F}^N$. Then $x^t = (-2t, -2t, -2t, 3t, 3t) \in \mathcal{PK}^{\boldsymbol{p}}(N, 0)$ for all $t \geq 0$. Indeed, the maximal $\boldsymbol{p}$-weighted excess at x^t is attained by the coalitions S with $p^N_S = 7$, and it is $7t$. However, the set of these coalitions is completely separating, i.e., for any $k, \ell \in N, k \neq \ell$, there exists a coalition $S \in \mathcal{F}^N$ with $p^N_S = 7$ and $\ell \notin S \ni k$ so that $s^{\boldsymbol{p}}_{k\ell}(x^t, v) = 7t$. Hence, this weighted prekernel is unbounded.

4 Reduced games according to weight systems

We first recall the well-accepted definition of the reduced game (Davis and Maschler 1965). Let (N, v) be a game, $S \subseteq N$ be a coalition, and $x \in \mathbb{R}^N$. The *reduced game* of (N, v) w.r.t. S and x, denoted by $(S, v_{S,x})$, is the game defined by $v_{S,x}(S) = v(N) - x(N \setminus S)$ and

$$v_{S,x}(T) = \max_{Q \subseteq N \setminus S} \big(v(T \cup Q) - x(Q)\big) \quad \text{for all } T \in \mathcal{F}^S. \tag{4.1}$$

A solution σ on a set Γ of games satisfies the *reduced game property* (RGP) if, for any $(N, v) \in \Gamma$, any $S \in \mathcal{F}^N$, and any $x \in \sigma(N, v)$, $(S, v_{S,x}) \in \Gamma$ and $x_S \in \sigma(S, v_{S,x})$.

Remark 4.1 The prekernel (Peleg 1986) and the prenucleolus (Sobolev 1975) satisfy RGP on the set of all games.

We now modify the reduced game such that the corresponding reduced game property is satisfied by the weighted versions of the foregoing solutions.

Definition 4.2 Let $\boldsymbol{p}$ be a weight system. The $\boldsymbol{p}$-*reduced game* of a game (N, v) w.r.t. a coalition $S \subseteq N$ and $x \in \mathbb{R}^N$ is the game $(S, v^{\boldsymbol{p}}_{S,x})$ defined by $v^{\boldsymbol{p}}_{S,x}(S) = v_{S,x}(S)$ and

$$v^{\boldsymbol{p}}_{S,x}(T) = \max_{Q \subseteq N \setminus S} \left(v(T \cup Q) - x(Q) + \frac{p^N_{T \cup Q} - p^S_T}{p^S_T} e(T \cup Q, x, v) \right) \quad \text{for all } T \in \mathcal{F}^S. \tag{4.2}$$

Note that (4.1) guarantees that the excess of a proper nonempty subcoalition T of S is the maximal excess of coalitions that arise from T by adding players of $N \setminus S$. Now, the modification (4.2) of (4.1) takes care of the weighted excesses instead, i.e., it guarantees a similar property when the excess is replaced by the weighted excess. Indeed, for $Q \subseteq N \setminus S$,

$$p^S_T \cdot \left(v(T \cup Q) - x(Q) + \frac{p^N_{T \cup Q} - p^S_T}{p^S_T} e(T \cup Q, x, v) - x(T) \right) = p^N_{T \cup Q} e(T \cup Q, x, v)$$

so that we have deduced the following proposition.

Proposition 4.3 *For any weight system* $\boldsymbol{p}$, *any game* (N, v), $S \in \mathcal{F}^N$, *and* $x \in \mathbb{R}^N$,

$$p^S_T e\big(T, x_S, v^{\boldsymbol{p}}_{S,x}\big) = \max_{Q \subseteq N \setminus S} p^N_{T \cup Q} e(T \cup Q, x, v) \quad \textit{for all } T \in \mathcal{F}^S. \tag{4.3}$$

This proposition is used to show that the $\boldsymbol{p}$-weighted prekernel satisfies the following weighted version of RGP, the *reduced game property w.r.t.* $\boldsymbol{p}$*-reduced games* ($\boldsymbol{p}$-RGP), defined by replacing $v_{S,x}$ by $v^{\boldsymbol{p}}_{S,x}$ wherever it occurs in the definition of RGP.

Corollary 4.4 *For any weight system* $\boldsymbol{p}$ *the* $\boldsymbol{p}$*-weighted prekernel satisfies* $\boldsymbol{p}$-RGP.

Proof Let (N, v) be a game, $S \in \mathcal{F}^N$, $x \in \mathcal{PK}^{\boldsymbol{p}}(N, v)$, and $k, \ell \in S$ with $k \neq \ell$. By Proposition 4.3

$$s^{\boldsymbol{p}}_{k\ell}\big(x_S, v^{\boldsymbol{p}}_{S,x}\big) = s^{\boldsymbol{p}}_{k\ell}(x, v) = s^{\boldsymbol{p}}_{\ell k}(x, v) = s^{\boldsymbol{p}}_{\ell k}\big(x_S, v^{\boldsymbol{p}}_{S,x}\big)$$

so that the proof is complete. □

The following lemma is useful to show that $\mathcal{PN}^{\boldsymbol{p}}$ satisfies $\boldsymbol{p}$-RGP.

