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Consistency, anonymity, and the core on the domain of convex games

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Abstract
We show that neither Peleg's nor Tadenuma's well-known axiomatizations of the core by *non-emptiness*, *individual rationality*, *super-additivity*, and *max consistency* or *complement consistency*, respectively, hold when only convex rather than balanced TU games are considered, even if *anonymity* is required in addition. Moreover, we show that the core and its relative interior are the only two solutions that satisfy Peleg's axioms together with *anonymity* and *converse max consistency* on the domain of convex games.

JEL Classification C71

1 Introduction

The core is one of the most important solutions for cooperative games. It is important mainly because it satisfies many desirable properties. In particular, it satisfies two kinds of reduced game properties, namely, "max consistency" (Peleg 1986 Davis and Maschler 1965) and "complement consistency" (Tadenuma 1992; Moulin 1985).[1] There are two well-known axiomatic characterizations of the core on the domain of

[1] For these two consistency axioms we use the terminology introduced by Thomson (1996) and call them max consistency and complement consistency because each name suggests how the underlying "reduced games" are defined in each case.

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balanced TU games based on each of these two axioms: (i) The core is the unique solution that satisfies *non-emptiness*, *individual rationality*, *super-additivity*, and *max consistency* (Peleg 1986); (ii) it is the unique solution that satisfies *non-emptiness*, *individual rationality* and *complement consistency* (Tadenuma 1992).[2]

In this note, we investigate what happens when the domain is restricted to the domain of convex TU games. Although the core satisfies Peleg's four axioms on this domain, it is not the only one.[3] It so happens that except for the core itself, all known examples of such solutions violate *anonymity*. So, one may conjecture that an axiomatic characterization of the core might be obtained by adding *anonymity* to Peleg's four axioms. In this note, we disprove this conjecture. Moreover, we show that there exist only two solutions, the core and its relative interior, that satisfy Peleg's four axioms together with *anonymity* and *converse max consistency*. We also consider a similar problem for *complement consistency*. In particular, we show that the core is not the only solution on the domain of convex games that satisfies Tadenuma's three axioms and *anonymity*.

2 Definitions and results

Let U be an arbitrary universe of at least three players, which is assumed to contain, for the ease of displaying examples and proofs, the elements 1, 2, and 3. We use $\subset$ for strict set inclusion, and $\subseteq$ for weak set inclusion. A **transferable utility (TU) game** (or a game, for short) is a pair (N, v), where N is a nonempty and finite subset of U and v is a function from 2^N to $\mathbb{R}$ with $v(\emptyset) = 0$. A game (N, v) is **convex** (Shapley 1971) if for all $S, T \in 2^N$, we have $v(S) + v(T) \leq v(S \cap T) + v(S \cup T)$. Let Γ^U and Γ^U_{vex} denote the sets of all games and all convex games, respectively. For all $x \in \mathbb{R}^N$ and all $S \in 2^N$, we write $x(S) := \sum_{i \in S} x_i$.

Given $(N, v) \in \Gamma^U$, the **core of** (N, v), denoted $C(N, v)$, is the set of vectors $x \in \mathbb{R}^N$ such that $x(N) = v(N)$ and for all $S \subset N$, $x(S) \geq v(S)$. A game has a nonempty core if and only if it is **balanced** in the sense of Bondareva (1963) and Shapley (1967). It is well-known that every convex game is balanced (Shapley 1971).

Given $\Gamma \subseteq \Gamma^U$, a **solution** on Γ is a mapping that assigns to all $(N, v) \in \Gamma$ a set of vectors $x \in \mathbb{R}^N$ with $x(N) \leq v(N)$. The core, as a mapping, may be regarded as a solution on any set of games. We use σ as a generic notation for solutions. Given two solutions σ and σ' on Γ, we say that σ is a **subsolution** of σ', and write $\sigma \subseteq \sigma'$, if for all $(N, v) \in \Gamma$, $\sigma(N, v) \subseteq \sigma'(N, v)$.

Next, we define *max consistency* (Peleg 1986) and *complement consistency* (Moulin 1985). Each of these axioms requires that the original choice in a game is "confirmed" by any subset of players in the corresponding "reduced game" obtained when the remaining players leave the game with their payoffs.

