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Decision Support

Characterizations of highway toll pricing methods[☆]

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ABSTRACT

A highway problem is a cost sharing problem that arises if the common resource is an ordered set of sections with fixed costs such that each agent demands consecutive sections. We provide axiomatizations of the core, the prenucleolus, and the Shapley value on the class of TU games associated with highway problems. However, the simple and intuitive properties employed in the results are exclusively formulated by referring to highway problems rather than games. The main axioms for the core and the nucleolus are consistency properties, while the Shapley value is characterized by requiring that the fee of an agent is determined by the highway problem when truncated to the sections she demands. An alternative characterization is based on the new contraction property. Finally it is shown that the games that are associated with generalized highway problems in which agents may demand non-connected parts are the positive cost games, i.e., nonnegative linear combinations of dual unanimity games.

1. Introduction

In this paper we analyze a particular kind of cost allocation problem in which some agents jointly produce and finance a common resource or facility. The peculiarity is that this resource can be separated into a number of ordered sections. Moreover, each agent requires some consecutive sections, and each section has a fixed cost. The issue of our present study is how to share the total cost of all sections among the users in an efficient and fair way. A simple example that illustrates this situation is a linear highway, where the sections are delimited by the entry and exit points, and each car only needs the highway sections between its entry and exit point.[1] This example motivates why these problems were called highway problems when they were introduced by Kuipers, Mosquera, and Zarzuelo (2013). The well-known airport problems (Littlechild & Thompson, 1977) can be considered as special highway problems, in which all agents' entries coincide. Çiftçy, Borm, and Hamers (2010) extend the class of highway problems to situations in which the sections are partially ordered. Highway problems form also a special subclass of realization problems introduced by Koster, Reijnierse, and Voorneveld (2003). Dong, Guo, and Wang (2012) study a situation where the cost of each section may depend on the number of cars using it.

The theory of cooperative games has proved to be very useful for solving cost allocation problems by first associating with each considered cost allocation problem the cooperative transferable utility (TU) game that assigns to every coalition the cost that accommodates all members of the coalition, and secondly applying some solution concept to the associated game. The cooperative game associated with a highway problem is, hence, called highway game. Here we will focus on three of the most important solution concepts, namely the core, the nucleolus, and the Shapley value, restricted to the class of highway games.

Our main goal is to provide characterizations of these three solutions by simple and intuitive properties that embody fairness criteria. It should be pointed out that the employed axioms are formulated exclusively with the help of the underlying highway problems, without any reference to the associated games. Nevertheless, they are inspired by properties that have been used in some traditional characterizations of the mentioned solutions on

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[1] This example is a simplification of a highway problem because there are other issues of primary importance, as congestion. In our context these issues will not appear explicitly, but at least some of them might be taken into account implicitly by the cost of each section.

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several classes of games. Only the *contraction property* introduced in Section 4 is, as far as we know, entirely new.

More specifically, the axiomatizations of the core and the nucleolus of highway problems are based on the consistency principle. According to this principle, if a group of agents pays its share and leaves the others in a renegotiation, then the shares of the remaining agents may remain unchanged in the subsequent reduced situation. On several classes of games consistency properties have proved to be very powerful in characterizing the prenucleolus (Sobolev, 1975), the core (Hwang & Sudhölter, 2001; Peleg, 1985; 1986; Tadenuma, 1992), the Shapley value of TU and NTU games (Hart & Mas-Colell, 1989), and the Harsanyi NTU value (Hinojosa, Romero, & Zarzuelo, 2012). An implementation of the Shapley value for airport problems (Albizuri, Echarri, & Zarzuelo, 2015) is also based on a consistency property. In general, a crucial issue is to identify the available alternatives for intermediate coalitions in a reduced situation. As a consequence, different kinds of reduced problems have been proposed in the literature. In the case of a highway problem, we require the consistency property only for an agent i, whose needs are minimal in the sense that they do not cover those of any other agent (up to an exception the explanation of which is postponed to Section 3). We assume that i's intention is to pay only for the part she uses, so it seems natural that her payment before leaving should be subtracted from the cost of her sections. So every agent sharing some of the segments used by i might benefit from this reduction, but it cannot harm her.

On the other hand the axiomatization of the Shapley value of a highway problem is based on principle stating that the share payed by agent i may only depend on the highway problem when restricted to the sections of the highway used by that agent. This property is related to a monotonicity property employed in a major characterization of the Shapley value by Young (1985).

The paper is organized as follows. In Section 2, we introduce two representations of highway problems and their corresponding highway games. Section 3 first presents the axioms employed in the subsequent characterizations of the core and the (pre)nucleolus.

The *reduced highway problem* is defined in such a way that its associated game coincides with the Davis–Maschler reduced game of the original highway game. However, neither the definition of the reduced highway problem nor of the corresponding *reduced highway problem property* (RHP) and its converse (CRHP) refer to the associated highway games. Secondly, resembling a result of Peleg (1986), we show that the core is the unique solution for highway problems that satisfies *individual rationality*, *unanimity for 2-person highway problems* (UTPH), RHP, and CRHP. Moreover, if a stronger version of CRHP is employed, then UTPH may be replaced by *non-emptiness*. Finally, we prove that the (pre)nucleolus is characterized by *single-valuedness*, the *equal treatment property*, *covariance under exclusive prolongations*, and RHP. In Section 4 we offer two characterizations of the Shapley value: (a) with the help of *individual independence of outside changes* and, alternatively, (b) with the help of the *contraction property* which is some kind of consistency property. As far as we know, this or a similar property has not been used or discussed in the literature before. In Section 5 we show that a generalized highway game in which the sections used by an agent may not be connected is a positive cost game, i.e., a nonnegative linear combination of dual unanimity games, and vice versa. Finally, Section 6 closes the main part of the paper by showing the logical independence of the axioms employed in the aforementioned characterizations and providing some remarks. The proofs of the main results are found in two appendices at the end of the paper.

2. Preliminaries

Let U be a set called the *universe of agents*.[2] A finite nonempty subset of U is called a *coalition*.

Definition 2.1. A *highway problem* is a pair (N, I) such that:

(1) N is a coalition.
(2) I is a mapping that assigns to each $i \in N$ a compact nonempty interval $I_i \subseteq \mathbb{R}_+$.
(3) $I_N = [0, b]$ for some $b \in \mathbb{R}_+$, where $I_S = \bigcup_{i \in S} I_i$ for every $S \subseteq N$.

Denote by $\mathcal{H}$ the set of highway problems. For the generic element $(N, I) \in \mathcal{H}$, we typically write $I_i = [a_i, b_i]$. As $I_i \neq \emptyset$, we have $a_i \leq b_i$.

The interpretation is as follows. The elements in N represent the agents involved in the problem. For each $i \in N$, the interval I_i represents the (connected) parts of the common facility that is used by agent i. This common facility is symbolized by I_N. Condition (3) says that the first part starts at 0, the last one finishes at certain real number b, and there are no gaps between the parts used by agents in N. The length of an interval represents its cost. Thus the cost of serving agent i is $b_i - a_i$, and accordingly the total cost of the common facility is b, that is the amount to be shared between all the agents.

Given a highway problem $(N, I) \in \mathcal{H}$, denote the set of *feasible cost allocations* and the set of *efficient feasible cost allocations (preimputations)* by $X^*(N, I)$ and $X(N, I)$ respectively, i.e.,

$$X^*(N, I) = \left\{x \in \mathbb{R}^N \mid x(N) \geqslant b\right\} \quad \text{and}$$
$$X(N, I) = \left\{x \in \mathbb{R}^N \mid x(N) = b\right\},$$

where $x(S) = \sum_{i \in S} x_i$ for all $S \subseteq N$ and $x \in \mathbb{R}^N$.

A *solution* σ assigns to each highway problem (N, I) a subset of $X^*(N, I)$. Its restriction to a set $\mathcal{H}' \subseteq \mathcal{H}$ is again denoted by σ. A solution on $\mathcal{H}'$ is the restriction to $\mathcal{H}'$ of a solution.

Thus solution concepts associate payoff vectors with highway problems, i.e. vectors x with components indexed by the members of N. Sometimes a solution concept selects several payoff vectors or none at all. Each solution concept usually represents a specific notion of stability, expected outcome, or the like.

It will be recalled that a cost TU game is a pair (N, c), such that N is a coalition and c is a function that associates a real number $c(S)$ with each subset S of N (by convention $c(\emptyset) = 0$.) The (cost) TU game associated with the highway problem (N, I) is the game (N, c^I) defined by

$$c^I(S) = \lambda^*(I_S) \text{ for all } S \subseteq N, \tag{2.1}$$

where λ^* denotes the Lebesgue measure on $\mathbb{R}$. That is, the real number $c^I(S)$ is the cost of serving the agents in S.

A TU game is a *highway game* if it is the TU game associated with a highway problem. Note that highway games are concave; that is $c^I(S) + c^I(T) \geqslant c^I(S \cap T) + c^I(S \cup T)$ for all $S, T \subseteq N$.

Taking the TU games (N, I) permit us the access to game theory concepts to address the problem of finding solutions for highway problems. We focus here on the core, the (pre)nucleolus, and the Shapley value. Thus we shall refer to the core, the (pre)nucleolus and the Shapley value of a highway problem as the respective solution concept of the associated cost TU game (N, c^I).