Lemma 4.5 *Let* $\boldsymbol{p}$ *be a weight system,* (N, v) *be a game,* $S \in \mathcal{F}^N$*, and* $x \in \mathbb{R}^N$*. Then the* $\boldsymbol{p}$*-weighted excess game w.r.t.* x_S *of the* $\boldsymbol{p}$*-reduced game of* (N, v) *w.r.t.* S *and* x *coincides with the classical reduced game w.r.t.* S *and* $0 \in \mathbb{R}^N$ *of the* $\boldsymbol{p}$*-weighted excess game w.r.t.* x *of* (N, v)*, i.e.,*

$$\big(v^{\boldsymbol{p}}_{S,x}\big)^{\boldsymbol{p}}_{x_S} = \big(v^{\boldsymbol{p}}_x\big)_{S,0}.$$

Proof By the definitions of the $\boldsymbol{p}$-weighted excess game, the classical and the $\boldsymbol{p}$-reduced games ((2.1), (4.1), and (4.2), respectively), we find

$$\begin{aligned}\big(v^{\boldsymbol{p}}_{S,x}\big)^{\boldsymbol{p}}_{x_S}(S) &= v^{\boldsymbol{p}}_{S,x}(S) - x(S) = \big(v(N) - x(N \setminus S)\big) - x(S) = e(N, x, v)\\ &= v^{\boldsymbol{p}}_x(N) = \big(v^{\boldsymbol{p}}_x\big)_{S,0}(S).\end{aligned}$$

Additionally applying Proposition 4.3 yields for all $T \in \mathcal{F}^S$

$$\begin{aligned}\big(v^{\boldsymbol{p}}_{S,x}\big)^{\boldsymbol{p}}_{x_S}(T) &= p^S_T e\big(T, x_S, v^{\boldsymbol{p}}_{S,x}\big) = \max_{Q \subseteq N \setminus S} p^N_{T \cup Q} e(T \cup Q, x, v) = \max_{Q \subseteq N \setminus S} v^{\boldsymbol{p}}_x(T \cup Q)\\ &= \big(v^{\boldsymbol{p}}_x\big)_{S,0}(T).\end{aligned}$$
□

Theorem 4.6 *For any weight system* $\boldsymbol{p}$ *the* $\boldsymbol{p}$*-weighted prenucleolus satisfies* $\boldsymbol{p}$-RGP.

Proof Let (N, v) be a game and denote $x = \nu^{\boldsymbol{p}}(N, v)$. By Proposition 2.2(1), we have $0 = \nu(N, v^{\boldsymbol{p}}_x)$. Let $S \in \mathcal{F}^N$. By Remark 4.1, $0 = \nu(S, (v^{\boldsymbol{p}}_x)_{S,0})$, which rewrites by Lemma 4.5 to $0 = \nu(S, (v^{\boldsymbol{p}}_{S,x})^{\boldsymbol{p}}_{x_S})$. Applying once more Proposition 2.2(1), we find that $x_S = \nu^{\boldsymbol{p}}(S, v^{\boldsymbol{p}}_{S,x})$. □

The following remark and lemma are used in subsequent sections.

Remark 4.7 Let $\boldsymbol{p}$ be a weight system and σ be a solution.

(1) If σ satisfies SIVA, COV, and $\boldsymbol{p}$-RGP, then it satisfies PO. Indeed, the proof in the classical case (Peleg and Sudhölter 2007, Lemma 6.2.11) may be literally copied because it suffices to apply RGP just to one-person reduced games.

(2) If σ satisfies NE, PO, COV, and ETP, then σ is a *standard solution* (Peleg 1986), i.e., for any $k, \ell \in U, k \neq \ell$,

$$\sigma(N, v) = \{y\}, \quad \text{where } y_k = \frac{v(\{k\}) - v(\{\ell\}) + v(N)}{2} \text{ and } N = \{k, \ell\},$$

for all games (N, v).

Hence, for any two-person game (N, v), $\sigma(N, v)$ is a singleton $\{y\}$. Moreover, $y_k = y_\ell$, where $N = \{k, \ell\}$, if and only if k and ℓ are substitutes.

Lemma 4.8 *Let* $|U| \geq 3$. *If there exists a solution that satisfies* NE, PO, COV, ETP, *and* $\boldsymbol{p}$-RGP, *then* $p^N_{\{k\}} = p^N_{\{\ell\}}$ *and* $p^N_{N\setminus\{k\}} = p^N_{N\setminus\{\ell\}}$ *for all* $N \in \mathcal{F}$ *with* $|N| \geq 2$ *and all* $k, \ell \in N$.