[2] Voorneveld and van den Nouweland (1998) provide an axiomatization of the core which is closely related to Peleg's result.

[3] Although this fact is widely known, we do not know any published or unpublished paper that mentions it.

Given $(N, v) \in \Gamma^U$, $N' \in 2^N \setminus \{N, \emptyset\}$, and $x \in \mathbb{R}^N$, the **max reduced game of** (N, v) **relative to** x **and** N' (Davis and Maschler 1965), denoted by $\left(N', v^{\max}_{N',x}\right)$, is defined by setting for all $S \in 2^{N'}$,

$$v^{\max}_{N',x}(S) := \begin{cases} \max_{T \subseteq N \setminus N'} \left[v\big(S \cup T\big) - x(T)\right] & \text{if } S \notin \{N', \emptyset\}, \\ v(N) - x(N \setminus N') & \text{if } S = N', \\ 0 & \text{if } S = \emptyset. \end{cases}$$

Max consistency: For all $(N, v) \in \Gamma$, all $x \in \sigma(v)$, and all $N' \in 2^N \setminus \{N, \emptyset\}$, we have $(N', v^{\max}_{N',x}) \in \Gamma$ and $x_{N'} \in \sigma(N', v^{\max}_{N',x})$.

Given $(N, v) \in \Gamma^U$, $N' \in 2^N \setminus \{N, \emptyset\}$, and $x \in \mathbb{R}^N$, the **complement reduced game of** (N, v) **relative to** x **and** N', denoted by $(N', v^{\text{comp}}_{N',x})$, is defined by setting for all $S \in 2^{N'}$,

$$v^{\text{comp}}_{N',x}(S) := \begin{cases} v\big(S \cup (N \setminus N')\big) - x(N \setminus N') & \text{if } S \neq \emptyset, \\ 0 & \text{if } S = \emptyset. \end{cases}$$

Complement consistency: For all $(N, v) \in \Gamma$, all $x \in \sigma(N, v)$, and all $N' \in 2^N \setminus \{N, \emptyset\}$, we have $(N', v^{\text{comp}}_{N',x}) \in \Gamma$ and $x_{N'} \in \sigma(N', v^{\text{comp}}_{N',x})$.

It should be noted that the core satisfies two further consistency properties for max reduced games, namely Peleg's *converse max consistency* (defined below) and an axiom called *reconfirmation property* that may be used to characterize the core on Γ^U and many other domains (Hwang and Sudhölter 2001). When replacing the max reduced game by the complement reduced game these properties are no longer satisfied by the core.

The following axioms apply to games with a fixed set of players.

Non-emptiness: For all $(N, v) \in \Gamma$, $\sigma(N, v) \neq \emptyset$;

Individual rationality: For all $(N, v) \in \Gamma$, all $x \in \sigma(N, v)$, and all $i \in N$, we have $x_i \geq v(\{i\})$;

Super-additivity: For all $(N, v), (N, w) \in \Gamma$ with $(N, v+w) \in \Gamma$, we have $\sigma(N, v) + \sigma(N, w) \subseteq \sigma(N, v + w)$.

As mentioned above, on the domain of balanced games, (i) the core is the unique solution satisfying *non-emptiness*, *individual rationality*, *super-additivity*, and *max consistency* (Peleg 1986); and (ii) the core is the unique solution satisfying *non-emptiness*, *individual rationality* and *complement consistency* (Tadenuma 1992). As the domain of convex games is closed under the reduction operation when the starting point is an allocation in the core, *max consistency* of the core is also guaranteed on the domain of convex games (Maschler et al. 1972). Moreover, the fact that the core satisfies *non-emptiness*, *individual rationality*, and *super-additivity* on the domain of convex games directly follows from the corresponding fact for the domain of balanced games. Given a total order $\preceq$ of U, define the following single-valued solution $\sigma^{\preceq}$ (slightly abusing the notation by identifying the unique element of $\sigma^{\preceq}(N, v)$ with $\sigma^{\preceq}(N, v)$), which assigns to each game (N, v) the "contribution vector" with respect to $\preceq$: for all $(N, v) \in \Gamma^U$, and all $i \in N$,

$$\sigma_i^{\preceq}(N, v) := v\left(\left\{j \in N \mid j \preceq i\right\}\right) - v\left(\left\{j \in N \mid j \prec i\right\}\right).$$

On the domain of convex games, this solution satisfies *max consistency* (Orshan 1994; Núñez and Rafels 1998; Hokari 2005). Moreover, it satisfies *non-emptiness*, *super-additivity*, and *individual rationality*. This means that on the domain of convex games, the core is **not** the only solution that satisfies Peleg's four axioms. Clearly, the above solution violates the following property.

