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Axiomatizations of Dutta-Ray's egalitarian solution on the domain of convex games

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ABSTRACT

We show that on the domain of convex games, Dutta-Ray's egalitarian solution is characterized by core selection, aggregate monotonicity, and bounded richness, a new property requiring that the poorest players cannot be made richer within the core. Replacing "poorest" by "poorer" allows to eliminate aggregate monotonicity. Moreover, we show that the egalitarian solution is characterized by constrained welfare egalitarianism and either bilateral consistency à la Davis and Maschler or, together with individual rationality, by bilateral consistency à la Hart and Mas-Colell.

1. Introduction

In the context of transferable utility cooperative games (games, for short), Dutta and Ray (1989) introduced the *egalitarian solution*, which combines individual interests with the Lorenz criterion to promote equality. Although this solution lacks general existence properties, on the domain of convex games it selects the unique Lorenz maximal imputation within the core. On this domain, the first axiomatizations of the egalitarian solution were provided by Dutta (1990) by means of *DM-consistency* or *HM-consistency*, that is, consistency with respect to (w.r.t) the reduced games proposed by Davis and Maschler (1965) or Hart and Mas-Colell (1989), respectively, together with *constrained egalitarianism* (CE), a prescriptive property that determines the solution for two-person games. Klijn et al. (2000) reformulated the above characterizations replacing CE by *efficiency* (EF), also known as Pareto optimality, that requires the solution to distribute the entire worth of the grand coalition, *equal division stability* (EDS), which forces the solution to select an allocation in the equal division core (Selten, 1972), and *bounded maximum payoff* (BMP) imposing an upper bound for the payoffs of the players receiving most, and only requiring DM-consistency and HM-consistency when these richest players leave with their assigned payoffs. Considering the anti-dual properties[1] in the characterizations of Dutta (1990) and Klijn et al. (2000), Oishi et al. (2016) and Dietzenbacher and Yanovskaya (2020a,b) obtain new axiomatizations.[2] Hougaard et al. (2001) described another axiomatization combining DM-consistency and EF with *individual rationality* (IR) and *rich are strong* (RS). IR guarantees that no single player is worse off compared to staying alone, and RS requires that the solution can only make a player i richer than another player j if the maximum surplus (in the sense of Davis and Maschler, 1965) of i over j is positive and larger than the maximum surplus of j over i. Arin et al. (2003) reinterpreted the egalitarian solution providing a characterization without making use of any consistency property and invoking *core selection* (CS) requiring that each coalition receives at least what it can get by itself, *continuity*, *equal treatment of equals*, and *independence of irrelevant core allocations*.[3] Recently, Llerena and Mauri (2017)

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1 See Section 2 for formal definitions of anti-dual solution and anti-dual property.

2 Dietzenbacher and Yanovskaya (2020a,b) also proposed new axiomatizations replacing DM-consistency by a weak version of *complement consistency* (Moulin, 1985), and its anti-dual *projection consistency* (Funaki, 1998), with EDS and BMP (or their anti-dual properties).

3 See Arin et al. (2003) for the precise definitions of these three properties.

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provided two characterizations imposing a weak version of DM-consistency and either IR together with *internal* and *external Lorenz stability* (over the core)[4] or CS and BMP.

In this paper, we provide several characterizations with and without consistency. For the latter, we use *aggregate monotonicity* (AM) defined by Megiddo (1974), a very natural property requiring that no player suffers if only the worth of the grand coalition increases,[5] and *bounded richness* (BR), imposing an upper bound for the payoffs of non-poorest players, together with CS. Up to our knowledge, AM has not been employed before in any of the existing characterizations of the egalitarian solution. Under EF, BR is equivalent to *bounded minimum payoff* introduced in Oishi et al. (2016), which is the anti-dual of BMP. This fact leads to a parallel characterization making use of CS, BMP, and the anti-dual of AM, called *adjusted aggregate monotonicity*. Strengthening BR, replacing "poorest" by "poorer", we introduce *strong bounded richness* (SBR) which allows to eliminate AM. At this point, it is worth to mention that Arin and Iñarra (2001) characterize the egalitarian solution on the domain of convex games by CS and RS. Therefore, RS and SBR are equivalent in the presence of CS. Hence, on convex games, a core allocation satisfies SBR if a positive difference in the payoffs between two players can only occur when any transfer from the richer to the poorer player produces an unstable allocation, i.e., an allocation that is not in the core.

Moreover, we prove that, on the domain of convex games, DM-consistency for two-person reduced games, called *bilateral* DM-consistency (2-DMC), implies CS. Furthermore, we show that bilateral HM-consistency (2-HMC) together with IR imply EF. These logical implications among properties allow us to revisit some of the well-established characterizations. We also show that the egalitarian solution can be characterized by *constrained welfare egalitarianism* (CWE) in the sense of Calleja et al. (2021) and either 2-DMC or 2-HMC and IR. We recall that CWE requires to distribute an additional amount obtained by the grand coalition to the poorer players making payoffs as equal as possible subject to nobody is worse off. Hence, CWE implies AM. However, egalitarianism may regard CWE, though stronger than AM, as even more appealing. Indeed, CWE prioritizes those players who received less before the grand coalition became richer.