Proof Let σ be a solution on Γ that satisfies the requested properties. Let $k, \ell \in N \in \mathcal{F}$ with $k \neq \ell$. Choose $t > \max\left\{\frac{p^N_R}{p^N_Q} \mid R, Q \in \mathcal{F}^N\right\}$ so that $t > 1$. We first show that

$$\frac{p^N_{\{k\}}}{p^N_{\{\ell\}}} = \frac{p^{\{k,\ell\}}_{\{k\}}}{p^{\{k,\ell\}}_{\{\ell\}}} = \frac{p^N_{N\setminus\{\ell\}}}{p^N_{N\setminus\{k\}}}. \tag{4.4}$$

To this end define the game (N, v) by $v(N) = 0$, $v(\{i\}) = -1$ for all $i \in N$, and $v(T) = -t$ for all other $T \in \mathcal{F}^N$. Let $y = 0 \in \mathbb{R}^N$. By NE, PO, and ETP, $\sigma(N, v) = \{y\}$. Let $S = \{k, \ell\}$. By $\boldsymbol{p}$-RGP, $y_S = 0 \in \sigma(S, v^{\boldsymbol{p}}_{S,y})$. By Remark 4.7(2), $v^{\boldsymbol{p}}_{S,y}(\{k\}) = v^{\boldsymbol{p}}_{S,y}(\{\ell\}) =: \alpha$. By Proposition 4.3, $s^{\boldsymbol{p}}_{k\ell}(y, v) = s^{\boldsymbol{p}}_{k\ell}(y_S, v^{\boldsymbol{p}}_{S,y})$ and $s^{\boldsymbol{p}}_{\ell k}(y, v) = s^{\boldsymbol{p}}_{\ell k}(y_S, v^{\boldsymbol{p}}_{S,y})$. By the choice of t, $s^{\boldsymbol{p}}_{k\ell}(y, v) = -p^N_{\{k\}}$ and $s^{\boldsymbol{p}}_{\ell k}(y, v) = -p^N_{\{\ell\}}$. Moreover, $s^{\boldsymbol{p}}_{k\ell}(y_S, v^{\boldsymbol{p}}_{S,y}) = \alpha p^S_{\{k\}}$ and $s^{\boldsymbol{p}}_{\ell k}(y_S, v^{\boldsymbol{p}}_{S,y}) = \alpha p^S_{\{\ell\}}$ so that the first equation in (4.4) is shown. The second equation in (4.4) is deduced similarly. Only the game (N, v) has to be replaced by the game (N, v') defined by $v'(N) = 0$, $v'(N \setminus \{i\}) = -1$ for all $i \in N$, and $v(T) = -t$ for all other $T \in \mathcal{F}^N$.

Hence, it suffices to prove that $p^M_{\{k\}} = p^M_{\{\ell\}}$ for any $M \subseteq U$ with $k, \ell \in M$ and $|M| = 3$. We may assume without loss of generality that $M = \{1, 2, 3\}$, $k = 1$, and $\ell = 2$. Choose $\beta > 2 \cdot \max\left\{\frac{p^M_R}{p^M_Q} \mid R, Q \in \mathcal{F}^M\right\}$ and define the game (M, w) by $w(M) = 0$, $w(\{1, 2\}) = w(\{3\}) = -1$, and $w(T) = -\beta$ for all other $T \in \mathcal{F}^M$. By NE there exists $z \in \sigma(M, w)$. By PO and ETP there exists $\alpha \in \mathbb{R}$ such that $z = (\alpha, \alpha, -2\alpha)$. If $\alpha \geq 0$, then $s^{\boldsymbol{p}}_{1,3}(z, w) < 0$, and if $\alpha < 0$, then $s^{\boldsymbol{p}}_{3,1}(z, w) < 0$. By $\boldsymbol{p}$-RGP and Proposition 4.3, Remark 4.7(2) guarantees that $e(\{1, 2\}, z, w) < 0$ and $e(\{3\}, z, w) < 0$, i.e., $-1 < 2\alpha < 1$. Hence by Remark 4.7(2), $e(\{1\}, z_{\{1,3\}}, w^{\boldsymbol{p}}_{\{1,3\}}) = e(\{3\}, z_{\{1,3\}}, w^{\boldsymbol{p}}_{\{1,3\}}) = -\delta$ for some $\delta > 0$. By Proposition 4.3 and the choice of β,

$$s^{\boldsymbol{p}}_{1,3}(z, w) = p^M_{\{1,2\}}(-1 - 2\alpha) = -p^{\{1,3\}}_{\{1\}}\delta \quad \text{and} \quad s^{\boldsymbol{p}}_{3,1}(z, w) = p^M_{\{3\}}(-1 + 2\alpha) = -p^{\{1,3\}}_{\{3\}}\delta,$$