Anonymity: For all $(N, v), (N', w) \in \Gamma$, if there exists a bijection $\pi : N \to N'$ such that for all $S \subseteq N$, $w\left(\left\{\pi(i) \mid i \in S\right\}\right) = v(S)$, then for all $x \in \sigma(N, v)$, we have $\pi(x) \in \sigma(N', w)$, where $\pi(x) = y \in \mathbb{R}^{N'}$ is defined by $y_j = x_{\pi^{-1}(j)}$ for all $j \in N'$. As far as we know, other than the core itself, no *anonymous* solution on the domain of convex games that satisfies Peleg's four axioms can been found in the literature. Here, we provide an example of such a solution.

For all $(N, v) \in \Gamma^U_{vex}$, let

$$\mathcal{S}(N, v) := \left\{S \in 2^N \,|\forall x \in C(N, v), x(S) = v(S)\right\},$$

and

$$\text{ri}\, C(N, v) := \left\{x \in C(N, v) \middle| \forall S \in 2^N \setminus \mathcal{S}(N, v), x(S) > v(S)\right\}.$$

Note that ri $C(N, v)$ is the relative interior of $C(N, v)$. Since the relative interior of a nonempty convex set is nonempty, ri $C(N, v)$ is nonempty. Note that ri C trivially satisfies *individual rationality* and *anonymity*. On the domain of balanced games, ri C satisfies *max consistency* (Orshan and Sudhölter 2010). Together with the fact that the core satisfies the property on the domain of convex games, the *max consistency* of ri C on the domain of balanced games implies the *max consistency* of ri C on the domain of convex games. We show that this solution also satisfies *super-additivity*.

Lemma 1 *On Γ^U_{vex}, ri C satisfies* super-additivity.

Proof Let $(N, v), (N, w) \in \Gamma_{vex}$, $x \in \text{ri}\, C(N, v)$, $y \in \text{ri}\, C(N, w)$, and $z \in \text{ri}\, C(N, v + w)$. Note that on the domain of convex games, the core is *additive* (Shapley 1971; Dragan et al. 1989).[4] Thus, $x + y \in C(N, v + w)$ and there exist $x' \in C(N, v)$ and $y' \in C(N, w)$ such that $z = x' + y'$. Let $S \in 2^N \setminus \mathcal{S}(N, v + w)$. Then, $z(S) = x'(S) + y'(S) > v(S) + w(S)$. Thus, $x'(S) > v(S)$ or $y'(S) > w(S)$. This implies that $S \in 2^N \setminus \mathcal{S}(N, v)$ or $S \in 2^N \setminus \mathcal{S}(N, w)$, hence $x(S) > v(S)$ or $y(S) > w(S)$, so that $x(S) + y(S) > v(S) + w(S)$. □

Thus, we have the following result:

Proposition 1 *On the domain of convex games, the core is* ***not*** *the only solution that satisfies* non-emptiness, individual rationality, super-additivity, max consistency, *and* anonymity.

[4] The definition of *additivity* is obtained by replacing $\subseteq$ with $=$ in the definition of *super-additivity*.

We now recall Peleg's (1986) definition of *converse max consistency*.
Converse max consistency: For all $(N, v) \in \Gamma$ with $|N| \geq 3$ and all $x \in \mathbb{R}^N$ with $x(N) = v(N)$: If, for all $N' \in 2^N$ with $|N'| = 2$, we have $(N', v^{\max}_{N',x}) \in \Gamma$ and $x_{N'} \in \sigma(N', v^{\max}_{N',x})$, then $x \in \sigma(N, v)$.

The following theorem shows that the core and ri C are the unique solutions that satisfy this axiom and the five axioms that appear in Proposition 1.