Remark 2.2. Kuipers et al. (2013) define a "highway problem" to be a quadruple (N, M, C, T) that satisfies the following properties: N is a coalition, M is a finite nonempty set with a strict total order $\prec$ (the set of sections), $C : M \to \mathbb{R}_+$ is a mapping that represents

[2] $|U| \geq 5$ is needed in Example 6.1, and we always assume that $\{1, \ldots, \ell\} \subseteq U$ if $|U| \geq \ell$.

the cost of each section, and T: $N \to 2^M\backslash\{\emptyset\}$ is a mapping, where $T(i)$ represents the set of sections used by agent i, satisfying for all $i \in N$

$$s, s' \in T(i), s'' \in M, \quad \text{and} \quad s \prec s'' \prec s', \quad \text{then} \quad s'' \in T(i). \tag{2.2}$$

The cost TU game $c^{M,C,T}$ of the 'highway problem' (N, M, C, T) is defined by

$$c^{M,C,T}(S) = \sum_{t \in T(S)} C(t) \quad \text{for all} \quad S \subseteq N, \tag{2.3}$$

where $T(S) = \bigcup_{i \in S} T(i)$.

Although "highway problems" in the sense of Kuipers et al. (2013) are formally different from those in Definition 2.1, they are conceptually equivalent. Indeed, given a highway problem $(N, I) \in \mathcal{H}$, it can be associated a 'highway problem' (N, M, C, T) as follows. Let $(\beta_0, \ldots, \beta_m)$ be a real sequence of minimal length that satisfies the following properties:

(i) $0 = \beta_0 \leqslant \cdots \leqslant \beta_m$.
(ii) For each $i \in N$ there exist r, r', $0 \le r < r' \le m$, such that $I_i = [\beta_r, \beta_{r'}]$.

Note that m and $(\beta_0, \ldots, \beta_m)$ are uniquely determined by the foregoing properties and minimality. Define

$$M^I = \{[\beta_r, \beta_{r+1}] \mid r = 0, \ldots, m\} \tag{2.4}$$

as the set of *sections* of (N, I). Note also that, if $\beta_r = \beta_{r+1}$ (i.e., if the $r+1$th section is a singleton), then there exists $i \in N$ with $I_i = [\beta_r, \beta_{r+1}]$, and if $r+1 < m$ in addition, then $\beta_{r+2} > \beta_{r+1}$. Finally note that M^I is totally ordered by $[0, \beta_1] \prec \cdots \prec [\beta_{m-1}, \beta_m]$, where $\prec = \prec^I$ is the strict total order relation. Also define

$$C^I\big([\beta_r, \beta_{r+1}]\big) = \beta_{r+1} - \beta_r, \tag{2.5}$$

$$T^I(i) = \{I \in M^I \mid I \subset I_i\}. \tag{2.6}$$

Then it is straightforward to show that (N, M^I, C^I, T^I) is a "highway problem" as defined above. Moreover $c^I = c^{M^I, C^I, T^I}$, and hence, we may say that a highway problem may be represented by (N, I) as well as by (N, M^I, C^I, T^I).

In what follows we shall use the representation (N, I) except in Section 5 where the representation (N, M, C, T) is more convenient.

3. Characterizations of the core and the nucleolus of highway problems

Next we introduce the axioms employed in the subsequent characterizations of the core and the nucleolus of highway problems defined in Sections 3.1 and 3.2, respectively.

A solution σ on $\mathcal{H}' \subseteq \mathcal{H}$ satisfies

(1) *non-emptiness* (NEM) if for all $(N, I) \in \mathcal{H}'$: $\sigma(N, I) \neq \emptyset$;
(2) *Pareto optimality* (PO) if for all $(N, I) \in \mathcal{H}'$: $\sigma(N, I) \subseteq X(N, I)$.
(3) *single-valuedness* (SIVA) if for all $(N, I) \in \mathcal{H}'$: $|\sigma(N, I)| = 1$;
(4) the *equal treatment property* (ETP) if for all $(N, I) \in \mathcal{H}'$, all i, $j \in N$, and all $x \in \sigma(N, I)$: $I_i = I_j$ implies $x_i = x_j$;
(5) *individual rationality* (IR) if for all $(N, I) \in \mathcal{H}'$, all $i \in N$, and all $x \in \sigma(N, I)$: $x_i \le \lambda^*(I_i)$;
(6) *reasonableness from below* (REASB) if for all $(N, I) \in \mathcal{H}'$, all $x \in \sigma(N, I)$, and all $i \in N$: $x_i \ge \lambda^*(I_i \backslash I_{N\backslash\{i\}})$;
(7) *covariance under exclusive prolongations* (PCOV) if for all $(N, I), (N, I') \in \mathcal{H}'$: If $i \in N$ and $t \ge 0$ such that $I_i = [a_i, b_i]$, $I'_i = [a_i, b_i + t]$, and
  (a) $b_i = \max I_N$ and $I'_j = I_j$ for all $j \in N\backslash\{i\}$, or
  (b) $a_i = 0$ and $I'_j = I_j + t$ for all $j \in N\backslash\{i\}$,
  then $\sigma(N, I') = \sigma(N, I) + x$, where $x_j = \begin{cases} t, & \text{if } j = i, \\ 0, & \text{if } j \in N \setminus \{i\}. \end{cases}$
(8) *unanimity of two-person highway problems* (UTPH) if, for any $(N, I) \in \mathcal{H}'$ with $|N| = 2$: $\sigma(N, I) = \big\{x \in X(N, I) \mid x_i \leqslant c^I(\{i\})$ for all $i \in N\big\}$.

NEM, PO, SIVA, ETP, IR, and REASB are standard in the literature and do not need further explanation. The interpretation of PCOV is simple: If an agent who is already using the last section and asks for prolonging the highway "to the right" just for herself or if she is already using the first section and asks for prolonging the highway "to the left" just for herself, then the cost of this modification is added to her payment, whereas the charges of the remaining agents are not changed. Finally, UTPH is a restatement of Peleg' (1989) notion (see also Sudhölter & Peleg, 2002) of 'unanimity for two-person games', requiring that in the particular case of two-person problems the solution coincides with the set of efficient and individually rational payoff vectors.

Now we address the consistency principle mentioned in the introduction. Assume that the agents in a highway problem have reached a final agreement. The consistency principle requires that if the agents of a subgroup renegotiate their shares under the assumption that the other agents have already paid their shares, then the agreement in the new reduced situation will not differ from the original one. To state this principle formally we shall define the concept of reduced highway problem. Prior we need some definitions.

Let $(N, I) \in \mathcal{H}$, $I_j = [a_j, b_j]$ for all $j \in N$, and i, $k \in N$. Say that I_k *starts* (respectively *ends*) in the interval I_i if $a_i < a_k < b_i$ (respectively $a_i < b_k < b_i$). Now, agent k is an *interior* agent of i if I_k starts and ends in I_i. Say that i is of *type p* if i has p interior agents, i.e., $\big|\{j \in N \mid a_i < a_j, b_j < b_i\}\big| = p$. Moreover, we say that i is a *left* (resp. *right*) agent if, for all $j \in N$ such that I_j starts (resp. ends) in I_i, j is an interior agent of i. Finally, agent i is *oriented* if i is a left or a right agent.

We use the term *left agent* because an agent i is a left agent if there is no other agent j who uses an interval that simultaneously starts in I_i and ends on the right hand side of I_i or exactly at b_i (i.e., $b_j \ge b_i$). The term *right agent* may be motivated similarly.

Definition 3.1. Let $(N, I) \in \mathcal{H}$ with $I_j = [a_j, b_j]$ for all $j \in N$ such that $|N| \ge 2$. An agent $i \in N$ is called *reducible* if i is of type 0 or if i is an oriented agent of type 1.

Let $i \in N$ be reducible, $|N| > 1$, and $x \in \mathbb{R}^N$. The *reduced problem* $(N \setminus \{i\}, I^{-i,x})$ with respect to (w.r.t.) x and when removing i is defined as follows, where $I_j^{-i,x} = [a'_j, b'_j]$ for all $j \in N\backslash\{i\}$:

If $j \in N\backslash\{i\}$ is not an interior agent of i, then

$$a'_j = \begin{cases} a_j, & \text{if } a_j \leqslant a_i, \\ \max\{a_i, a_j - x_i\}, & \text{if } a_i < a_j, \end{cases} \tag{3.1}$$

$$b'_j = \begin{cases} \min\{b_j, b_i - x_i\}, & \text{if } b_j < b_i, \\ b_j - x_i, & \text{if } b_j \geqslant b_i. \end{cases} \tag{3.2}$$

If $j = k$ is an interior agent of i, i.e., i is of type 1 and k is the unique interior agent of i, then the definition differs from (3.1) and (3.2) only inasmuch as

$$a'_k = \max\big\{a_i, \min\{a_k, b_i - x_i - b_k + a_k\}\big\}, \quad \text{if } i \text{ is a left agent, and} \tag{3.3}$$

$$b'_k = \min\big\{b_i - x_i, \max\{b_k - x_i, b_k - a_k + a_i\}\big\}, \quad \text{if } i \text{ is a right but not left agent.} \tag{3.4}$$

According to Definition 3.1, in the reduced situation every agent $j \neq i$ considers that the charge x_i of the leaving agent i has been used to pay some suitable parts of I_i, but in such a way that agent i finances firstly the parts that are disjoint from I_j. This is illustrated in the following examples that are rendered in Fig. 1 (we have not