To conclude, we investigate if our results are valid for some extensions of the egalitarian solution and on larger domains than convex games. Some incompatibilities arise when imposing the properties we adopt in our axiomatizations, highlighting the limits of the characterizations of the egalitarian solution presented in the literature as well.

The remainder of the paper is organized as follows. Section 2 contains preliminaries on games. In Section 3 we introduce properties of solutions. Section 4 contains our main results. Section 4.1 is devoted to the characterization results of the egalitarian solution with AM and without consistency. In Section 4.2 we provide alternative axiomatizations for a variable society of agents making use of 2-DMC and 2-HMC. In Section 5, we study the compatibility of the groups of properties used in our characterization results on some subdomains of balanced games.

2. Preliminaries

Let U be a set (the universe of potential players) and $\mathcal{N}$ be the set of coalitions in U (a *coalition* is a nonempty finite subset of U). Given $S, T \in \mathcal{N}$, we use $S \subset T$ to indicate strict inclusion, that is, $S \subseteq T$ and $S \neq T$. By $|S|$ we denote the cardinality of the coalition $S \in \mathcal{N}$. We assume that $|U| \geq 3$. Given $N \in \mathcal{N}$, let $\mathbb{R}^N$ stand for the set of all real functions on N. An element $x \in \mathbb{R}^N$, $x = (x_i)_{i \in N}$, is a payoff vector for N. For all $S \subseteq N$, $x(S) = \sum_{i \in S} x_i$, with the convention $x(\emptyset) = 0$. For each $x \in \mathbb{R}^N$ and $T \subseteq N$, x_T denotes the restriction of x to T: $x_T = (x_i)_{i \in T} \in \mathbb{R}^T$. Given $N \in \mathcal{N}$, for all $x, y \in \mathbb{R}^N$, $x \geq y$ if $x_i \geq y_i$ for all $i \in N$. For all $\alpha \in \mathbb{R}$, $\alpha_+ = \max\{0, \alpha\}$. For any two vectors $y, x \in \mathbb{R}^N$ with $y(N) = x(N)$, we say that *y weakly Lorenz dominates x*, denoted by $y \succeq_{\mathcal{L}} x$, if $\min\{y(S) \mid S \subseteq N, \ |S| = k\} \geq \min\{x(S) \mid S \subseteq N, \ |S| = k\}$, for all $k = 1, 2, \ldots, n-1$. We say that *y Lorenz dominates x*, denoted by $y \succ_{\mathcal{L}} x$, if at least one of the above inequalities is strict.

A *transferable utility cooperative game (a game)* is a pair (N, v) where $N \in \mathcal{N}$ is the set of players and $v : 2^N \longrightarrow \mathbb{R}$ is the characteristic function that assigns to each $S \subseteq N$ a real number $v(S)$, with $v(\emptyset) = 0$. Given a game (N, v) and $\emptyset \neq N' \subset N$, the *subgame* associated to N' is denoted by $\left(N', v\right)$. The *anti-dual* of a game (N, v) is the game (N, v^{ad}) defined by $v^{ad}(S) = v(N \setminus S) - v(N)$ for each $S \subseteq N$. For a game (N, v), define

$$X^*(N, v) = \{x \in \mathbb{R}^N \mid x(N) \leq v(N)\} - \text{the set of } \textit{feasible} \text{ payoff vectors},$$
$$C(N, v) = \{x \in X^*(N, v) \mid x(S) \geq v(S) \ \forall\, S \subseteq N\} - \text{the } \textit{core}.$$

A game (N, v) is *balanced* if it has a non-empty core. We denote by Γ_{bal} the set of balanced games. A well-known subset of balanced games is the set of convex games. A game (N, v) is *convex* if, for every $S, T \subseteq N$, $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$. We denote by Γ_{conv} the set of all convex games and, for any coalition $N \in \mathcal{N}$, by Γ^N_{bal} and Γ^N_{conv} the subsets of games in Γ_{bal} and Γ_{conv} with N as player set, respectively. Oishi et al. (2016) showed that the domains of balanced games and convex games are closed under anti-duality. For $t \in \mathbb{R}$ and any game (N, v), we define the game (N, v^t) as $v^t(S) = v(S)$ for all $S \subset N$ and $v^t(N) = v(N) + t$. Note that (N, v^t) remains balanced (convex) if (N, v) is balanced (convex) and $t > 0$. Any $x \in \mathbb{R}^N$ defines the *inessential* game $(N, v) \in \Gamma_{conv}$ by $v(S) = \sum_{i \in S} x_i$.