i.e., $\frac{p^{\{1,3\}}_{\{1\}}}{p^{\{1,3\}}_{\{3\}}} = \frac{p^M_{\{1,2\}}(1-2\alpha)}{p^M_{\{3\}}(1+2\alpha)}$. Similarly, by considering $w^{\boldsymbol{p}}_{\{2,3\}}$, we receive $\frac{p^{\{2,3\}}_{\{2\}}}{p^{\{2,3\}}_{\{3\}}} = \frac{p^M_{\{1,2\}}(1-2\alpha)}{p^M_{\{3\}}(1+2\alpha)}$. By (4.4),

$$\frac{p^M_{\{1\}}}{p^M_{\{3\}}} = \frac{p^{\{1,3\}}_{\{1\}}}{p^{\{1,3\}}_{\{3\}}} = \frac{p^{\{2,3\}}_{\{2\}}}{p^{\{2,3\}}_{\{3\}}} = \frac{p^M_{\{2\}}}{p^M_{\{3\}}},$$

so that the proof is complete. □

5 Axiomatization of the symmetrically weighted (pre)nucleolus

This section is devoted to the generalization of Sobolev's (1975) famous axiomatization of the prenucleolus that makes the following assertion: If $|U| = \infty$, then the prenucleolus is the unique solution that satisfies SIVA, AN, COV, and RGP. A careful inspection of the proof shows that, instead of AN, in fact the weaker SYM is used. Moreover, SYM and SIVA imply ETP, and, in fact, Orshan (1993) shows that AN may even be replaced by ETP. In view of Theorem 3.3 and Remark 3.4, we shall modify the aforementioned result by employing SIVA, COV, SYM and $\boldsymbol{p}$-RGP for a symmetric weight system $\boldsymbol{p}$.

Moreover, the logical independence of each of the employed axioms of the remaining axioms will be discussed, and Snijders' (1995) result on the nucleolus will be generalized.

5.1 Symmetrically weighted prenucleoli

First, we show a relation between SIVA, SYM, COV, and $\boldsymbol{p}$-RGP and the symmetry of the weight system.

Theorem 5.1 *Let $|U| \geq 3$ and $\boldsymbol{p}$ be a weight system. Then there exists a solution that satisfies* SIVA, SYM, COV, *and* $\boldsymbol{p}$-RGP *if and only if $\boldsymbol{p}$ is symmetric.*

Proof If $\boldsymbol{p}$ is symmetric, then $\mathcal{PN}^{\boldsymbol{p}}$ satisfies SIVA and COV as is known (see Sect. 2), SYM by Remark 3.4, and $\boldsymbol{p}$-RGP by Theorem 4.6. In order to show the "only if"-part, let σ be a solution on Γ that satisfies the requested properties. Let $N \in \mathcal{F}$, $n = |N| \geq 2$, $S, S' \in \mathcal{F}^N$ such that $s = |S| = |S'|$ and $S \neq S'$. It remains to show that $p_S^N = p_{S'}^N$. By Remark 4.7, σ satisfies PO. Moreover, SIVA and SYM together imply NE and ETP. Hence, by Lemma 4.8 we may assume that $|N| \geq 4$ and $2 \leq s \leq n-2$.

As S' arises from S by a sequence of replacements of one player by one other player, we may assume that $|S \setminus S'| = 1$. Let $t > \max\{\frac{p_R^N}{p_Q^N} \mid R, Q \in \mathcal{F}^M\}$. Hence, $t > 1$. Without loss of generality we may assume that

$$N = \{1, \ldots, n\}, \quad S = \{1, \ldots, s\}, \quad \text{and} \quad S' = \{1, \ldots, s-1, n-1\}.$$

Let π be the "cyclic" permutation of N defined by $\pi(i) = i - 1$ for all $i \in N \setminus \{1\}$ and $\pi(1) = n$, and let π' be the permutation of N defined by $\pi'(j) = \pi(j)$ for all $j \in N \setminus \{n-1, n\}$, $\pi'(n-1) = n-1$, and $\pi'(n) = n-2$. Now, let the game (N, v) be defined by $v(N) = 0$, $v(\pi^j(S)) = -1$ for all $j \in N$, and $v(T) = -t$ for all other $T \in \mathcal{F}^N$ (here π^j denotes the j-fold composition of π). Moreover, let the game (N, v') be defined by $v'(N) = v'(\{n-1\}) = v'(N \setminus \{n-1\}) = 0$, $v(\pi'^j(S')) = -1$ for all $j \in N \setminus \{n\}$ and $v(T) = -t$ for all other $T \in \mathcal{F}^N$.

By SIVA, $\sigma(N, v) = \{z\}$ and $\sigma(N, v') = \{z'\}$ for some $z, z' \in \mathbb{R}^N$. By Remark 4.7(1), $z(N) = z'(N) = 0$. By construction $\pi \in \mathcal{SYM}(N, v)$ and $\pi' \in \mathcal{SYM}(N, v')$ so that, by SYM, $z_n = z_{n-1} = \cdots = z_1$ and $z'_n = z'_{n-2} = \cdots = z'_1$. Hence, $z = 0 \in \mathbb{R}^N$ and there exists $\alpha \in \mathbb{R}$ such that $z' = (\underbrace{\alpha, \ldots, \alpha}_{n-2}, -(n-1)\alpha, \alpha)$. Let $w = v'^{\boldsymbol{p}}_{Q,z'}$, where $Q = \{n-1, n\}$.