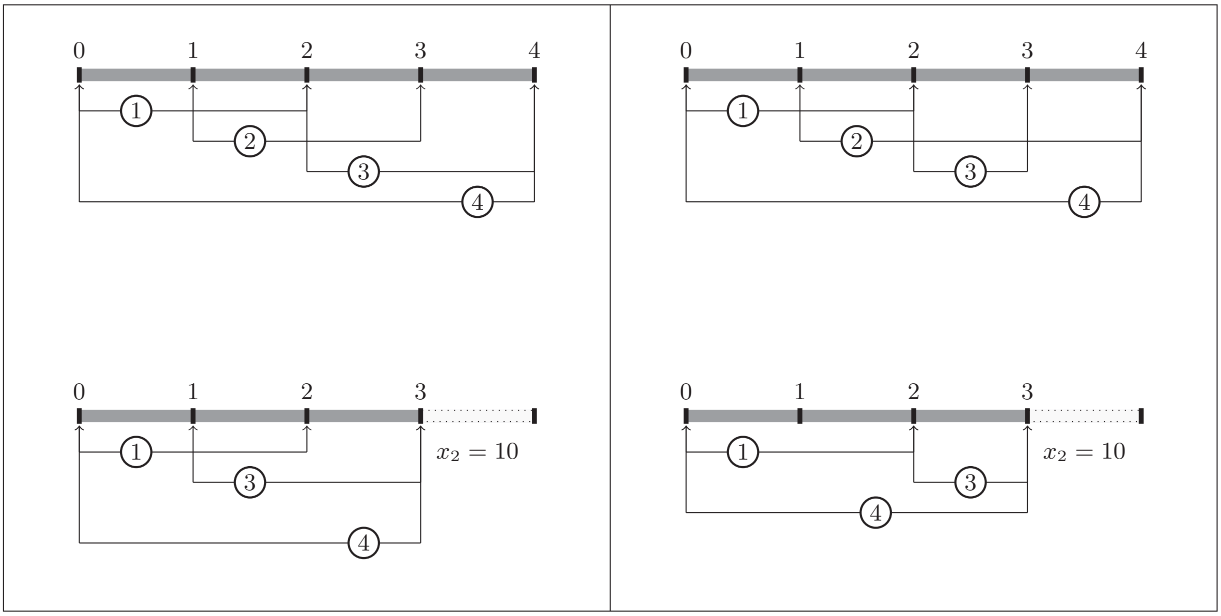

Fig. 1. Examples of reduction when removing agent 2 when she is an agent of type 0 (left panel) and a left-oriented agent of type 1 (3 is interior of 2) (right panel).

set down the case of right but not left agents, because it mirrors the one for left agents).

Example 3.2. (1) Let (N, I) be the highway problem with $N = \{1, \ldots, 4\} \subseteq U$, such that $I_1 = [0, 2]$, $I_2 = [1, 3]$, $I_3 = [2, 4]$, and $I_4 = [0, 4]$. Observe that 2 is an agent of type 0.

Now let $x = (1, 1, 1, 1)$ and consider the reduced problem $\left(\{1, 3, 4\}, I^{-2,x}\right)$. Applying the definition we get $I_1^{-2,x} = [0, 2]$, $I_3^{-2,x} = [1, 3]$, and $I_4^{-2,x} = [0, 3]$. Hence, the reduced highway problem assigns to each remaining player j an interval $I_j^{-2,x}$ whose cost, i.e., length, arises from the cost of the original interval I_j when the share x_2 of the leaving agent 2 is used to first pay for the part of 2's highway that is disjoint from I_j. Only if something of x_2 is left, i.e., if the difference of x_2 and the length of $I_2 \cap I_j$ is positive, then the length of I_j is reduced by this positive number which is exclusively the case for $j = 4$ in the current example. For a sketch of (N, I) and $(N \setminus \{2\}, I^{-2,x})$ see the left part of Fig. 1.

(2) Consider the slightly modified highway problem (N, J) given by $J_1 = [0, 2]$, $J_2 = [1, 4]$, $J_3 = [2, 3]$, and $J_4 = [0, 4]$. In this case agent 2 is a left-oriented agent of type 1, and player 3 is interior of agent 2.

Using the definition again we get $J_1^{-2,x} = [0, 2]$, $J_3^{-2,x} = [2, 3]$, and $J_4^{-2,x} = [0, 3]$.

This situation is graphed in the left part of Fig. 1. As in (1), in the reduced situation the charge of agent 2 fully subsidizes the part of the highway used by agent 4 because $I_2 \subseteq I_4$, and does not subsidize the parts of the highway used by agents 1 and 3 because the length of $I_2 \setminus I_1$ and of $I_2 \setminus I_3$ is not smaller than the charge x_2.

Note that reduced highway problems are not necessarily highway problems. Indeed, if x_i in Definition 3.1 is small enough, then the "highway" may receive a gap, i.e., $I_{N\setminus\{i\}}^{-i,x}$ may be not an interval; and if x_i is large enough, then some "intervals" may have a negative length, i.e., are empty because $a'_j > b'_j$ may occur (see Example 3.11.) However, according to Proposition 3.3 below, if reducible agents are removed w.r.t. to individually rational payoffs that in addition are reasonable from below, then reducing yields highway problems indeed. This is the rationale behind the definition of the following consistency properties.

A solution σ on $\mathcal{H}' \subseteq \mathcal{H}$ satisfies

(9) the *reduced highway problem property* (RHP) if for any $(N, I) \in \mathcal{H}'$ with $|N| > 1$, any reducible agent $i \in N$, and any $x \in \sigma(N, I)$: $(N \setminus \{i\}, I^{-i,x}) \in \mathcal{H}'$ and $x_{N\setminus\{i\}} \in \sigma(N \setminus \{i\}, I^{-i,x})$;

(10) the *converse reduced highway problem property* (CRHP) if for any $(N, I) \in \mathcal{H}'$ with $|N| \geq 3$ and any $x \in X(N, I)$ the following condition holds: If, for each reducible agent $i \in N$, $(N \setminus \{i\}, I^{-i,x}) \in \mathcal{H}'$ and $x_{N\setminus\{i\}} \in \sigma(N \setminus \{i\}, I^{-i,x})$, then $x \in \sigma(N, I)$;

(11) the *strong converse reduced highway problem property* (SCRHP) if for any $(N, I) \in \mathcal{H}'$ with $|N| \geq 2$, and any $x \in X(N, I)$ the following condition holds: If, for each reducible agent $i \in N$, $(N \setminus \{i\}, I^{-i,x}) \in \mathcal{H}'$ and $x_{N\setminus\{i\}} \in \sigma(N \setminus \{i\}, I^{-i,x})$, then $x \in \sigma(N, I)$.

Next we recall the definition of the Davis–Maschler reduced game. Let (N, c) be a cost TU game, $x \in X(N, c)$, and $\emptyset \neq S \subsetneq N$. The *reduced game* w.r.t. S and x is the TU game $(S, c_{S,x})$ defined by

$$c_{S,x}(T) = \begin{cases} c(N) - x(N \setminus S), & \text{if } T = S, \\ \min_{P \subseteq N\setminus S}\left(c(T \cup P) - x(P)\right), & \text{if } \emptyset \neq T \subsetneq S. \end{cases}$$

Proposition 3.3. *Let $(N, I) \in \mathcal{H}$ with $|N| \geq 2$, $i \in N$ be a reducible agent, and $x \in X(N, I)$. If $\lambda^*(I_i \setminus I_{N\setminus\{i\}}) \leq x_i \leq \lambda^*(I_i)$ (i.e., x_i is reasonable from below and individually rational for i), then $(N \setminus \{i\}, I^{-i,x}) \in \mathcal{H}$, $x_{N\setminus\{i\}} \in X(N \setminus \{i\}, I^{-i,x})$, and $c^{I^{-i,x}} = (c^I)_{N\setminus\{i\},x}$.*

With the help of Proposition 3.3, proved in Appendix A , we show now that the definitions by Peleg (1986) of the reduced game property and its converse are parallel to the definitions of our RHP and CRHP axioms. Recall that a solution σ on a set Γ of (cost) TU games satisfies

(8′) the *reduced game property* (RGP) if for all $(N, c) \in \Gamma$, $\emptyset \neq S \subsetneq N$, and $x \in \sigma(N, c)$: $(S, c_{S,x}) \in \Gamma$ and $x_S \in \sigma(S, c_{S,x})$;

(9′) the *converse reduced game property* (CRGP) if the following condition is satisfied for $(N, c) \in \Gamma$ with $|N| \geq 3$, $x \in X(N, c) = \{x \in \mathbb{R}^N \mid x(N) = c(N)\}$: If, for any $S \subseteq N$ with $|S| = 2$, $(S, c_{S,x}) \in \Gamma$ and $x_S \in \sigma(S, c_{S,x})$, then $x \in \sigma(N, c)$.

If we add the prerequisite "$S = N \setminus \{i\}$ for some $i \in N$" in RGP, then we receive an equivalent property because reducing is transitive in the sense that $c_{S,x} = (c_{T,x})_{S,x_T}$ for all $\emptyset \neq S \subseteq N$ and

$x \in X(N, c)$. Hence, in view of Proposition 3.3, RHP may be translated to the weaker version of RGP where "for some $i \in N$" is replaced by "for some reducible $i \in N$".

If σ is a solution on Γ that satisfies RGP, then we may also compare CRHP with CRGP. Indeed, in this case CRGP is equivalent to the following property, called CRGP': If $(N, c) \in \Gamma$, $|N| \geq 3$, $x \in X(N, c)$, $(N\backslash\{i\}, c_{N\backslash\{i\}, x}) \in \Gamma$, and $x_{N\backslash\{i\}} \in \sigma(N\backslash\{i\}, c_{N\backslash\{i\}, x})$, then $x \in \sigma(N, c)$. Hence, CRHP may be translated to the stronger version of CRGP' that only requires to consider removing reducible agents $i \in N$.