A *solution* on Γ_{bal} is a mapping σ that assigns an element $\sigma(N, v)$ of $X^*(N, v)$ to any $(N, v) \in \Gamma_{bal}$. The restriction of a solution σ to a set $\Gamma \subseteq \Gamma_{bal}$ is again denoted by σ. Moreover, a solution on $\Gamma \subseteq \Gamma_{bal}$ is the restriction to Γ of some solution. Notice that we do not consider multi-valued solutions. Let σ be a solution on Γ and σ' be a solution on $\Gamma' = \{(N, v^{ad}) \mid (N, v) \in \Gamma\}$. Then σ and σ' are *anti-dual* to each other if, for all $(N, v) \in \Gamma$, $\sigma'(N, v^{ad}) = -\sigma(N, v)$. If, moreover, $\Gamma = \Gamma'$ and $\sigma = \sigma'$, then σ is *self anti-dual*. A solution σ on Γ satisfies

- *Efficiency* (EF) if for all $(N, v) \in \Gamma$, $\sum_{i \in N} \sigma_i(N, v) = v(N)$.

EF (or Pareto optimality) requires to distribute the entire worth of the grand coalition. Two *properties* P and P' are *anti-dual* to each other if a solution satisfies P (on a domain Γ) if and only if its anti-dual solution (on the domain Γ') satisfies P'. If, moreover, P coincides with P', then P is *self anti-dual*.

The following well-known lemma (the proof of which is included below for the benefit of the reader) allows to recall the formula that determines the egalitarian solution of Dutta and Ray (1989) on the domain of convex games.

Lemma 1. *Let $(N, v) \in \Gamma_{conv}$ and denote $\mu = \max_{\emptyset \neq S \subseteq N} \frac{v(S)}{|S|}$. If $\emptyset \neq S, T \subseteq N$ are such that $v(S) = \mu|S|$ and $v(T) = \mu|T|$, then $v(S \cup T) = \mu|S \cup T|$ and $v(S \cap T) = \mu|S \cap T|$.*

Proof. Note that, by definition of μ, $v(S \cup T) \leq \mu|S \cup T|$ and $v(S \cap T) \leq \mu|S \cap T|$. Hence, by convexity of (N, v), $\mu(|S \cup T| + |S \cap T|) = \mu|S \cup T| + \mu|S \cap T| \geq v(S \cup T) + v(S \cap T) \geq v(S) + v(T) = \mu(|S| + |T|) = \mu(|S \cup T| + |S \cap T|)$ so that the inequalities must be equalities. □

[4] These properties are inspired by the notion of stable sets, but changing the usual order in $\mathbb{R}^N$ for the Lorenz order. See Llerena and Mauri (2017) for formal definitions.

[5] See also Hougaard et al. (2005) for a generalization of Dutta-Ray's egalitarian solution on the domain of convex games satisfying monotonicity properties.

Let $(N, v) \in \Gamma_{conv}$ and denote

$$\mu(v) = \max_{\emptyset \neq S \subseteq N} \frac{v(S)}{|S|} \text{ and } S(v) = \bigcup \{S \in 2^N \setminus \{\emptyset\} \mid v(S) = \mu(v)|S|\}.$$

By Lemma 1, $\mu(v)|S(v)| = v(S(v))$. Now, we are able to introduce the egalitarian solution of (N, v), denoted by $E(N, v)$. Namely, let $(S_1, \ldots, S_m)$ be the ordered partition of N that is recursively determined by the requirement that $S_k = S(v_k)$, where $S_0 = \emptyset$ and for all $k = 1, \ldots, m$, $N_k = N \setminus \bigcup_{j=0}^{k-1} S_j$ and (N_k, v_k) is defined by $v_k(T) = v(T \cup (N \setminus N_k)) - v(N \setminus N_k)$ for all $T \subseteq N_k$. Note that $N_1 = N$, $v_1 = v$, and $(N_k, v_k) \in \Gamma_{conv}$ so that S_k is well defined. The egalitarian solution $E(N, v) = \{x^*\}$ is given by

$$x_i^* = \mu(v_k) = \frac{v_k(S_k)}{|S_k|} \text{ for all } i \in S_k \text{ and all } k = 1, \ldots, m. \tag{1}$$

Remark 1. Let $(N, v) \in \Gamma_{conv}$ and $x^* = E(N, v)$. Let $(S_1, \ldots, S_m)$ be the ordered partition of N *induced* by x^* that is defined by

$$\begin{aligned} x_i^* &= x_j^* \text{ for all } i, j \in S_k \text{ and all } k = 1, \ldots, m; \\ x_i^* &> x_j^* \text{ for all } i \in S_t, j \in S_k \text{ and all } 1 \leq t < k \leq m; \\ \bigcup_{t=1}^m S_t &= N. \end{aligned}$$

Then the allocation x^* satisfies

$$\sum_{t=1}^{k} x^*(S_t) = v\left(\bigcup_{t=1}^{k} S_t\right) \text{ for all } k = 1, \ldots, m.$$

Moreover, according to Theorem 3 of Dutta and Ray (1989), the egalitarian solution E selects the unique core element that Lorenz dominates every other core element. That is, $x^* \in C(N, v)$ and $x^* \succ_{\mathcal{L}} y$ for all $y \in C(N, v) \setminus \{x^*\}$.