Theorem 1 *On the domain of convex games, the core, C, and its relative core, ri C, are the only solutions that satisfy* non-emptiness, individual rationality, anonymity, super-additivity, max consistency, *and* converse max consistency.

As we have already seen, the core satisfies the axioms of Theorem 1, and ri C satisfies the first five of the axioms. The following lemma shows that ri C satisfies *converse max consistency* as well.

Lemma 2 *On Γ^U_{vex}, ri C satisfies* converse max consistency.

In the proof of this lemma, we use the following remark that follows from the definitions of a convex game and the core.

Remark 1 Let $(N, v) \in \Gamma^U_{vex}$, $x \in C(N, v)$, and $S, T \in 2^N$. If $x(S) = v(S)$ and $x(T) = v(T)$, then $x(S \cap T) = v(S \cap T)$ and $x(S \cup T) = v(S \cup T)$.

Proof of Lemma 2 Let $(N, v) \in \Gamma^U_{vex}$ with $|N| \geq 3$, and $x \in \mathbb{R}^N$ be such that for all $N' \in 2^N$ with $|N'| = 2$, we have $(N', v^{\max}_{N',x}) \in \Gamma^U_{vex}$ and $x_{N'} \in$ ri $C(N', v^{\max}_{N',x})$.

Since ri C is a subsolution of the core and the core satisfies *converse max consistency*, $x \in C(N, v)$. Suppose, on the contrary, that there exists $S \in 2^N \setminus \mathcal{S}(N, v)$ such that $x(S) = v(S)$. Let $i \in S$. Note that for all $j \in N \setminus S$,

$$v^{\max}_{\{i,j\},x}(\{i\}) = \max_{T \subseteq N \setminus \{i,j\}} [v(\{i\} \cup T) - x(T)] \geq v(S) - x(S \setminus \{i\}) = x_i.$$

Since $(x_i, x_j) \in$ ri $C(\{i, j\}, v^{\max}_{\{i,j\},x})$ and ri C satisfies *individual rationality*, $x_i = v^{\max}_{\{i,j\},x}(\{i\})$. This implies $x_j = v^{\max}_{\{i,j\},x}(\{j\})$. Thus, there exists $T_{ij} \subset N$ such that $j \in T_{ij}$, $i \notin T_{ij}$, and $x(T_{ij}) = v(T_{ij})$. Let $T_i := \bigcup_{j \in N \setminus S} T_{ij}$. Then, by Remark 1, $x(T_i) = v(T_i)$.

Note that $N \setminus S = \bigcap_{i \in S} T_i$. Again by Remark 1, $x(N \setminus S) = v(N \setminus S)$. This implies

$$v(S) + v(N \setminus S) = x(S) + x(N \setminus S) = x(N) = v(N).$$

Thus, for all $y \in C(N, v)$, we have $y(S) = v(S)$, which contradicts our assumption that $S \in 2^N \setminus \mathcal{S}(N, v)$. □

We postpone the uniqueness part of the proof and first show that the axioms in Theorem 1 imply the following two properties.
Translation covariance: For all $(N, v), (N, w) \in \Gamma$ such that there exists $b \in \mathbb{R}^N$ with $w(S) = v(S) + b(S)$ for all $S \in 2^N$, we have $\sigma(N, w) = \sigma(N, v) + b$;
Pareto optimality: For all $(N, v) \in \Gamma$ and all $x \in \sigma(N, v)$, $x(N) = v(N)$.

Lemma 3 *If* σ *on* Γ^U_{vex} *satisfies* non-emptiness, individual rationality, *and* super-additivity, *then it satisfies* translation covariance.

Proof Let $b \in \mathbb{R}^N$ and $(N, v), (N, w) \in \Gamma^U_{vex}$ be such that for all $S \in 2^N$, $w(S) = v(S) + b(S)$. Let $x \in \sigma(N, v)$. It remains to show that $x + b \in \sigma(N, w)$. Now, the additive game (N, b) is convex and, by *individual rationality* and *non-emptiness*, $\sigma(N, b) = \{b\}$. By *super-additivity*, $x + b \in \sigma(N, v + b)$. □

The following remark can be proved by literally copying Peleg's (1986) proof of the corresponding statement for balanced games.