Therefore, the interpretations of RHP and CRHP are similar to the well-known interpretations of RGP and CRGP for TU games. Finally note that the modification of CRHP that leads to SCRHP is crucial. Indeed the strong version of CRHP is similar to a modification of CRGP that has been employed by Serrano and Volij (1998) in an axiomatization of the core and by Sudhölter and Potters (2001) in the axiomatization of the semi-reactive prebargaining set.

3.1. Characterization of the core

The *core* of $(N, I) \in \mathcal{H}$ is the set

$$\mathcal{C}(N,I) = \left\{x \in X(N,I) \mid x(S) \leqslant \lambda^*(I_S) \text{ for all } S \subseteq N\right\},$$

i.e., $\mathcal{C}(N,I)$ is the core of the highway game (N, C^I).

In Appendix A we prove the following results for the core of a highway problem. Theorem 3.6 resembles an alternative characterization of Peleg (1986) of the core for TU games.

Proposition 3.4. *The core on $\mathcal{H}$ satisfies NEM, PO, IR, REASB, PCOV, RHP, CRHP, SCRHP,* and *UTPH.*

Theorem 3.5. *The core is the unique solution that satisfies NEM, IR, RHP, and SCRHP.*

Theorem 3.6. *The core is the unique solution that satisfies IR, UTPH, RHP, and CRHP.*

3.2. Characterization of the nucleolus

We now recall the definition of the (pre)nucleolus (Schmeidler, 1969).

Let (N, c) be a cost TU game, $x \in \mathbb{R}^N$, $S \subseteq N$, and $i, j \in N$, $i \neq j$. The *excess* of S at x is $e(S,x,c) = x(S) - c(S)$, and the *maximum surplus* of i over j at x is $s_{ij}(x,c) = \max\{e(S,x,c) \mid i \in S \subseteq N \setminus \{j\}\}$.

Write $X(N,c) = \{x \in \mathbb{R}^N \mid x(N) = c(N)\}$. The *prekernel* of (N, c) is the set

$$\mathcal{PK}(N,c) = \Big\{x \in X(N,c) \mid s_{ij}(x,c) = s_{ji}(x,c) \text{ for all } i \in N,\; j \in N \setminus \{i\}\Big\}.$$

The *prenucleolus* of (N, c) is the subset of elements of $X(N, c)$ that lexicographically minimize the non-increasingly ordered vector $(e(S, x, c))_{S \subseteq N}$ of excesses. The *nucleolus* is obtained similarly by restricting the attention to individually rational elements of $X(N, c)$ called imputations. According to Schmeidler (1969), the prenucleolus of (N, c) is a singleton whose unique element is denoted by $\nu(N, c)$.

Remark 3.7. Maschler, Peleg, and Shapley (1972) show that for concave cost games the prekernel is individually rational and consists of a single point, namely the prenucleolus. Hence, the prenucleolus is individually rational on concave games, and therefore it is the nucleolus of the game.

We define the *nucleolus* of the highway problem (N, I) to be the nucleolus of the associated cost game (N, c^I), that is denoted $\nu(N,I) = \nu(N,c^I)$.

Now we turn to establish the following results for the nucleolus proved in Appendix A.

Proposition 3.8. *The nucleolus on $\mathcal{H}$ satisfies NEM, PO, SIVA, ETP, IR, REASB, PCOV, RHP and CRHP.*

Theorem 3.9. *The nucleolus on $\mathcal{H}$ is the unique solution that satisfies SIVA, ETP, PCOV, and RHP provided $|U| \geq 2$.*

Remark 3.10.

(1) A careful inspection of the proof of Theorem 3.9 in Appendix A shows that the axiom SIVA may be replaced by NEM and PO.
(2) The nucleolus does neither satisfy UTPH nor SCRHP.

By means of the following example we show that a reduced game w.r.t. the nucleolus of a highway game may not be a highway game if (1) a non-oriented agent of type 1 is removed or if (2) an oriented agent of type 2 is removed (provided $|U| \geq 4$).

Example 3.11. (1) Let (N, I) be the highway problem defined in Example 3.2 (1). With the help of Kohlberg's (1971) balancedness condition that characterizes the nucleolus it can be checked that $\nu(N,I) = \nu(N,c^I) = x = (1,1,1,1)$. Let $N' = \{1,2,3\}$ and $c = c^I_{N',x}$. Then, for any $\emptyset \neq S \subseteq N'$,

$$c(S) = \begin{cases} 2, & \text{if } |S| = 1, \\ 3, & \text{if } |S| \geqslant 2. \end{cases}$$

We now show that (N', c) is not strategically equivalent to a highway game. Assume the contrary. As each positive multiple of a highway game is a highway game, there exist $(N', I') \in \mathcal{H}$ and $y \in \mathbb{R}^{N'}$ such that $c + y = c^{I'}$. Let $I'_i = [a'_i, b'_i]$ for $i \in N'$, choose j, k, $\ell \in N'$ such that $a'_j = \min_{i \in N'} a'_i$, and choose $k \in N'\backslash\{j\}$ such that $b'_k = \max_{i \in N'\backslash\{j\}} b'_i$, and $N' = \{j,k,\ell\}$. As $c(\{j\}) = 2$, $b'_j = a'_j + 2 + y_j$. As $a'_i \geqslant a'_j$ and $c(\{j,i\}) = 3$ for $i \in N'\backslash\{k\}$, $b'_i = a'_j + 3 + y_j + y_i$. Moreover, as $c(\{i\}) = 2$, $a'_i = a'_j + 1 + y_j$. Finally, as $c(\{k,\ell\}) = 3$, $a'_k = a'_\ell$, and $b'_k \geqslant b'_\ell$,

$$b'_k = a'_k + 3 + y_k + y_\ell = a'_j + 4 + y(N') = a'_j + 3 + y_j + y_k$$

so that $y_\ell = -1$, i.e., $c^{I'}(N) = c^{I'}(N' \setminus \{\ell\}) - 1$, and the desired contradiction has been obtained. Note that I_3 starts in I_4 and that 3 is not an interior agent of 4. Similarly, I_1 ends in I_4, and 1 is also not an interior agent of 4. Hence, agent 4 is neither a left nor a right agent. Finally, agent 2 is an interior agent of 4. Hence, agent 4 is of type 1, but not oriented.

(2) Now we consider $(N, I'') \in \mathcal{H}$ defined by $N = \{1,\ldots,4\} \subseteq U$, $I''_1 = [0,2]$, $I''_2 = [1,3]$, $I''_3 = [2,4]$, and $I''_4 = [0,5]$. Then $\nu(N,I'') = (1,1,1,2)$ and the reduced game of $(N, c^{I''})$ w.r.t. N' and $\nu(N,I'')$ is again (N', c) that does not correspond to any highway problem, as we have just seen in (1). Moreover, agent 4 is a left agent of type 2.

4. The Shapley value of highway problems

Dual unanimity games are useful to provide a formula for the Shapley value of a highway problem. Recall that for any coalition $S \subseteq N$, the *dual unanimity game* (N, u^*_S) is defined by

$$u^*_S(T) = \begin{cases} 0, & \text{if } T \subseteq N \setminus S, \\ 1, & \text{if } T \subseteq N \text{ and } T \cap S \neq \emptyset. \end{cases}$$

Note that $\{(N,u^*_S) \mid \emptyset \neq S \subseteq N\}$ is a vector space basis of the Euclidean space of all TU games with player set N, i.e., of $\mathbb{R}^{2^N \backslash \{\emptyset\}}$. Hence, if (N, c) is a TU game there are unique real coefficients λ_S, $\emptyset \neq S \subseteq N$, such that $c = \sum_{\emptyset \neq S \subseteq N} \lambda_S u^*_S$. Since the Shapley value is additive, we have

$$\phi_i(N,c) = \sum_{\substack{S \subseteq N \\ i \in S}} \frac{\lambda_S}{|S|} \quad \text{for all } i \in N. \tag{4.5}$$

Let $(N, I) \in \mathcal{H}$ and consider its representation (N, M^I, T^I, C^I) as defined in expressions (2.4)–(2.6). Denote $(T^I)^{-1}(j) = \{i \in N \mid j \in T^I(i)\}$ for all $j \in M$. Then we obtain $c^I = \sum_{j \in M^I} \lambda^*(j) u^*_{(T^I)^{-1}(j)}$ so that, by (4.5),

$$\phi_i(N, I) = \sum_{j \in T^I(i)} \frac{\lambda^*(j)}{|(T^I)^{-1}(j)|} \text{ for all } i \in N. \tag{4.6}$$

That is, according to the Shapley value the cost of each section has to be shared equally by its users.

For $(N, I) \in \mathcal{H}$ with $I_j = [a_j, b_j]$ for all $j \in N$ and $\alpha, \beta \in [0, b]$, $\alpha \leq \beta$, the $[\alpha, \beta]$-*truncated* highway problem $(N, I^{[\alpha,\beta]})$ is defined by $I_i^{[\alpha,\beta]} = [\min\{(a_i - \alpha)_+, \beta - \alpha\}, \min\{(b_i - \alpha)_+, \beta - \alpha\}]$ for all $i \in N$, where $a_+ = \max\{a, 0\}$ for $a \in \mathbb{R}$. Hence, $(N, I^{[\alpha,\beta]})$ is the highway problem that arises from (N, I) if the highway is restricted to the interval $[\alpha, \beta]$.

In what follows, by slightly abusing notation, for a single-valued solution σ on $\mathcal{H}$ the unique element of the singleton $\sigma(N, I)$ is also denoted by $\sigma(N, I)$ and, conversely, we use $\phi(N, I)$ for $\{\phi(N, v)\}$ so that ϕ becomes a solution.