The equal split solution, ED, is defined by $ED_i(N, v) = v(N)/|N|$ for all $i \in N$ and all games $(N, v) \in \Gamma_{bal}$. Note that if $ED(N, v) \in C(N, v)$, then[6] $E(N, v) = ED(N, v)$. The egalitarian solution and the equal split solution are self anti-dual on the domain of convex games.

3. Properties of solutions

In this section we introduce the properties employed in our new characterizations of E. Except for the axiomatic approach in Arin et al. (2003), the other characterizations found in the literature impose one of the properties in at least two of the following four groups of axioms.

The following stability properties used to axiomatize E are prominent. They aim to avoid providing players with incentives for blocking cooperation. So far, all existing characterizations of E impose, directly or indirectly, one of these stability requirements. Let $\Gamma \subseteq \Gamma_{bal}$. A solution σ on Γ satisfies

- *Individual rationality* (IR) if for all $(N, v) \in \Gamma$ and all $i \in N$, $\sigma_i(N, v) \geq v(\{i\})$;
- *Core selection* (CS) if for all $(N, v) \in \Gamma$, $\sigma(N, v) \in C(N, v)$;
- *Equal division stability* (EDS) if for all $(N, v) \in \Gamma$ and all $\emptyset \neq S \subseteq N$, there exists $i \in S$ with $\sigma_i(N, v) \geq \frac{v(S)}{|S|}$.

IR imposes that no single player can improve the payoff proposed by the solution without cooperating, while CS is a sort of secession-proofness property that extends this requirement to any coalition, i.e., no coalition worth is higher than the payoff to that coalition proposed by the solution. Under EF, EDS is equivalent to imposing the solution to select a payoff vector from the *equal division core* (Selten, 1972),[7] which implies that the proposed payoff vector cannot be blocked by any coalition using its equal division allocation. Note that since for balanced games the core is non-empty and, moreover, it is a subset of the equal division core, on $\Gamma \subseteq \Gamma_{bal}$, CS implies EDS and EDS implies IR. Except for the first characterizations of Dutta (1990), all remaining axiomatic approaches impose one of the foregoing stability properties or its anti-dual.

The second group, the "egalitarian bounds" properties, which can be interpreted as solidarity requirements, establishes certain egalitarian bounds that fix thresholds for the payoffs of some particular coalitions. Let $\Gamma \subseteq \Gamma_{bal}$. A solution σ on Γ satisfies

- *Rich are strong* (RS) if for all $(N, v) \in \Gamma$, $x_i < x_j$ implies $s_{ji}(x, v) \geq (s_{ij}(x, v))_+$ where $x = \sigma(N, v)$ and, for all $k, \ell \in N$, $s_{k\ell}(x, v) = \max\{v(S) - x(S) \mid k \in S \subseteq N \text{ and } \ell \notin S\}$ is called the *maximum surplus* of k over ℓ at x;
- *Bounded maximum payoff* (BMP) if for all $(N, v) \in \Gamma$, $\sum_{i \in S^{max}} \sigma_i(N, v) \leq v(S^{max})$ where $S^{max} = \arg\max_{j \in N} \sigma_j(N, v)$;
- *Bounded richness* (BR) if for all $(N, v) \in \Gamma$, $\sum_{i \in N \setminus S^{min}} \sigma_i(N, v) \leq v(N \setminus S^{min})$, where $S^{min} = \arg\min_{j \in N} \sigma_j(N, v)$;
- *Strong bounded richness* (SBR) if for all $(N, v) \in \Gamma$, $\sum_{i \in N \setminus S} \sigma_i(N, v) \leq v(N \setminus S)$ for all $\alpha \in \mathbb{R}$, where $S = \{i \in N \mid \sigma_i(N, v) < \alpha\}$.