By $\boldsymbol{p}$-RGP, $z'_Q \in \sigma(Q, w)$. By Remark 4.7(2), $s^{\boldsymbol{p}}_{n-1,n}(z'_Q, w)$ and $s^{\boldsymbol{p}}_{n,n-1}(z'_Q, w)$ have the same signum. As $v'(\{n-1\}) = v'(N \setminus \{n-1\}) = 0$, $s^{\boldsymbol{p}}_{n-1,n}(z', v') \geq (n-1) p^N_{\{n-1\}} \alpha$ and $s^{\boldsymbol{p}}_{n,n-1}(z', v') \geq -(n-1) p^n_{N \setminus \{n-1\}} \alpha$. By Proposition 4.3, $s^{\boldsymbol{p}}_{n-1,n}(z'_Q, w) = s^{\boldsymbol{p}}_{n-1,n}(z', v')$ and $s^{\boldsymbol{p}}_{n,n-1}(z'_Q, w) = s^{\boldsymbol{p}}_{n,n-1}(z', v')$ so that $\alpha = 0$, i.e., $z' = 0 \in \mathbb{R}^N$.

Let $T = \pi^s(S)$ and $T' = \pi'^{s-1}(S')$. Then T is the unique coalition in $\{\pi^j(S) \mid j \in N\}$ that contains n and does not contain 1, and T' is the unique coalition in $\{\pi'^j(S') \mid j \in N \setminus \{n\}\}$ with $1 \notin T' \ni n$. Moreover, $T = T'$. By the choice of t,

$$s^{\boldsymbol{p}}_{1,n}(0, v) = -p^N_S, \quad s^{\boldsymbol{p}}_{1,n}\big(0, v'\big) = -p^N_{S'}, \quad s^{\boldsymbol{p}}_{n,1}(0, v) = -p^N_T, \quad \text{and} \quad s^{\boldsymbol{p}}_{n,1}\big(0, v'\big) = -p^N_T.$$

Let $R = \{1, n\}$. By $\boldsymbol{p}$-RGP, $z_P = 0 \in \sigma(P, v^{\boldsymbol{p}}_{P,0})$ and $0 \in \sigma(P, v'^{\boldsymbol{p}}_{P,0})$ so that, by Remark 4.7, $v^{\boldsymbol{p}}_{P,0}(\{1\}) = v^{\boldsymbol{p}}_{P,0}(\{n\}) =: \beta$ and $v'^{\boldsymbol{p}}_{P,0}(\{1\}) = v'^{\boldsymbol{p}}_{P,0}(\{n\}) = \gamma$. Hence, by Proposition 4.3,

$$-p^N_S = p^P_{\{1\}}\beta, \quad -p^N_{S'} = p^P_{\{1\}}\gamma, \quad \text{and} \quad -p^N_T = p^P_{\{2\}}\beta = p^P_{\{2\}}\gamma$$

so that $\beta = \gamma$ and $p^N_S = p^N_{S'}$ and the proof of this case is complete. □

Now, the main result of this section can be proved.

Theorem 5.2 *Let $|U| = \infty$ and $\boldsymbol{p}$ be a symmetric weight system. Then the weighted prenucleolus $\mathcal{PN}^{\boldsymbol{p}}$ is the unique solution that satisfies* SIVA, SYM, COV, *and* $\boldsymbol{p}$-RGP.

Proof By Theorem 5.1 only the uniqueness part has to be verified. To this end, let σ be a solution that satisfies the desired axioms. By Remark 4.7(1), σ satisfies PO.

Let (N, v) be a game. In order to show $\sigma(N, v) = \mathcal{PN}^{\boldsymbol{p}}(N, v)$, by COV we may assume that $\nu^{\boldsymbol{p}}(N, v) = 0$. By Proposition 2.2, $0 = \nu(N, v^{\boldsymbol{p}}_0)$ (for the definition of the $\boldsymbol{p}$-weighted excess game $(N, v^{\boldsymbol{p}}_0)$ at 0 see (2.1)). In the main step of his proof of the axiomatization of the classical prenucleolus, Sobolev (see, e.g., Peleg and Sudhölter (2007, pp. 112–114) for an English version or Kleppe (2010) for an adaptation to the per capita weight system of his proof) shows that, as $|U| = \infty$, it is possible to construct a game (M, w') with the following properties:

- $N \subseteq M$ and $w'(M) = 0$.
- $w'_{N,0} = v^{\boldsymbol{p}}_0$.
- (M, w') is *transitive*, i.e., for any $k, \ell \in M$ with $k \neq \ell$ there exists a permutation π of M such that $\pi(k) = \ell$ and $\pi w' = w'$.