We say that a single-valued solution σ (on $\mathcal{H}' \subseteq \mathcal{H}$) satisfies

(12) *individual independence of outside changes* (IIOC) if for all $(N, I), (N, I') \in \mathcal{H}'$ and all $i \in N$: If $I^{I_i} = I'^{I'_i}$, then $\sigma_i(N, I) = \sigma_i(N, I')$.

IIOC means that the charge of an agent i may only depend on the highway problem truncated to her used interval.

Remark 4.1. The property of *strong monotonicity* (Young, 1985) of a single-valued solution σ on the set of games requires that if $(N, c), (N, c') \in \Gamma$, $i \in N$, and $c(S \cup \{i\}) - c(S) \geqslant c'(S \cup \{i\}) - c'(S)$ for all $S \subseteq N$, then $\sigma_i(N, c) \geq \sigma_i(N, c')$. From (4.6), it is straightforward that strong monotonicity of σ implies IIOC of the corresponding solution on $\mathcal{H}$.

We have the following characterization result the proof of which can be found in Appendix B, as the proofs of Theorems 4.3 and 4.4.

Theorem 4.2. *The Shapley value on $\mathcal{H}$ is the only solution that satisfies SIVA, PO, ETP, and IIOC.*

4.1. The Shapley value of airport problems

A highway problem $(N, I) \in \mathcal{H}$ with $I_j = [a_j, b_j]$ for all $j \in N$ is an *airport problem* if $a_i = 0$ for all $i \in N$, i.e., every agent has the same starting point. Let $\mathcal{A}$ denote the set of airport problems. Let $(N, I) \in \mathcal{A}$ and let $N = \{i_1, \ldots, i_n\}$ so that $b_{i_1} \leqslant \cdots \leqslant b_{i_n}$. By (4.6), the Shapley value can be recursively computed as

$$\phi_{i_1}(N, I) = \frac{b_{i_1}}{n} \text{ and } \phi_{i_{j+1}}(N, I) = \phi_{i_j} + \frac{b_{i_{j+1}} - b_{i_j}}{n - j} \quad \text{for all } j = 1, \ldots, n-1. \tag{4.7}$$

Let $|N| \geq 2$, $i \in N$ and $x \in \mathbb{R}^N$. The *contracted problem* w.r.t. i and x, denoted $(N \setminus \{i\}, I^{-i,x,\text{ctr}})$, is defined as follows. For $j \in N \setminus \{i\}$, $I_j^{-i,x,\text{ctr}} = [0, b_j - \min\{x_j, x_i\}]$.

The contracted problem can be interpreted as a kind of reduced problem in the following way. Assume that a payoff vector is at stake, say x, and everybody accepts the payoff assigned to a certain agent, say i. In the contracted problem the remaining agents are assuming that the cost of the runway used by agent $j \neq i$ is decreased by x_i, unless this discount were higher than x_j, in which case the discount would be x_j.

Note that $(N, I^{-i,x,\text{ctr}}) \in \mathcal{A}$ if and only if $b_j - \min\{x_j, x_i\} \geqslant 0$ for all $j \in N \setminus \{i\}$.

We say that a solution σ on $\mathcal{A}$ satisfies the

(13′) *contraction property* (CONTR) if it is consistent w.r.t. contracted problems, i.e., if, for all $(N, I) \in \mathcal{A}$ with $|N| > 1$, all $x \in \sigma(N, I)$, and all $i \in N$: $(N \setminus \{i\}, I^{-i,x,\text{ctr}}) \in \mathcal{A}$ and $x_{N \setminus \{i\}} \in \sigma(N \setminus \{i\}, I^{-i,x,\text{ctr}})$.

Theorem 4.3. *On $\mathcal{A}$ the Shapley value is the unique solution that satisfies NEM, PO, and CONTR.*

4.2. The contraction property on highway games

Let $(N, I) \in \mathcal{H}$, $I_j = [a_j, b_j]$ for all $j \in N$, and assume $|N| \geq 2$.

For any left agent $i \in N$ of type 0 (i.e., $a_j > a_i$ implies $a_j \geq b_i$) and any $y \in \mathbb{R}^N$ we define the *contracted problem* w.r.t. i and y, $(N \setminus \{i\}, I^{-i,y,\text{ctr}})$, for any $j \in N \setminus \{i\}$, by $I_j^{-i,y,\text{ctr}} = [a'_j, b'_j]$, where

$$a'_j = \begin{cases} a_j, & \text{if } a_j \leqslant a_i, \\ a_j - y_i, & \text{if } a_j > a_i, \end{cases} \quad \text{and}$$

$$b'_j = \begin{cases} b_j, & \text{if } b_j < a_i, \\ b_j - \min\{y_j, y_i\}, & \text{if } b_j \geqslant a_i \geqslant a_j, \\ b_j - y_i, & \text{if } a_j > a_i. \end{cases}$$

Note that a contracted problem may not be a highway problem.

We say that a solution σ on $\mathcal{H}$ satisfies the

(13) *contraction property* (CONTR) if, for all $(N, I) \in \mathcal{H}$ with $|N| \geq 2$, all left agents $i \in N$ of type 0, with $I_i = [a_i, b_i]$, $b = \max_{j \in N} b_j$, and $x \in \sigma(N, I)$: $(N \setminus \{i\}, I^{-i,y,\text{ctr}}) \in \mathcal{H}$ and $x_{N \setminus \{i\}} \in \sigma(N \setminus \{i\}, I^{-i,y,\text{ctr}})$ for all $y \in \sigma(N, I^{[a_i,b]})$.

Hence, CONTR requires that σ is consistent w.r.t. any contraction of a highway problem according to the solution applied to the truncated highway problem the highway of which starts at the interval used by a left agent of type 0.

In an airport problem each agent i is a left agent of type 0 and $a_i = 0$ so that the current CONTR coincides with the former CONTR on airport problems – the only further requirement that must be satisfied on $\mathcal{H}$ is that consistency must be satisfied w.r.t. contracted highway problems defined with the help of any element of the solution applied to the truncated highway problem.

It should be noted that this kind of "reduction" that depends on the solution applied to certain derived problems (here certain truncated highways) is not new for axiomatizations of the Shapley value—Hart and Mas-Colell (1989) also define their consistency property only for solutions that satisfy SIVA so that their "reduced game" is defined with the help of the solution applied to subgames. Note, however, that the TU game corresponding to a contracted highway problem w.r.t. the Shapley value does typically not coincide with the corresponding Hart–Mas-Colell "reduced game" of the initial highway game (which may be illustrated by any 4-person airport problem with equal positive demands).

Theorem 4.4. *On $\mathcal{H}$ the Shapley value is the unique solution that satisfies NEM, PO, and CONTR.*

5. Generalized highway problems

The definition of a *generalized highway problem* (N, I) differs from Definition 2.1 only inasmuch as (2) is weakened to "I is a mapping that assigns to each $i \in N$ a finite union of compact nonempty intervals in $\mathbb{R}_+$". Hence, in a generalized highway problem the customers may use disconnected sections of the highway. The associated TU cost game (N, c^I) is still defined by (2.1). For a generalized highway problem it is convenient to use the representation of Kuipers et al. (2013): a tuple (N, M, C, T) as in Remark 2.2 that not necessarily satisfies condition 2.2 is a generalized highway problem. Thus the cost function $c^{M,C,T}$ is defined by (2.3). In particular we do not need a strict ordering $\prec$ of M. Hence, we denote by $\mathcal{GH}$ the set of generalized highway games (N, M, C, T).

Let $(N, M, C, T) \in \mathcal{GH}$. We now show that $(N, c^{M,C,T})$ is a positive game. A game (N, c) is called *positive* if $c = \sum_{\emptyset \neq S \subseteq N} \lambda_S u_S^*$, where the unique coefficients λ_S, $\emptyset \neq S \subseteq N$, are nonnegative. For each section $j \in M$ let $T^{-1}(j) = \{i \in N \mid j \in T(i)\}$, i.e., the set of users of j. Therefore, we have

$$c^{M,C,T} = \sum_{j \in M} C(j) u^*_{T^{-1}(j)}. \tag{5.8}$$

As $C(j) \geq 0$ for all $j \in M$, $(N, c^{M,C,T})$ is a positive game. The following theorem shows that the converse is also true.

Theorem 5.1. *A TU cost game is a positive game if and only if it is a highway game.*

Proof. By (5.8) we only have to show the only-if part. Let N be a coalition, $\lambda_S \geq 0$ for all $\emptyset \neq S \subseteq N$, and let $c = \sum_{\emptyset \neq S \subseteq N} \lambda_S u_S^*$. We define the *direct* generalized highway problem (N, M, C, T) (corresponding to (N, c)) by

$$M = 2^N \setminus \{\emptyset\}, T(i) = \{S \in M \mid i \in S\} \text{ for all } i \in N, \text{ and } C(S) = \lambda_S \text{ for all } S \in M.$$

Then $T^{-1}(S) = S$ and, by (5.8), $C^{M,C,T} = c$. □

It should be remarked that the game (N', c) defined in Example 3.11 is a generalized highway game. Indeed, with $I_1 = [0, 2], I_2 = [1, 3]$, and $I_3 = [0, 1] \cup [2, 3]$, $c^I = c$. Note also, that this example is not pathological. Thus, the set of generalized highway game strictly contains the set of highway games.