RS and BMP (or its anti-dual) are used in many characterizations of E. BR and SBR are new. RS applies to any pair of agents $i, j \in N$, and it requires that the solution may only assign a larger payoff to player j compared to player i if j is *stronger* than i, i.e., if the maximum surplus of j over i, $s_{ji}(x, v)$, is positive and larger than i's maximum surplus over j. BMP imposes that the payoff to the coalition of players with the highest payoff does not exceed the worth of this coalition, while BR imposes this upper bound for the payoff to the coalition of all non-poorest players. SBR strengthens BR (and also implies BMP) by replacing poorest players by players who are poorer than any wealth level $\alpha \in \mathbb{R}$. This axiom admits an interpretation in terms of individual complaints. Indeed, a solution σ satisfies SBR if, for all $(N, v) \in \Gamma$ and all $i \in N$, $x(S^i) \leq v(S^i)$, where $x = \sigma(N, v)$ and $S^i = \{j \in N \mid x_j > x_i\}$, because the inequality $x(N) \leq v(N)$ induced by $\alpha \leq \min_{i \in N} x_i$ is automatically satisfied by feasibility of σ. This means that no player can request an amount from the coalition of richer players because this coalition already does not receive more than its coalitional worth. Alternatively, under EF, it is the same to impose that the coalition of players who receive at most the payoff of some player $i \in N$ obtains already an amount that is not smaller than its contribution to the coalition of all remaining (richer) players, i.e., $v(N) - v(S^i) \leq x(N \setminus S^i)$. For games with at most two person, BMP, BR, and SBR are equivalent, and RS implies each of the three mentioned properties. On a domain of balanced games with at most two person, under EF, all properties are equivalent. Due to the imposed upper bounds all these properties prioritize the social goal of equality over selfishness.

In the original characterizations of E, Dutta (1990) uses a property that applies to two-person games only. Let $\Gamma \subseteq \Gamma_{bal}$. A solution σ on Γ satisfies

6 For the definition of $E(N, v)$ for an arbitrary game (N, v) see Dutta and Ray (1989).

7 For any game (N, v), the equal division core is defined by

$$EDC(N, v) = \left\{ x \in \mathbb{R}^N \,\middle|\, \begin{array}{l} x(N) = v(N) \text{ and } \forall\, \emptyset \neq S \subseteq N \\ \text{there is } i \in S \text{ such that } x_i \geq \frac{v(S)}{|S|} \end{array} \right\}.$$

- *Constrained egalitarianism* (CE) if for all $(N, v) \in \Gamma$ with $N = \{i, j\}$, $i \neq j$, such that $v(\{i\}) \leq v(\{j\})$, $\sigma_j(N, v) = \max\left\{\frac{v(N)}{2}, v(\{j\})\right\}$ and $\sigma_i(N, v) = v(N) - \sigma_j(N, v)$;

CE forces to select the egalitarian solution E for two-person games, which divides the worth of the grand coalition as equal as possible preserving IR. Therefore, it can be interpreted as a stability property as well as an egalitarian bounds property.

The third group, extensively used in the literature, are consistency properties, which require internal stability when some players leave, thereby referring to suitable notions of reduced games. Let $\Gamma \subseteq \Gamma_{bal}$. A solution σ on Γ satisfies

- *DM-consistency* (DMC) if for all $(N, v) \in \Gamma$ and all $\emptyset \neq S \subset N$, $(S, v_{S,x}) \in \Gamma$ and $\sigma(S, v_{S,x}) = x_S$, where $x = \sigma(N, v)$ and $(S, v_{S,x})$ is the game defined by

  $$v_{S,x}(T) = \max_{Q \subseteq N \setminus S} \{v(T \cup Q) - x(Q)\} \text{ for all } \emptyset \neq T \subset S$$

  and[8]

  $$v_{S,x}(S) = v(N) - x(N \setminus S).$$

- *HM-consistency* (HMC) if for all $(N, v) \in \Gamma$ and all $\emptyset \neq S \subset N$, $(S, v_{S,\sigma}) \in \Gamma$ and $\sigma(S, v_{S,\sigma}) = x_S$, where $x = \sigma(N, v)$ and $(S, v_{S,\sigma})$ is the game defined by[9]

$$v_{S,\sigma}(T) = v(T \cup (N \setminus S)) - \sum_{i \in N \setminus S} \sigma_i(T \cup (N \setminus S), v) \text{ for all } \emptyset \neq T \subseteq S.$$

  The *bilateral DM-consistency* (2-DMC) and *bilateral HM-consistency* (2-HMC) only require DMC and HMC when $|S| = 2$, respectively.

Note that if the solution is consistent and assigns x to a game (N, v), then, for every coalition $S \subset N$, the payoff allocation x_S solves the corresponding reduced game w.r.t. S and, therefore, it is consistent with the expectation of the members of S as reflected by this reduced game. The above definitions are due to Sobolev (1975) and Hart and Mas-Colell (1989), respectively. The egalitarian solution E satisfies DMC on Γ_{conv}. However, as was shown by Hokari (2002), it satisfies 2-HMC but violates HMC on Γ_{conv}.[10] So far, except for those of Arin and Iñarra (2001) and Arin et al. (2003), all remaining characterizations of E make use of one consistency property. Most of them impose DMC, HMC, weak versions (or variations) of these two,[11] or the corresponding anti-dual consistency property, but also complement consistency (Moulin, 1985) or its anti-dual, projection consistency (Funaki, 1998).