Now, by SYM and PO of the classical prenucleolus, $\nu(M, w') = 0 \in \mathbb{R}^M$. Let (M, w) be the game defined by $w(M) = w'(M) = 0$ and

$$w(S) = \frac{w'(S)}{p^M_S} \quad \text{for all } S \in \mathcal{F}^M.$$

As $\boldsymbol{p}$ is symmetric, the game (M, w) inherits transitivity from (M, w'). Hence, by SIVA and SYM, $\sigma(M, w)$ consists of a unique element y that satisfies $y_k = y_\ell$ for all $k, \ell \in M$. By PO, $y = 0$. By Lemma 4.5 we have

$$v^{\boldsymbol{p}}_0 = w'_{N,0} = \big(w^{\boldsymbol{p}}_0\big)_{N,0} = \big(w^{\boldsymbol{p}}_{N,0}\big)^{\boldsymbol{p}}_0,$$

which implies $v = w^{\boldsymbol{p}}_{N,0}$. Finally, by $\boldsymbol{p}$-RGP and SIVA, $\sigma(N, v) = 0 \in \mathbb{R}^N$. □

Remark 5.3 It should be noted that a weighted prenucleolus according to a symmetric weight system satisfies SYM, but it may not satisfy AN. E.g., one may partition $\mathcal{F}$ into two nonempty subsets $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ and define the weight system $\boldsymbol{p}$ by the requirement $p^N_S = 1$

for all $N \in \mathcal{F}_1$, $S \in \mathcal{F}^N$, and $p_{S'}^{N'} = \frac{1}{|S'|}$ for all $N' \in \mathcal{F}_2$, $S' \in \mathcal{F}^{N'}$. This weight system is symmetric but violates AN and its weighted prenucleolus is the classical prenucleolus when applied to a game (N, v) with $N \in \mathcal{F}_1$, and it is the per capita prenucleolus when applied to a game (N', v') with $N' \in \mathcal{F}_2$.

5.2 Symmetrically weighted nucleoli

We now consider the class Γ_I. As in the classical case, the weighted nucleolus according to any weight system $\boldsymbol{p}$ does not satisfy $\boldsymbol{p}$-RGP. Indeed, consider the 3-person game (N, v) defined by $v(\{i\}) = 0$ for all $i \in N$ and $v(S) = 1$ for all other coalitions and let $x = \nu_I^{\boldsymbol{p}}(N, v)$. Then there exists $T \subseteq N$ with $|T| = 2$ and $e(T, x, v) > 0$. Let $k \in T$ and $\ell \in N \setminus T$ and define $S = \{k, \ell\}$. Then $v_{S,x}^{\boldsymbol{p}}(\{k\}) > x_k$ so that x_S is not individually rational for this reduced game. Hence, we modify the imputation saving reduced game property introduced by Snijders (1995).

Definition 5.4 Let (N, v) be a game, $S \in \mathcal{F}^N$, and $x \in \mathbb{R}^N$. The *imputation saving* $\boldsymbol{p}$-*reduced game* $(S, \widetilde{v}_{S,x}^{\boldsymbol{p}})$ is defined by the following requirement: If $|S| = 1$, then $\widetilde{v}_{S,x}^{\boldsymbol{p}} = v_{S,x}^{\boldsymbol{p}}$, and if $|S| > 1$, then

$$\widetilde{v}_{S,x}^{\boldsymbol{p}}(T) = \begin{cases} v_{S,x}^{\boldsymbol{p}}(T), & \text{if } T \subseteq S, |T| > 1, \\ \min\{x_i, v_{S,x}^{\boldsymbol{p}}(\{i\})\}, & \text{if } T = \{i\}, i \in S. \end{cases} \tag{5.1}$$

Now the *imputation saving reduced game property w.r.t. imputation saving* $\boldsymbol{p}$-*reduced games* ($\boldsymbol{p}$-ISRGP) arises from $\boldsymbol{p}$-RGP by replacing $v_{S,x}^{\boldsymbol{p}}$ by $\widetilde{v}_{S,x}^{\boldsymbol{p}}$ wherever it occurs. Modifying (4.3) suitably yields, for any weight system $\boldsymbol{p}$, any game (N, v) with $|N| > 1$, for all $S \in \mathcal{F}^N$ and all $x \in \mathbb{R}^N$, that

$$p_T^S e\big(T, x, \widetilde{v}_{S,x}^{\boldsymbol{p}}\big) = \max_{Q \subseteq N \setminus S} p_{T \cup Q}^N e(T \cup Q, x, v) \quad \text{for all } T \in \mathcal{F}^S, |T| \geq 2, \quad \text{and} \tag{5.2}$$

$$p_{\{i\}}^S e\big(\{i\}, x, \widetilde{v}_{S,x}^{\boldsymbol{p}}\big) = \min\Big\{0, \max_{Q \subseteq N \setminus S} p_{\{i\} \cup Q}^N e\big(\{i\} \cup Q, x, v\big)\Big\} \quad \text{for all } i \in S. \tag{5.3}$$

In the proofs of Lemma 4.8 and of Theorem 5.1 only games in Γ_I are used and the proofs remain valid if the employed $\boldsymbol{p}$-reduced games are replaced by their imputation saving versions. Hence, these results remain valid if we consider a solution on Γ_I and replace $\boldsymbol{p}$-RGP by $\boldsymbol{p}$-ISRGP.