6. Discussion and remarks

By means of examples we show that each of the axioms is logically independent of the remaining axioms in the foregoing characterizations of the core, the nucleolus, and the Shapley value of highway problems. Note that the empty solution violates NEM, UTPH, and SIVA, but satisfies all other axioms in Theorems 3.5, 3.6, 3.9, 4.2, 4.3/4.4, and in Remark 3.10(1). Define, for a highway problem (N, I), $\sigma_1(N, I) = \mathcal{C}(N, I)$ if $|N| \geq 2$, and $\sigma_1(N, I) = X^*(N, I)$ if $|N| = 1$. Then σ_1 exclusively violates IR and PO in Theorems 3.5 and 3.6. The solution that assigns $\{x \in X(N, I) | x \text{ is individually rational}\}$ to each highway problem (N, I) exclusively violates RHP, and the solution that assigns the core whenever $|N| \leq 2$ and the nucleolus whenever $|N| > 2$ to each $(N, I) \in \mathcal{H}$ exclusively violates CRHP in the aforementioned theorems.

The solution that assigns the nucleolus to any highway problem with at least two agents and the set of feasible allocations to any one-person problem shows the logical independence of PO in Remark 3.10 (1). The Shapley value exclusively violates RHP in Theorem 3.9 and the foregoing remark. The solution σ defined in Section 6.3.2 of Peleg and Sudhölter (2007) that selects a lexicographical extreme point of the positive core of (N, c^I) may be chosen to show that ETP is logically independent of the remaining axioms in the characterizations of the nucleolus. Moreover, the egalitarian solution (Dutta, 1990) may be chosen to show the independence of PCOV.

The nucleolus satisfies SIVA, NEM, ETP, and PO, but violates CONTR and IIOC provided that $|U| \geq 3$. The solution σ_2 that differs from the Shapley value only in as much as $\sigma_2(\{i\}, I) = \{\phi(\{i\}, I), \phi(\{i\}, I) + 1\}$ for one-person highway problems $(\{i\}, I)$ satisfies NEM and CONTR, but violates PO. Hence, each of the axioms employed in Theorem 4.3 as well as in Theorem 4.4 is logically independent of the remaining axioms. The solution that differs from σ_2 only inasmuch as it assigns $\{\phi(\{i\}, I) + 1\}$ to each one-person highway problem $(\{i\}, I)$ exclusively violates PO in Theorem 4.2, and a non-trivially weighted Shapley value (Kalai & Samet, 1988) exclusively violates ETP.

The first part of Theorem 5.5 of Potters and Sudhölter (1999) provides a characterization of the nucleolus on airport problems that employs properties similar to those that occur in our Theorem 3.9. However, our properties are defined without mentioning the games associated with the corresponding cost sharing problems (here highway problems) whereas in the mentioned paper, e.g., the covariance property refers to the associated games rather than directly to airport problems.

The first part of the aforementioned Theorem 5.5 characterizes the modiclus (Sudhölter, 1996) on airport problems. However, for an airport game (N, c) the modiclus coincides with the prenucleolus of the dual game (N, c^*) (defined by $c^*(S) = c(S) - c(N \setminus S)$ for all $S \subseteq N$) and it is a member of $\mathcal{C}(N, c)$. By means of the following 5-person example we show that the prenucleolus of the dual of a highway game (N, c) may not be a member of the core of this game. In fact, we show that the *least core*[3] of the dual of the highway game does not intersect the core. (Another 5-person example of a general convex game the modiclus of which does not belong to the least core of the dual game was already found by Sudhölter (1997), Example 3.2(iii).)

Example 6.1. Let $(N, I) \in \mathcal{H}$ be defined by $N = \{1, \dots, 5\}$, $I_1 = [0, 6], I_2 = [0, 4], I_3 = [3, 9], I_4 = [5, 10]$, and $I_5 = [6, 10]$, and let $c = c^I$. With $x = (4, 3, 1, 1, 1)$, $\mu = \max_{S \subseteq N}(c(S) - x(S)) = 5$. Hence, for any $y \in \mathcal{LC}(N, c^*)$, $c(S) - y(S) \leqslant 5$ for all $S \subseteq N$.

Claim 1: $y_1 \geq 3$. Assume, on the contrary, that $y_1 = 3 - \varepsilon$ for some $\varepsilon > 0$. As $c(\{1, 4\}) = c(\{1, 5\}) = 10$, $y_4, y_5 \geqslant 2 + \varepsilon$. As $c(\{1, 3\}) = 9, y_3 \geqslant 1 + \varepsilon$. By Pareto optimality of y, $y_2 \leqslant 2 - 2\varepsilon$. We conclude that $y_3 \geqslant 2 + 2\varepsilon$, and a contradiction to Pareto optimality has been obtained.

Claim 2: $y_2 \geq 3$. Assume, on the contrary, $y_2 = 3 - \varepsilon$ for some $\varepsilon > 0$. Then $y_3, y_4 \geqslant 1 + \varepsilon, y_4 \geqslant \varepsilon$, and, hence, $y_1 \leqslant 5 - 2\varepsilon$. Thus, $y_5 \geq 2\varepsilon$, and the desired contradiction has been obtained.

Claim 3: $y_1 + y_2 > 6 = c(\{1, 2\})$. Assume the contrary. By Claims 1 and 2, $y_1 = y_2 = 3$. Then $y_3 \geq 1$, y_4, $y_5 \geq 2$, which is in contradiction to Pareto optimality. Thus, $\mathcal{LC}(N, c^*) \cap \mathcal{C}(I, c) = \emptyset$.

We may define the Shapley value for generalized highway problems and characterize it similarly to Theorem 4.2 by conveniently adapting the proof. The other characterizations on highway games proposed in this paper do not possess straightforward generalizations on generalized highway games. In particular, it may be checked that the symmetric 5-person game (N, c) defined by

$$c(S) = \begin{cases} 4, & \text{if } |S| = 1, \\ 7, & \text{if } |S| = 2, \\ 9, & \text{if } |S| = 3, \\ 10, & \text{if } |S| \geqslant 4, \end{cases}$$

for all $\emptyset \neq S \subseteq N$ is a positive game, hence a generalized highway game, but that none of its 4-person reduced games w.r.t. the nucleolus (2, 2, 2, 2, 2) is a positive game.

Appendix A

Here we prove the results of Section 3 concerning the characterizations of the core and the nucleolus on the set of highway problems.

The following lemmas will be useful in the sequel.

Lemma A.1. *For any highway problem (N, I) with $|N| > 1$ there exist at least two distinct reducible agents.*

[3] The *least core* of a TU game (N, c), $\mathcal{LC}(N, c)$ is the intersection of all ε-cores $\{x \in X(N, c) | e(S, x, c) \leq \varepsilon$ for all $S \subseteq N\}$ that are nonempty.

Proof. Call $i \in N$ minimal if for all $j \in N\backslash\{i\}$ such that $I_j \neq I_i$ it holds $I_j \backslash I_i \neq \emptyset$. There exists at least one minimal agent $i \in N$, and moreover a minimal agent is of type 0, hence reducible. Therefore, the proof is finished in the case that there are two distinct minimal agents. Hence we may now assume that there is a unique minimal agent, say i. Then $I_i \subsetneq I_j$ for all $j \in N\backslash\{i\}$. Choose $j \in N\backslash\{i\}$ such that $a_j = \max_{k \in N\backslash\{i\}} a_k$, where $I_i = [a_i, b_i]$ for all $i \in N$. Then j is a left agent of type 1 or 0, i.e., the second reducible agent has been found. □

Proof of Proposition 3.3. Let $I_j = [a_j, b_j]$ for all $j \in N$, denote $I' = I^{-i,x}$, and let $I'_j = [a'_j, b'_j]$ for all $j \in N\backslash\{i\}$. From $\lambda^*(I_i \backslash I_{N\backslash\{i\}}) \leq x_i \leq \lambda^*(I_i)$ it follows that $(N \setminus \{i\}, I') \in \mathcal{H}$ and, moreover, that $\max_{\ell \in N} b_\ell - x_i = \max_{j \in N\backslash\{i\}} b'_j$, hence $x_{N\backslash\{i\}} \in X(N\backslash\{i\}, I')$ and $c^{I'}(N \setminus \{i\}) = c_{N\backslash\{i\},x}(N \setminus \{i\})$. Now, let $\emptyset \neq T \subsetneq N\backslash\{i\}$. Denote $T_1 = \{j \in T \mid a_j \leqslant a_i < b_j\}$ and $T_2 = \{j \in T \mid a_j < b_i \leqslant b_j\}$. If i is an agent of type 0, then

$$c^I(T \cup \{i\}) - c^I(T) = \max\left\{0, \min\left(\{a_j \mid j \in T_2\} \cup \{b_i\}\right) - \max\left(\{b_j \mid j \in T_1\} \cup \{a_i\}\right)\right\}.$$

A careful inspection of (3.1) and (3.2) finishes the proof in this case. If i is a left agent of type 1 and k is the unique interior agent of i, i.e., $a_i < a_k \leq b_k < b_i$, then the case $k \notin T$ can be treated as before. If $k \in T$, then the cases $T_2 \neq \emptyset$ or ($T_1 \neq \emptyset$ and $\max\{b_j | j \in T_1\} \leq b_i$) are straightforward as well as the case that $x_i \geqslant (b_i - a_i) - (b_k - a_k)$. In the remaining case, $I'_k = I_k - \varepsilon$, where $\varepsilon = b_k + x_i - b_i$, again a careful inspection (3.1) and (3.2) completes the proof. Finally, if i is a right agent of type 1, but not a left agent, then we may argue similarly. □

Proof of Proposition 3.4. NEM follows from the concavity of highway games, PO, IR, REASB, and UTPH are immediate consequences of the definition of the core, and PCOV follows from the well-known scale covariance and translation covariance of the core on any set of games. As the core is reasonable and satisfies RGP on the set of games with nonempty cores (Peleg, 1986), Lemma 3.4 shows RHP.