Some of our new characterization results combine properties of these three groups, but others invoke monotonicity properties w.r.t. the worth of the grand coalition, another type of requirements that, up to our knowledge, have not been employed before to characterize E. Let $\Gamma \subseteq \Gamma_{bal}$. A solution σ on Γ satisfies

- *Aggregate monotonicity* (AM) if for all $(N, v) \in \Gamma$ and all $t > 0$ such that $(N, v^t) \in \Gamma$, $\sigma(N, v^t) \geq \sigma(N, v)$;
- *Constrained welfare egalitarianism* (CWE) if for all $(N, v) \in \Gamma$, all $t > 0$ such that $(N, v^t) \in \Gamma$, and all $i \in N$, $\sigma_i(N, v^t) = \sigma_i(N, v) + (\lambda - \sigma_i(N, v))_+$, where λ is determined by $\sum_{k \in N}(\lambda - \sigma_k(N, v))_+ = t$.

Table 1
Solutions and properties on Γ_{conv}.

	E	Sh	ν	$\bar{\nu}$	ψ	CC	τ	ED	CIS	$ENSC$
IR	Yes	Yes	Yes	Yes	Yes	Yes	Yes	No	Yes	No
CS	Yes	Yes	Yes	Yes	Yes	Yes	No	No	No	No
EDS	Yes	Yes	Yes	Yes	Yes	Yes	No	No	No	No
CE	Yes	No	No	No	No	No	No	No	No	No
RS	Yes	No	No	No	No	No	No	Yes	No	No
BMP	Yes	No	No	No	No	No	No	Yes	No	No
BR	Yes	No	No	No	No	No	No	Yes	No	No
SBR	Yes	No	No	No	No	No	No	Yes	No	No
2-DMC	Yes	No	Yes	No	No	No	No	No	No	No
2-HMC	Yes	Yes	No	No	No	No	No	No	No	No
AM	Yes	Yes	No	Yes	No	No	No	Yes	Yes	Yes
CWE	Yes	No	No	No	No	No	No	Yes	No	No

AM requires that no player is worse off if only the worth of the grand coalition is increased. CWE implies to distribute an additional amount to the poorer players so that their final payoffs become equal but not larger than the remaining players' original payoffs. Though CWE is stronger than AM, it may be regarded as even more appealing from an egalitarian point of view.

In the following Table 1, the properties used to characterize E are listed, together with a number of well-established efficient solution concepts.[12] It shows which properties are satisfied by each solution on the domain of convex games. It also indicates that egalitarian bounds properties are clearly oriented to "egalitarian based" solutions in the sense of Lorenz, E and ED. In particular, a solution that satisfies the well-known property of covariance under strategic equivalence[13] automatically violates all egalitarian bounds properties. It also makes clear that stability conditions play a central role to distinguish the egalitarian solution E among these two. Regarding monotonicity properties, although many solution concepts satisfy AM, only E and ED satisfy CWE as well. We employ both to characterize E, the difference is that when imposing AM, a strong stability property and an egalitarian bounds property are needed additionally, while if we apply CWE we demand consistency only or consistency combined with a weak stability property.

Concerning AM, it should be noted that, on the domain of convex games, the nucleolus and the core-center satisfy this property if $|U| \leq 3$ and violate it, provided $|U| \geq 4$ (Hokari, 2000; Mirás-Calvo et al., 2021, respectively). Hokari and van Gellekom (2002) show that the τ-value is not AM either if $|U| \geq 7$. The proof that the modiclus violates AM on the domain of convex games, provided $|U| \geq 7$, is available upon demand from each of the authors.

[8] The game $(S, v_{S,x})$ is called *DM-reduced game* of (N, v) w.r.t. S at x and was introduced by Davis and Maschler (1965).

[9] The game $(S, v_{S,\sigma})$ is called the *HM-reduced game* of (N, v) w.r.t. S at σ and was introduced by Hart and Mas-Colell (1989). Note that the set of convex games Γ_{conv} is closed under taking subgames.

[10] Indeed, Hokari's (2002) Example 1 shows that the HM-reduced game (of 3 or more players) of a convex game w.r.t. E may not be convex.

[11] Usually the internal stability requirement is imposed only when some particular coalition(s) leave.

[12] In particular, we consider the Shapley value (Shapley, 1953), Sh, the nucleolus (Schmeidler, 1969), ν, the per-capita nucleolus (Grotte, 1970), $\bar{\nu}$, the modified nucleolus also known as modiclus (Sudhölter, 1996, 1997), ψ, the core-center (González-Díaz and Sánchez-Rodríguez, 2007), CC, the τ-value (Tijs, 1981), τ, the equal split solution, ED, the center of imputations (Driessen and Funaki, 1991), CIS, and the egalitarian non-separable contribution solution (Moulin, 1985), $ENSC$. We invite the reader to contact the authors for a detailed explanation (counterexamples) of the results in the table.