Theorem 5.5 *Let $|U| \geq 3$ and $\boldsymbol{p}$ be a weight system. Then there exists a solution that satisfies* SIVA, SYM, COV, *and* $\boldsymbol{p}$-ISRGP *if and only if $\boldsymbol{p}$ is symmetric.*

A careful inspection of Snijders' (1995) proof in the classical case (cf. Peleg and Sudhölter (2007, Sect. 6.3.1)) and an application of Proposition 2.2(2), Lemma 4.8, and (5.2) and (5.3) yields a proof of the following theorem.

Theorem 5.6 *Let $|U| = \infty$ and $\boldsymbol{p}$ be a symmetric weight system. Then the weighted nucleolus $\mathcal{N}^{\boldsymbol{p}}$ is the unique solution on Γ_I that satisfies* SIVA, SYM, COV, *and* $\boldsymbol{p}$-ISRGP.

5.3 Logical independence of the axioms

Examples of solutions are presented that exclusively violate one of the axioms in Theorems 5.2 or 5.6, respectively.

It is well-known that the Shapley value (Shapley 1953) satisfies SIVA, COV, and SYM on any class of games. It violates, however, $\boldsymbol{p}$-RGP and $\boldsymbol{p}$-ISRGP on Γ_I, provided $|U| \geq 3$. Indeed, there are 3-person games whose core is nonempty and whose Shapley value does not belong to the core, while weighted (pre)nucleoli always belong to a nonempty core.

If $|U| \geq 4$, according to Sect. 6 (see Example 6.2), $\mathcal{PK}^{\boldsymbol{p}}$ or $\mathcal{K}^{\boldsymbol{p}}$ is a nonempty solution that satisfies all axioms of Theorem 5.2 or Theorem 5.6, respectively, with the exception of SIVA.

The "equal split solution" exclusively violates COV in Theorem 5.2 provided $|U| \geq 2$.

The following modification of the equal split solution exclusively violates COV in Theorem 5.6. The modified solution assigns $\max\{\lambda, v(\{i\})\}$ to each player $i \in N$ of game $(N, v) \in \Gamma_I$ where λ is determined by Pareto optimality. As its definition is similar to the definition of the "constrained equal award solution" for bankruptcy problems (Aumann and Maschler 1985), we could call it the "constrained equal split solution".

In order to show that SYM is logically independent of the remaining axioms in Sobolev's axiomatization, first an auxiliary solution, the "positive core", is defined (Orshan and Sudhölter 2010). A preimputation x of a game (N, v) belongs to its *positive core* if it lexicographically minimizes the positive parts of the excesses, i.e., if the excess of a coalition at x coincides with the excess of the coalition at the prenucleolus if the former excess is positive. Hence, the positive core is a convex polytope, and it is easily seen that the positive core satisfies COV, SYM, and RGP. Fixing some total order on U and selecting the lexicographic smallest element of the positive core results in a solution that exclusively violates SYM (cf. Peleg and Sudhölter (2007, Sect. 6.3.2)). One may similarly define the "weighted positive core according to $\boldsymbol{p}$" of a game (N, v) by the requirement that a preimputation x belongs to this solution if the weighted excesses at x coincide with the weighted excesses at $\nu^{\boldsymbol{p}}(N, v)$ if positive. In order to show that SYM is logically independent of the remaining axioms in Theorem 5.6 one may further modify the weighted positive core by requiring that an imputation x belongs to the modified auxiliary solution if the excess of a coalition at x coincides with the excess of this coalition at the nucleolus point, if it is positive.

It should be mentioned that the infinity assumption on the cardinality of U cannot be deleted in either Theorem 5.2 or Theorem 5.6. Indeed, if $4 \leq |U| < \infty$, then together with a suitable modification of the game given in Exercise 6.3.2, suitable modifications of the solution defined in Remark 6.3.3 of Peleg and Sudhölter (2007) satisfy all properties of Theorem 5.2 or Theorem 5.6, respectively.

Finally, it is remarked that whether Orshan's (1993) result that ETP replaces SYM in Sobolev's axiomatization is still valid for, e.g., Theorem 5.2, is an open question.

6 Axiomatization of the symmetrically weighted prekernel

We show that Peleg's (1986) axiomatization of the prekernel may be generalized to the weighted prekernel according to any symmetric weight system.