In order to show CRHP and SCRHP, let $(N, I) \in \mathcal{H}$ such that $|N| \geq 2$. Let $x \in X(N, I)$ such that $(N \setminus \{i\}, I^{-i,x}) \in \mathcal{H}$ and $x_{N\backslash\{i\}} \in \mathcal{C}(N \setminus \{i\}, I^{-i,x})$ for each reducible agent $i \in N$. Assume that $x \notin \mathcal{C}(N, I)$ and let $\emptyset \neq S \subsetneq N$ such that $x(S) > c^I(S)$. Two cases may occur:

(a) If $|N| > 2$, by Lemma A.1 one of the following subcases must occur: (a1) There exists a reducible $i \in S$ and $|S| \geq 2$. In this case $x(S \setminus \{i\}) > c^I(S) - x_i \geqslant c^{I^{-i,x}}(S \setminus \{i\})$. (a2) There exists a reducible $i \in N\backslash S$ and $|S| \leqslant |N| - 2$. In this case $x(S) > c^I(S) \geqslant c^{I^{-i,x}}(S)$. Hence, in both subcases $x_{N\backslash\{i\}} \notin \mathcal{C}(N \setminus \{i\}, I^{-i,x})$ and the desired contradiction has been obtained. Thus, CRHP has been verified.

(b) If $|N| = 2$, $I_j = [a_j, b_j]$ for $j \in N, N = \{k, \ell\}$, then both agents are reducible by Lemma A.1. If $I_k \backslash I_\ell \neq \emptyset \neq I_\ell \backslash I_k$, then we may assume that $a_k = 0$. We conclude that $x_\ell \geqslant b_\ell - b_k$ (otherwise $x_k \notin X(\{k\}, I^{-\ell,x})$). Moreover, $x_k \geqslant a_\ell - a_k$ (otherwise $(\{\ell\}, I^{-k,x}) \notin \mathcal{H}$ because $a'_\ell > 0$, where $I_\ell^{-k,x} = [a'_\ell, b'_\ell]$) so that $x \in \mathcal{C}(N, I)$. In the remaining case, we may assume that $I_\ell \subseteq I_k$, i.e., $a_k = 0 \leqslant a_\ell \leqslant b_\ell \leqslant b_k$. Then $x_k \leq b_k$ (otherwise $b'_\ell < 0$). If $x_k < b_k + a_\ell - b_\ell$, then $a_\ell = 0$ (otherwise $a'_\ell > 0$) and $b_\ell = b_k$ (otherwise $x_\ell < \lambda^*(I_\ell^{-k,x})$). If, however, $I_k = I_\ell$, then $x_k < b_k + a_\ell - b_\ell = 0$ would imply $x_\ell > b_\ell$, hence $b''_k < 0$, where $I_k^{-\ell,x} = [a''_k, b''_k]$ which is impossible. Thus, $x \in \mathcal{C}(N, I)$ and SCRHP has been verified. □

Proof of Theorem 3.5. By Proposition 3.4 the core satisfies NEM, IR, RHP, and CRHP. In order to show the opposite implication, let σ be a solution that satisfies the desired properties. Let $(N, I) \in \mathcal{H}$. If $|N| = 1$, then $\sigma(N, I) = \mathcal{C}(N, I)$ by NEM and IR. Assume that $\sigma(N, I) = \mathcal{C}(N, I)$ whenever $|N| < k$ for some $k > 1$. If $|N| = k$ and $x \in \mathcal{C}(N, I)$, then, by RHP of the core, $x_{N\backslash\{i\}} \in \mathcal{C}(N \setminus \{i\}, I^{-i,x}) = \sigma(N \setminus \{i\}, I^{-i,x})$ for each reducible $i \in N$ so that by CRHP of σ, $x \in \sigma(N, I)$. The other inclusion follows by exchanging the roles of σ and $\mathcal{C}$. □

Proof of Theorem 3.6. By Proposition 3.4 the core satisfies the required axioms. In order to show uniqueness, let σ be a solution that satisfies IR, UTPH, RHP, and CRHP. Let $(N, \mathcal{I}) \in \mathcal{H}$. If $|N| \leq 2$, then by IR, UTPH, and RHP, $\sigma(N, \mathcal{I}) = \mathcal{C}(N, \mathcal{I})$. We proceed by induction on $|N|$ and assume that $\sigma(N, \mathcal{I}) = \mathcal{C}(N, \mathcal{I})$ whenever $|N| < t$ for some $t > 2$. Now, if $|N| = t$, let $x \in \sigma(N, \mathcal{I})$ and $y \in \mathcal{C}(N, \mathcal{I})$. By RHP of σ and CRHP of $\mathcal{C}$, $x \in \mathcal{C}(N, \mathcal{I})$. By RHP of $\mathcal{C}$ and CRHP of σ, $y \in \sigma(N, \mathcal{I})$. □

The following lemma will be used in the proof of Proposition 3.8.

Lemma A.2. *For any $(N, I) \in \mathcal{H}$ that has exactly two distinct reducible agents k and ℓ, $(I_k \cup I_\ell)$ is contained in the interior of I_i for all $i \in N\backslash\{k, \ell\}$.*

Proof. Let $I_j = [a_j, b_j]$ for $j \in N$, $a = \min\{a_k, a_\ell\}$, say $a_k = a$, and $b = \max\{b_k, b_\ell\}$. Let $i_1, i_2 \in N\backslash\{k, \ell\}$ such that $a_{i_1} = \max\{a_i \mid i \in N \setminus \{k, \ell\}\}$ and $b_{i_2} = \min\{b_i \mid i \in N \setminus \{k, \ell\}\}$. Note that w.r.t. the highway subproblem $(N\backslash\{k, \ell\}, (I_j)_{j \in N\backslash\{k, \ell\}})$, agent i_1 is a left agent of type 0 and i_2 is a right agent of type 0. Assume that $a_{i_1} \geqslant a$. As i_1 is not reducible, she is not an agent of type 0, i.e., I_ℓ is contained in the interior of I_{i_1}. But then i_1 is a left agent of type 1, i.e., still reducible, which was excluded. Similarly it is seen that $b_{i_2} > b$: Assuming that, on the contrary, $b_{i_2} \leqslant b$ yields a contradiction because on the one hand side i_2 cannot be of type 0 and on the other hand she cannot be a right agent of type 1. □

Proof of Proposition 3.8. The prenucleolus on cost games satisfies the properties corresponding to SIVA (hence NEM), and ETP so that these properties are also satisfied by the nucleolus of highway problems. Moreover, it satisfies translation covariance which implies PCOV, and, by definition, if satisfies PO. The prenucleolus always selects a core element if the core is nonempty. By Lemma 3.3 the reduced problems w.r.t. reducible agents are highway problems, the associated games of which are Davis–Maschler reduced games. According to Sobolev (1975) the prenucleolus satisfies RGP which implies that our nucleolus satisfies RHP. In order to show CRHP, let $(N, I) \in \mathcal{H}$ with $|N| \geq 3$ and let $x \in X(N, I)$ such that $x_{N\backslash\{i\}} = \nu(N \setminus \{i\}, I^{-i,x})$ for all reducible agents $i \in N$. Let $k, \ell \in N$, $k \neq \ell$. By Remark 3.7 it suffices to show that $s_{k\ell}(x, c^I) = s_{\ell k}(x, c^I)$. If there is a reducible agent $i \in N\backslash\{k, \ell\}$, then the game $(N\backslash\{i\}, c)$ associated with the reduced highway problem $(N \setminus \{i\}, I^{-i,x})$ is the Davis–Maschler reduced game of (N, c^I) so that $s_{k\ell}(x, c^I) = s_{k\ell}(x_{N\backslash\{i\}}, c) = s_{\ell k}(x_{N\backslash\{i\}}, c) = s_{\ell k}(x, c^I)$. Otherwise, k and ℓ are the unique reducible agents and we know that $s_{ij}(x, c^I) = s_{ji}(x, c^I)$ for all $\{i, j\} \subseteq N$ with $i \neq j$ except $\{k, \ell\}$. As the nucleolus selects a member of the core, $x \in \mathcal{C}(N, I)$ by CRHP of the core. Let $\mu = \max\{e(S, x, c^I) \mid \emptyset \neq S \subsetneq N\}$ and define $\mathcal{D} = \{S \subsetneq N \mid S \neq \emptyset, e(S, x, c^I) = \mu\}$. It suffices to show that $s_{k\ell}(x, c^I) = \mu$. Assume the contrary. We claim that $\mathcal{D} = \{N \setminus \{k\}\}$. Let $S \in \mathcal{D}$. If $S \cap (N\backslash\{k, \ell\}) \neq \emptyset$, choose $i \in S \cap (N\backslash\{k, \ell\})$. As $x_k \geq 0$ and as $I_k \subseteq I_i$ by Lemma A.2, $e(S \cup \{k\}, x, c^I) \geq \mu$ so that $\ell \in S$ by our assumption. If there exists $j \in N\backslash(S \cup \{k, \ell\})$, then $\mu = s_{\ell j}(x, c^I) = s_{j\ell}(x, c^I)$ so that there exists $S' \in \mathcal{D}$ with $\ell \notin S' \ni j$ which cannot be true by the former argument. Hence, $S = N \setminus \{k\}$ in this case. If $S \cap N \setminus \{k, \ell\} = \emptyset$, then $\ell \in S$ because $e(\{k\}, x, c^I) < \mu$. Therefore, for $i \in N\backslash\{k, \ell\}$, $s_{\ell i}(x, c^I) = \mu = s_{i\ell}(x, c^I)$, and hence there exists $S' \in \mathcal{D}$ with $\ell \notin S' \ni i$ which is impossible by the former argument. Now the proof can be finished. By our claim, $s_{ik}(x, c^I) = \mu = s_{ki}(x, c^I)$ for $i \in N\backslash\{k, \ell\}$ so that we have derived a contradiction to our claim that $N\backslash\{k\}$ is the unique coalition in $\mathcal{D}$. □