[13] A solution σ on $\Gamma \subseteq \Gamma_{bal}$ satisfies covariance under strategic equivalence if for all $(N, v) \in \Gamma$, all $\alpha > 0$ and all $d \in \mathbb{R}^N$, if $(N, w) \in \Gamma$ is such that $w(S) = \alpha\, v(S) + d(S)$ for all $S \subseteq N$, then $\sigma(N, w) = \alpha\, \sigma(N, v) + d$.

4. Characterization results

This section is devoted to characterize E on Γ_{conv} and it is divided into two parts. First, we characterize the egalitarian solution by means of AM and for a fixed society of agents. In the second part, we use consistency properties. Remarkably, we show that 2-DMC implies CS and 2-HMC together with IR imply EF. These logical consequences allow us to obtain new and compact axiomatic approaches.

4.1. Characterizations of the egalitarian solution without consistency

In this subsection, we connect AM with stability properties. It turns out that, combined with CS and either BR or SBR, AM yields new characterizations of E. Moreover, BMP is not strong enough to select E, while using RS provides the already known characterization of Arin and Iñarra (2001). Our first result makes use of CS, AM, and BR.

Theorem 1. *Let $N \in \mathcal{N}$. The egalitarian solution E on Γ^N_{conv} is the unique solution that satisfies* CS, AM, *and* BR.

Proof. It is well known that the egalitarian solution E satisfies CS and AM. Note that CS implies EF. By Remark 1 it also satisfies BR. To show uniqueness, let σ be a solution satisfying these properties. Let $(N, v) \in \Gamma^N_{conv}$. Denote $x = \sigma(N, v)$. By CS, $x \in C(N, v)$. Let $x^* = E(N, v)$ and $(S_1, \ldots, S_m)$ be the ordered partition of N such that $v(S_1 \cup \cdots \cup S_k) = x^*(S_1 \cup \cdots \cup S_k)$ for all $k = 1, \ldots, m$ (see Remark 1 and the preceding paragraph). It remains to show that $x = x^*$. Let $\alpha = \min\{x_i \mid i \in N\}$ and $S = \{i \in N \mid x_i = \alpha\}$. We proceed by induction on m.

If $m = 1$ then, by EF of x and x^*, $\alpha \leq \frac{v(N)}{|N|} = x^*_j$ for all $j \in N$. Hence, by BR and CS, $v(N \setminus S) = x(N \setminus S) = v(N) - x(S) \geq v(N) - x^*(S) = x^*(N \setminus S) \geq v(N \setminus S)$. We conclude that $x(S) = x^*(S)$, $S = N$ and $x = x^*$.

Induction hypothesis: $\sigma(N, v) = E(N, v)$ whenever $m < \ell$ for some $\ell \in \mathbb{N}$, $\ell > 1$.

We now assume that $m = \ell$. Put

$$t = |S_m| \left(\frac{v_{m-1}(S_{m-1})}{|S_{m-1}|} - \frac{v_m(S_m)}{|S_m|} \right) = |S_m| \left(x^*_k - x^*_h \right) > 0,$$

where $k \in S_{m-1}$ and $h \in S_m$.

Observe that $y^* \in \mathbb{R}^N$ defined by $y^*_i = \max\{x^*_i, v_{m-1}(S_{m-1})/|S_{m-1}|\}$ for all $i \in N$ is the egalitarian solution of (N, v^t). Hence, by induction hypothesis, $\sigma(N, v^t) = y^*$. By AM, $x \leq y^*$. By CS, Remark 1 implies $x_i = x^*_i$ for all $i \in N \setminus S_m$. By EF, $\alpha \leq x^*_i$ for all $i \in N$. Hence, by BR and CS, $v(N \setminus S) = x(N \setminus S) = v(N) - x(S) \geq v(N) - x^*(S) = x^*(N \setminus S) = v(N \setminus S)$. We conclude that $\alpha = \min\{x^*_i \mid i \in N\}$ and, hence, $S = S_m$ and $x = x^*$. □