Remark 2 If σ on Γ^U_{vex} satisfies *individual rationality* and *max consistency*, then it satisfies *Pareto optimality*.

Proof of Theorem 1 It has been already shown that the core and ri C satisfy the desired properties.

To show the uniqueness part, let σ be a solution that satisfies the properties. By Lemma 3 and Remark 2, σ satisfies *translation covariance* and *Pareto optimality*. Also, by *non-emptiness*, *individual rationality*, and *max consistency*, σ is a nonempty subsolution of the core. Let $(N, v) \in \Gamma^U_{vex}$.

Claim 1: ri $C(N, v) \subseteq \sigma(N, v)$. If $|N| = 1$ then *non-emptiness* and *Pareto optimality* finish the proof. By *converse max consistency* of ri C, we may assume $|N| = 2$. If (N, v) is inessential (additive), then the proof is finished because the core is a singleton. For coalitions N and S with $\emptyset \neq S \subseteq N$, let (N, u^S_N) denote the **unanimity game of** S **with player set** N, which is defined, for all $T \in 2^N$, by

$$u^S_N(T) := \begin{cases} 1 & \text{if } S \subseteq T, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, by *translation covariance* and *anonymity*, we may assume that v is of the form αu^N_N for some $\alpha > 0$ of the unanimity game of N on $N = \{1, 2\}$, i.e., $v(\{i\}) = 0$ for $i = 1, 2$ and $v(N) = \alpha$. Again, by *anonymity*, it suffices to show that $(\alpha - t, t) \in \sigma(N, v)$ for all $t \in (0, \alpha/2]$. Let $M{:} = \{1, 2, 3\}$.
Claim A: If $\alpha > 0$ and $(\alpha - t, t) \in \sigma(N, \alpha u^N_N)$, then $(\alpha - t, 0, t) \in \sigma(M, \alpha u^{\{1,3\}}_M)$. Indeed, up to renaming players 2 and 3, the reduced game of $(M, \alpha u^{\{1,3\}}_M)$ relative to $(\alpha - t, 0, t)$ and $\{1, 3\}$ coincides, with $(N, \alpha u^N_N)$, and as $0 \leq t \leq \alpha$, the reduced games of $(M, \alpha u^{\{1,3\}}_M)$ relative to $(\alpha - t, 0, t)$ and $\{1, 2\}$ or $\{2, 3\}$ are the corresponding additive games so that *anonymity* and *converse max consistency* show Claim A.
Claim B: For all $\alpha > 0$, $(\alpha, \alpha, \alpha) \in \sigma\left(M, 3\alpha u^M_M\right)$ and $(\alpha, \alpha) \in \sigma(N, 2\alpha u^N_N)$. Indeed the 2nd statement follows from the 1st statement by *max consistency*. In order to show the 1st statement, note that by *non-emptiness* there exists $x \in \sigma(M, \alpha u^M_M)$ and, by *Pareto optimality*, $x(M) = \alpha$. By *anonymity*, $y = (x_3, x_1, x_2)$ and $z = (x_2, x_3, x_1)$ are also members of $\sigma(M, \alpha u^M_M)$ so that, by *super-additivity*, $x + y + z = (\alpha, \alpha, \alpha) \in \sigma\left(M, 3\alpha u^M_M\right)$.
Claim C: If $\alpha > 0$ and $(\alpha - t, t) \in \sigma(N, \alpha u^N_N)$, then $(\alpha - t, t, t) \in \sigma\left(M, (\alpha + t)u^M_M\right)$. Indeed, up to renaming players 2 and 3 the reduced game of $\left(M, (\alpha + t)u^M_M\right)$ relative

to $(\alpha - t, t, t)$ and $\{1, 2\}$ is $(N, \alpha u_N^N)$, the reduced game of $\big(M, (\alpha + t)u_M^M\big)$ relative to $(\alpha - t, t, t)$ and $\{1, 3\}$ is, $(N, \alpha u_N^N)$, and the reduced game of $\big(M, (\alpha + t)u_M^M\big)$ relative to $(\alpha - t, t, t)$ and $\{2, 3\}$ is $(\{2, 3\}, 2tu_{\{2,3\}}^{\{2,3\}})$. Thus, Claim C follows from *converse max consistency*, *anonymity*, and Claim B.