Proof of Theorem 3.9. By Proposition 3.8 the nucleolus satisfies these properties. In order to show the opposite implication, let σ be a solution that satisfies the desired axioms. Let $(N, I) \in \mathcal{H}$ and let x be the unique element of $\sigma(N, I)$. We have to show that $x = \nu(N, c^I)$. If $|N| = 1$, say $N = \{i\}$, then by PCOV we may assume that $I_i = [0, 0]$. Choose $j \in U \backslash \{i\}$, define $I_j = I_i$, and let $y = \sigma(\{i, j\}, I)$. By ETP, $y_i = y_j$. By RHP, $y_j = 0$ because otherwise $I_i^{-j,y} = \emptyset$. Hence, $x = y_i = 0$. If $|N| = 2$, then by PCOV we may assume that $I_i = I_j$ for i, $j \in N$, and, hence, $x_i = x_j$ by ETP. By RHP, $x \in X(N, I)$, hence $x = \nu(N, I)$. Now we proceed by induction on $|N|$ and assume that the unique element of $\sigma(N, I)$ coincides with $\nu(N, I)$ whenever $|N| < r$ for some $r > 2$. If $|N| = r$, then by RHP, $x_{N\backslash\{i\}} = \nu(N \setminus \{i\}, c^{I^{i,x}})$ for each reducible agent so that, by CRHP of ν, $x = \nu(N, I)$. □

Appendix B

Proof of Theorem 4.2. The Shapley value satisfies the four axioms by (4.6). In order to prove the other implication, let σ be a solution that satisfies SIVA, PO, ETP, and IIOC. Let $(N, I) \in \mathcal{H}$ and $b = \max I_N$. By induction on $|M^I|$ we prove that $\sigma(N, I) = \phi(N, I)$. If $\left|M^I\right| = 1$, then $I_i = I_j$ for every $i, j \in N$, and the result follows from SIVA, PO, and ETP. Assume that $\sigma(N, I) = \phi(N, I)$ whenever $|M^I| < k$ for some $k \geq 2$. If $\left|M^I\right| = k$, let α, β be determined by $\alpha \neq b$, $\beta \neq 0$, and $[0, \beta]$, $[\alpha, b] \in M^I$. Define

$$P = \{i \in N \mid [0, \beta[\cap I_i = \emptyset\} \text{ and } Q = \{i \in N \mid \ ]\alpha, b] \cap I_i = \emptyset\}.$$

Note that, for any $i \in N \backslash (P \cup Q)$, $I_i = I_N$. Hence, by SIVA, PO, and ETP it suffices to show that $\sigma_{P\cup Q}(N, I) = \phi_{P\cup Q}(N, I)$. With $I' = I^{[\beta,b]}$ we have $\left|M^{I'}\right| < k$ and $I'_i = I_i$ for all $i \in P$, and with $I' = I^{[0,\alpha]}$ we have $\left|M^{I'}\right| < k$ and $I'_j = I_j$ for all $j \in Q$ so that the inductive hypothesis finishes the proof. □

Proof of Theorem 4.3. By definition the Shapley value is a singleton, hence satisfies NEM. By (4.7) it satisfies PO and CONTR as well. In order to show the uniqueness part, let σ be a solution on $\mathcal{A}$ that satisfies NEM, PO, and CONTR. Let $(N, I) \in \mathcal{A}$, $I_j = [a_j, b_j]$ for $j \in N$, and $x \in \sigma(N, I)$. By NEM it suffices to show that $x = \phi(N, I)$. We proceed by induction on $|N|$. If $|N| = 1$, then $x = \phi(N, I)$ by NEM and PO. Now assume that $x = \phi(N, I)$ whenever $|N| < k$ for some $k \geq 2$. If $|N| = k$, then choose $N = \{i_1, \ldots, i_n\}$ so that $b_{i_1} \leqslant \cdots \leqslant b_{i_n}$.

Claim: $x_j \leqslant x_{i_n}$ for all $j \in N$. Indeed, assume on the contrary that there exists $j \in N$ such that $x_j > x_{i_n}$. Then $c^{I^{-j,x,\text{ctr}}}\left(N \setminus \{i\}\right) \geqslant b_{i_n} - x_{i_n} = x(N) - x_{i_n} > x(N \setminus \{j\})$ so that $x_{N\backslash\{j\}}$ is not feasible for the reduced airport problem $(N \setminus \{j\}, I^{-j,x,\text{ctr}})$.

Now let $I' = I^{-i_n,x,\text{ctr}}$, $I'_\ell = [a'_\ell, b'_\ell]$ for $\ell \in N \backslash \{i\}$. By our claim, $b'_j = b_j - x_j$. By the inductive hypothesis, $x_{N\backslash\{i_n\}} = \phi(N \setminus \{i_n\}, I')$ so that we conclude from (4.7) that $x_i \leq x_j$ if and only if $b'_i \leqslant b'_j$ for all $i, j \in N \backslash \{i_n\}$. Hence, $b'_{i_1} \leqslant \cdots \leqslant b'_{i_{n-1}}$. By (4.7), $x_{i_1} = \frac{b'_{i_1}}{n-1} = \frac{b_{i_1} - x_{i_1}}{n-1}$ so that $x_{i_1} = \frac{b_{i_1}}{n} = \phi_{i_1}(N, I)$. We proceed recursively and assume that $x_{i_j} = \phi_{i_j}(N, I)$ for $j = 1, \ldots, t$. If $t < n - 1$, then, by (4.7),

$$x_{i_{t+1}} = x_{i_t} + \frac{b'_{i_{t+1}} - b'_{i_t}}{n - i_t} = x_{i_t} + \frac{b_{i_{t+1}} - x_{i_{t+1}} - (b_{i_t} - x_{i_t})}{n - i_t},$$

hence $x_{i_{t+1}} = x_{i_t} + \frac{b_{i_{t+1}} - b_{i_t}}{n+1-i_t} = \phi_{i_{t+1}}(N, I)$. Finally, by PO, $x_{i_n} = \phi_{i_n}(N, I)$. □

Proof of Theorem 4.4. The Shapley value satisfies NEM and PO. In order to show CONTR let $(N, I) \in \mathcal{H}, I_j = [a_j, b_j]$ for $j \in N$, $|N| \geq 2$, and i a left agent of type 0. Let $I' = I^{[0,a_i]}$, $I'' = I^{I_i}$, and $I''' = I^{[b_i, \max I_N]}$, i.e., I' represents the first part of the highway from 0 to a_i, I'' is the middle part from a_i to b_i, and $I''{}'$ represents the rest, namely the part from b_i to $\max I_N$. Moreover, let y', y'', y'' be the Shapley values of (N, I'), (N, I''), $(N, I''{}')$ respectively. As $c^I = c^{I'} + c^{I''} + c^{I'''}$, by the well-known additivity of ϕ (see (4.6)), $x := \phi(N, I) = y' + y'' + y'''$. Moreover, agent i is a null-player of $(N, c^{I'})$ and $(N, c^{I'''})$ (an agent $k \in N$ is a *null-player* of a TU game (N, c) if $c(S \cup \{k\}) = c(S)$ for all $S \subseteq N$) so that $x'_i = x'''_i = 0$ by definition of ϕ. Now, $(N, I'') \in \mathcal{A}$ so that $y''_{N\backslash\{i\}} = \phi(N \setminus \{i\}, I''^{-i,y'',\text{ctr}})$ by Theorem 4.3. Also, it is well-known that the Shapley value satisfies the strong null-player property, i.e., $\phi_{N\backslash\{i\}}(N, c) = \phi(N \setminus \{i\}, c)$ (where $(N\backslash\{i\}, c)$ denotes the subgame of (N, c) with player set $N\backslash\{i\}$). Let $(N \setminus \{i\}, I'^{-i})$ and $(N \setminus \{i\}, I'''^{-i})$ denote the corresponding highway subproblems of (N, I') and $(N, I''{}')$. As $c^{I^{-i,x,\text{ctr}}} = c^{I'^{-i}} + c^{I''^{-i,y',\text{ctr}}} + c^{I'''^{-i}}$, additivity of the Shapley value yields $\phi(N \setminus \{i\}, I^{-i,x,\text{ctr}}) = y'_{N\backslash\{i\}} + y''_{N\backslash\{i\}} + y'''_{N\backslash\{i\}} = x_{N\backslash\{i\}}$.

To prove uniqueness, let σ be a solution that satisfies NEM, PO, and CONTR. Let $(N, I) \in \mathcal{H}$, $I_j = [a_j, b_j]$ for all $j \in N$, $x \in \sigma(N, I)$. It remains to show that $x = \phi(N, I)$. We proceed by induction on $|N|$. If $|N| = 1$, then $x = \phi(N, I)$ by NEM and PO. Now assume that $x = \phi(N, I)$ whenever $|N| < k$ for some $k \geq 2$. If $|N| = k$, choose $i \in N$ such that $a_i \geq a_j$ for all $j \in N$. Then i is not only a left agent of type 0, but the truncated highway problem $(N, I^{[a_i, \max I_N]})$ is an airport problem so that, by Theorem 4.3, $\sigma\left(N, I^{[a_i, \max I_N]}\right) = \phi\left(N, I^{[a_i, \max I_N]}\right)$. CONTR of ϕ and the inductive hypothesis complete the proof. □

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