We first define the weighted version of Peleg's "converse reduced game property". Let $\boldsymbol{p}$ be a weight system. A solution σ satisfies the *converse reduced game property* w.r.t. $\boldsymbol{p}$-reduced games ($\boldsymbol{p}$-CRGP) if the following property holds for all games (N, v) with $|N| \geq 2$ and all $x \in X(N, v)$: If $x_S \in \sigma(S, v^{\boldsymbol{p}}_{S,x})$ for all $S \subseteq N$ with $|S| = 2$, then $x \in \sigma(N, v)$.

Theorem 6.1 *Let* $\boldsymbol{p}$ *be a weight system and* $|U| \geq 3$. *Then the following statements are valid.*

(1) *There exists a solution that satisfies* NE, PO, COV, ETP, $\boldsymbol{p}$-RGP, *and* $\boldsymbol{p}$-CRGP *if and only if* $\boldsymbol{p}$ *is symmetric.*
(2) *If* $\boldsymbol{p}$ *is symmetric, then the weighted prekernel* $\mathcal{PK}^{\boldsymbol{p}}$ *is the unique solution that satisfies* NE, PO, COV, ETP, $\boldsymbol{p}$-RGP, *and* $\boldsymbol{p}$-CRGP.

Proof If $\boldsymbol{p}$ is symmetric, then, by Remark 2.4, the definition of $\mathcal{PK}^{\boldsymbol{p}}$, and Corollary 4.4, $\mathcal{PK}^{\boldsymbol{p}}$ satisfies the first five desired properties. By Proposition 4.3, $s^{\boldsymbol{p}}_{k\ell}(x, v) = s^{\boldsymbol{p}}_{k\ell}(x_{\{k,\ell\}}, v^{\boldsymbol{p}}_{\{k,\ell\},x})$ for any game (N, v) and all $k, \ell \in N$ with $k \neq \ell$. Hence, $\mathcal{PK}^{\boldsymbol{p}}$ satisfies $\boldsymbol{p}$-CRGP.

In order to show the remaining only-if part of (1) and the uniqueness part of (2), let σ be a solution that satisfies the desired properties. We may copy Peleg's proof for the classical prekernel, just the reduced game has to be replaced by the $\boldsymbol{p}$-reduced game wherever it occurs: Let (N, v) be a game. The properties $\boldsymbol{p}$-RGP and $\boldsymbol{p}$-CRGP cannot be distinguished from RGP and CRGP if $|U| \leq 2$ so that $\sigma(N, v) = \mathcal{PK}(N, v)$ if $|N| \leq 2$. By Remark 4.7(2), $\mathcal{PK}(N, v) = \mathcal{PK}^{\boldsymbol{p}}(N, v)$ for $|N| \leq 2$. If $|N| \geq 3$ and $x \in \sigma(N, v)$, then by $\boldsymbol{p}$-RGP of σ, $x_S \in \sigma(S, v^{\boldsymbol{p}}_{S,x}) = \mathcal{PK}^{\boldsymbol{p}}(S, v^{\boldsymbol{p}}_{S,x})$ for all $S \subseteq N$ with $|S| = 2$. By $\boldsymbol{p}$-CRGP of $\mathcal{PK}^{\boldsymbol{p}}$, $x \in \mathcal{PK}^{\boldsymbol{p}}(N, v)$. The opposite inclusion may be proved by interchanging the roles of σ and $\mathcal{PK}^{\boldsymbol{p}}$. Now, by Theorem 3.3, $\boldsymbol{p}$ is symmetric. □

Suitable modifications of the examples that show that each of the six axioms in Peleg's (1986) axiomatization of the prekernel is logically independent of the remaining axioms, provided that $|U| \geq 4$, may be used to show that the axioms employed in Theorem 6.1(2) are logically independent as well.

As in the classical case, it is straightforward to apply the characterization of the $\boldsymbol{p}$-weighted prenucleolus of a game by balanced collections of coalitions (Kohlberg 1971) in order to show that $\mathcal{PK}^{\boldsymbol{p}}(N, v)$ is a singleton for all 3-person games (N, v). Hence, if $|U| = 3$, $\boldsymbol{p}$-CRGP is implied by the remaining five properties of Theorem 6.1(2).

For completeness we present an example of a 4-person game of which none of the weighted (pre)kernels is a singleton.

Example 6.2 Let $N = \{1, 2, 3, 4\}$ and (N, v) be a game such that $v(\{1, 2\}) = v(\{2, 3\}) = v(\{3, 4\}) = v(\{1, 4\}) = 1$, $v(N) = 0$, $v(\{i\}) = -1$ for all $i \in N$ and $v(S) \leq -2$ for all other coalitions S of N. Then $(t, -t, t, -t) \in \mathcal{K}^{\boldsymbol{p}}(N, v) \cap \mathcal{PK}^{\boldsymbol{p}}(N, v)$ for all $-1 \leq t \leq 1$ and all weight systems $\boldsymbol{p}$ that are symmetric.

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