Now the proof of Claim 1 is finished as soon as we show that, for all $k \in \mathbb{N}$ and all $t > 0$,

$$(\beta, t) \in \sigma(N, (\beta + t)u_N^N), \text{ if } kt < \beta \le (k + 1)t. \tag{1}$$

We proceed by induction on k.

For $t < \beta \le 2t$, $(t, t, t) \in \sigma(M, 3tu_M^M)$ and $(\beta - t, 0, \beta - t) \in \sigma(M, 2(\beta - t)u_M^{\{1,3\}})$ by Claims A and B. By *super-additivity*, $(\beta, t, \beta) \in \sigma(M, w)$ where $w = 3tu_M^M + 2(\beta - t)u_M^{\{1,3\}}$. Now, as $\beta \ge 2(\beta - t)$, the reduced game $(N, w_{N,(\beta,t,\beta)}^{\max})$ is $(N, (\beta + t)u_N^N)$ so that the base case $k = 1$ follows.

If $k > 1$, then, by the inductive hypothesis, for all $t > 0$, $(kt, t) \in \sigma(N, (k + 1)tu_N^N)$, hence, $(kt, t, t) \in \sigma(M, (k+2)tu_M^M)$ by Claim C and, for all β with $kt < \beta \le (k+1)t$, $(\beta - kt, 0, \beta - kt) \in \sigma(M, 2(\beta - kt)u_M^{\{1,3\}})$ by Claim B. Therefore, by *super-additivity*, $(\beta, t, \beta - (k - 1)t) \in \sigma(M, w)$ where $w = (k + 2)tu_M^M + 2(\beta - kt)u_M^{\{1,3\}}$. As $2(\beta - kt) \le \beta - (k - 1)t$, the reduced game $(N, w_{N,(\beta,t,\beta)}^{\max})$ is $(N, (\beta + t)u_N^N)$ so that the inductive step is finished by *max consistency*.

Claim 2: If $\sigma \ne \text{ri } C$, then $\sigma = C$. Hence, we assume that there exists a convex game (N', v') and $x \in \sigma(N', v') \setminus \text{ri } C(N', v')$. By *max consistency* of σ and *converse max consistency* of ri C, we may assume that $|N'| = 2$. By *anonymity*, we may assume that $N' = N$ and $x_1 = v'(\{1\})$. By *translation covariance*, we may assume that there exists $\beta > 0$ such that $v' = \beta u_N^N$, i.e., $x = (0, \beta)$. By *translation covariance*, *anonymity*, and *converse max consistency*, it suffices to show that $(0, \gamma) \in \sigma(N, \gamma u_N^N)$ for all $\gamma > 0$. Now, let $k \in \mathbb{N}$ be such that $k > \gamma/\beta$. By applying *super-additivity* k times, we obtain $(0, k\beta) = kx = \underbrace{x + \cdots + x}_{k} \in \sigma(N, \underbrace{v' + \cdots + v'}_{k}) = \sigma(N, kv')$. Therefore, we may assume that $\beta > \gamma$.

We claim that $(\beta, \beta, 0) \in \sigma(M, \beta(u_M^{\{1,3\}} + u_M^{\{2,3\}}))$. To prove this claim, note that the reduced games of $(M, \beta(u_M^{\{1,3\}} + u_M^{\{2,3\}}))$ relative to $(\beta, \beta, 0)$ and coalitions $\{1, 2\}$, $\{1, 3\}$, and $\{2, 3\}$, respectively, are the additive game $(N, (\beta, \beta))$, the game $(\{1, 3\}, \beta u_{\{1,3\}}^{\{1,3\}})$, and the game $(\{2, 3\}, \beta u_{\{1,3\}}^{\{1,3\}})$, respectively. The restriction of $(\beta, \beta, 0)$ to each of these 2-person coalitions belongs to the solution of the corresponding reduced game by *anonymity*. Hence, the proof of this claim is finished by *converse max consistency*.

By Claim 1, ri C is a subsolution of σ. Thus, $(\beta - \gamma, 0, \gamma) \in \sigma(M, \beta u_M^{\{1,3\}})$.

Now the proof can be completed. Let $y = (\beta - \gamma, 0, \gamma) + (\beta, \beta, 0) = (2\beta - \gamma, \beta, \gamma)$. By *super-additivity*, $y \in \sigma(M, \beta(2u_M^{\{1,3\}} + u_M^{\{2,3\}}))$. Now, the reduced game of $(M, \beta(2u_M^{\{1,3\}} + u_M^{\{2,3\}}))$ relative to y and N is $(N, \gamma u_N^N + (2\beta - \gamma, \beta - \gamma))$. Hence, by *max consistency* and *translation covariance*, $(0, \gamma) = (2\beta - \gamma, \beta) - (2\beta - \gamma, \beta - \gamma) \in \sigma(N, \gamma u_N^N)$ so that Claim 2 is proved. □

We now show that each axiom in Theorem 1 is logically independent of the remaining axioms.
(i) Without *non-emptiness*, the empty solution becomes admissible.
(ii) Without *individual rationality*, the solution that selects $\{x \in \mathbb{R} \mid x \leq v(\{i\})\}$ in the one-person case and coincides with the core otherwise becomes admissible.
(iii) Without *super-additivity*, the kernel (Davis and Maschler 1965), which in fact coincides with Schmeidler's (1969) nucleolus for convex games (Maschler et al. 1972), becomes admissible.
(iv) Without *anonymity*, the solution $\sigma^{\preceq}$ defined above becomes admissible.
(v) Without *max consistency*, the solution that coincides with the nucleolus in the two-person case and with the core otherwise becomes admissible.
(vi) Without *converse max consistency*, the solution that coincides with the core in the two-person case and with ri C otherwise, becomes admissible.

Now, let us consider *complement consistency*. It turns out that the core satisfies this axiom on the domain of convex games. Indeed, this fact follows from the facts that the core enjoys this property on the domain of balanced games and that the domain of convex games is closed under the reduction operation when starting from an allocation in the core. The proof that the complement reduced game of a convex game relative to a core element is convex is similar to the proof of the corresponding statement where complement reduced game is replaced by max reduced game.

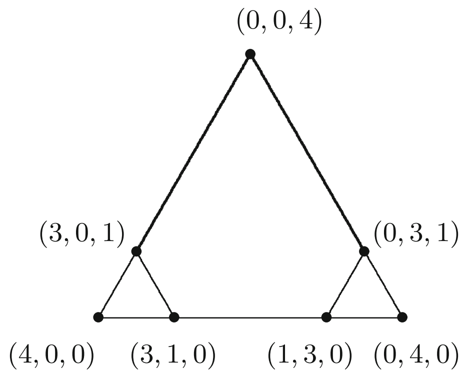
Hence, on the domain of convex games, the core satisfies Tadenuma's three axioms and *anonymity*. We construct another solution that satisfies these four axioms.

Our starting point is the solution $\sigma^{\preceq}$, defined above, that picks for each game the contribution vector with respect to a given ordering $\prec$ of players. Although $\sigma^{\preceq}$ itself does not satisfy *complement consistency*, we can enlarge it so that the resulting solution satisfies the axiom. Then we endogenize the total order $\preceq$ to make the resulting solution *anonymous*.

Consider the following solution σ^* on the domain of convex games: for all $(N, v) \in \Gamma^U_{vex}$ and all $x \in C(N, v)$, $x \in \sigma^*(N, v)$ if and only if there exists a total order $\preceq$ on N such that
(i) for all $i, j \in N$, if $v(N) - v(N\backslash\{i\}) < v(N) - v(N\backslash\{j\})$, then $i \prec j$;
(ii) for all $i \in N$, if $\{j \in N \mid j \prec i\} \neq \emptyset$, then

$$x_i \leq v\left(\left\{j \in N \mid j \preceq i\right\}\right) - v\left(\left\{j \in N \mid j \prec i\right\}\right).$$

Since the contribution vectors are in the core on this domain, σ^* satisfies *non-emptiness*. Note that it coincides with the core when $|N| \leq 2$. The following example illustrates a case in which $\sigma^*(N, v)$ does not coincides with the core, and there are two total orders that satisfy condition (i) above.



Example 1 (Figure 1) Let $N \equiv \{1, 2, 3\}$ and $(N, v) \in \Gamma^U_{vex}$ be such that $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$, $v(\{1, 2\}) = 0$, $v(\{1, 3\}) = v(\{2, 3\}) = 1$, and $v(N) = 4$. Then, $v(N) - v(\{2, 3\}) = v(N) - v(\{1, 3\}) < v(N) - v(\{1, 2\})$. So, there are two total orders on N that satisfy condition (i) in the definition of $\sigma^*(N, v)$: $1 \prec 2 \prec 3$ and $2 \prec' 1 \prec' 3$. Thus, $\sigma^*(N, v) = \left\{x \in C(N, v) \mid x_2 = 0 \text{ or } x_1 = 0\right\}$.

Note that σ^* trivially satisfies *anonymity* and *individual rationality*. We show that it also satisfies *complement consistency*.

Lemma 4 *On* Γ^U_{vex}, σ^* *satisfies* complement consistency.

Proof Let $(N, v) \in \Gamma^U_{vex}$, $x \in \sigma^*(N, v)$, and $N' \in 2^N \backslash \{N, \emptyset\}$. By the definition of $\sigma^*(N, v)$, there exists a total order $\preceq$ on N such that
(i) for all $i, j \in N$, if $v(N) - v(N\backslash\{i\}) < v(N) - v(N\backslash\{j\})$, then $i \prec j$;
(ii) for all $i \in N$, if $\left\{j \in N \mid j \prec i\right\} \neq \emptyset$, then

$$x_i \leq v\left(\left\{j \in N \mid j \prec i\right\} \cup \left\{i\right\}\right) - v\left(\left\{j \in N \mid j \prec i\right\}\right).$$

Since $x \in C(N, v)$ and the core is *complement consistent*, we have $(N, v^{\text{comp}}_{N',x}) \in \Gamma_{vex}$ and $x_{N'} \in C(N', v^{\text{comp}}_{N',x})$. We want to show that $x_{N'} \in \sigma^*(N', v^{\text{comp}}_{N',x})$.

If $|N'| \leq 2$, then $\sigma^*(N', v^{\text{comp}}_{N',x}) = C(N', v^{\text{comp}}_{N',x})$, and we are done.

Suppose that $|N'| \geq 3$. Note that, for all $i \in N'$, since $N'\backslash\{i\} \neq \emptyset$, by the definition of $v^{\text{comp}}_{N',x}$,

$$v^{\text{comp}}_{N',x}(N') - v^{\text{comp}}_{N',x}(N'\backslash\{i\}) = v(N) - v(N\backslash\{i\}).$$

Thus, if $i, j \in N'$ are such that

$$v^{\text{comp}}_{N',x}(N') - v^{\text{comp}}_{N',x}(N'\backslash\{i\}) < v^{\text{comp}}_{N',x}(N') - v^{\text{comp}}_{N',x}(N'\backslash\{j\}),$$

then $i \prec j$.

Let $i \in N'$ and $S := \{j \in N \mid j \prec i\}$. If $S \cap N' \neq \emptyset$, then, by the definition of $v^{\text{comp}}_{N',x}$ and the convexity of (N, v),

$$\begin{aligned} &v^{\text{comp}}_{N',x}((S \cup \{i\}) \cap N') - v^{\text{comp}}_{N',x}(S \cap N') \\ &\quad = v((S \cup \{i\}) \cup (N \backslash N')) - v(S \cup (N \backslash N')) \\ &\quad \geq v(S \cup \{i\}) - v(S) \\ &\quad \geq x_i. \end{aligned}$$

Thus, $x_{N'} \in \sigma^*(N', v^{\text{comp}}_{N',x})$. □

Thus, we have the following result:

Proposition 2 *On the domain of convex games, the core is* ***not*** *the unique solution that satisfies* non-emptiness*,* individual rationality*,* complement consistency*, and* anonymity.

Although we have shown that two well-known axiomatizations break down if the domain is restricted to the class of convex games, we should mention that there is another axiomatization of the core on the domain of all TU games provided by Peleg (1986), which remains valid even on the domain of convex games. It says that on this domain, the core is the unique solution that satisfies *max consistency*, *converse max consistency*, and coincides with the core in the two-person case.

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