Let $\Gamma \subseteq \Gamma_{bal}$. An efficient solution σ on Γ satisfies BR if and only if it satisfies the *bounded minimum payoff* property (Oishi et al., 2016), that is, for all $(N, v) \in \Gamma$,

$$\sum_{i \in S^{min}} \sigma_i(N, v) \geq v(N) - v(N \setminus S^{min}),$$

which is the anti-dual of BMP. Hence, one might expect that, on Γ_{conv}, CS, AM, and BMP characterize the egalitarian solution E. However, we now show that BR cannot be replaced by BMP in Theorem 1. To this end, define, for any $(N, v) \in \Gamma_{conv}$ with $|N| \geq 2$, the *convex root game* of (N, v), denoted by (N, v_r), as the convex game with smallest worth of the grand coalition such that $v_r(S) = v(S)$ for all $S \subset N$. That is, if $\Gamma_{conv}(v) = \{w \in \Gamma_{conv} | w(S) = v(S) \text{ for all } S \subset N\}$, $(N, v_r) \in \Gamma_{conv}(v)$ is such that $v_r(N) \leq w(N)$ for all $w \in \Gamma_{conv}(v)$. Note that (N, v_r) is well defined since $v_r(N) = \max\{v(S) + v(T) - v(S \cap T) \mid S, T \subset N,\ S \cup T = N\}$. Moreover, $v = v^t_r$ with $t = v(N) - v_r(N)$. In the following example we introduce a solution σ on Γ_{conv}, different from E, that satisfies CS, AM, and BMP.

Example 1. Define σ as follows. Let $(N, v) \in \Gamma_{conv}$ and $n = |N|$. If $n = 1$, then $\sigma(N, v) = E(N, v)$. If $n \geq 2$, put $t^* = n \max_{i \in N} E_i(N, v_r) - v_r(N) \geq 0$. Now, define $\sigma(N, v) = E(N, v) = ED(N, v)$ if $v(N) \geq v_r(N) + t^*$. Otherwise, if $v_r(N) \leq v(N) < v_r(N) + t^*$, define $\sigma(N, v) = (1 - \lambda)E(N, v_r) + \lambda E(N, v^{t^*}_r)$ where λ is determined by $\sum_{i \in N} \sigma_i(N, v) = v(N)$. Hence, if $t^* > 0$, $\lambda = {}^{(v(N) - v_r(N))}/_{t^*}$ so that $\sigma(N, v)$ is the vector on the line from $E(N, v_r)$ to $E(N, v^{t^*}_r)$ whose components sum up $v(N)$.

CS and AM follow from the observations that $E(N, v_r) \in C(N, v_r)$, E satisfies AM and λ is non-decreasing w.r.t. $v(N)$. In order to check BMP, we may assume $n \geq 2$ and $v_r(N) < v(N) < v_r(N) + t^*$ because E satisfies BMP. Now, $\sigma_i(N, v^{t^*}_r) = ED_i(N, v^{t^*}_r) = \max_{j \in N} E_j(N, v_r)$ for all $i \in N$ and, consequently, S^{max} according to $\sigma(N, v_r)$ is the same as S^{max} according to $\sigma(N, v)$. Thus, for every player $i \in S^{max}$, we have $\sigma_i(N, v) = \max_{j \in N} E_j(N, v_r) = E_i(N, v_r)$.

To see that $\sigma \neq E$, let $n = 3$, say $N = \{1, 2, 3\}$, $x = (2, 1, 0) \in \mathbb{R}^N$, and (N, v) the inessential game generated by x. Then, $v = v_r$ and $\sigma(N, v) = E(N, v) = x$. Moreover, $t^* = 6 - 3 = 3$ and $\sigma(N, v^3) = E(N, v^3) = (2, 2, 2)$. By construction, $\sigma(N, v^1) = (2, 4/3, 2/3) \neq (2, 1, 1) = E(N, v^1)$. In case $n > 3$, a similar example can be constructed adding null players.[14]

Nevertheless, a new characterization of E emerges combining CS and BMP with the anti-dual property of AM. To this end, for each $(N, v) \in \Gamma_{bal}$ and $t > 0$, we first introduce the *t-diminished game* $(N, v_{-t}) \in \Gamma_{bal}$ defined by $v_{-t}(S) = v(S) - t$, for all $\emptyset \neq S \subseteq N$. Let $\Gamma \subseteq \Gamma_{bal}$. A solution σ on Γ satisfies

- *Adjusted aggregate monotonicity* (AAM) if for all $(N, v) \in \Gamma$ and all $t > 0$ such that $(N, v_{-t}) \in \Gamma$, $\sigma(N, v_{-t}) \leq \sigma(N, v)$.

AAM requires that if the worth of each coalition decreases by the same amount $t > 0$, then no player in the t-diminished game should be better off.

Proposition 1. *AM and* AAM *are anti-dual to each other.*

Proof. Let σ be a solution on a domain $\Gamma \subseteq \Gamma_{bal}$ of games and σ' its anti-dual solution on $\Gamma' = \{(N, v^{ad}) \mid (N, v) \in \Gamma\}$. Let $(N, v) \in \Gamma'$ and $t > 0$ such that $(N, v_{-t}) \in \Gamma'$. Let $w = v^{ad}$. Then $v = w^{ad}$ and $(w^t)^{ad} = v_{-t}$, hence $(N, w), (N, w^t) \in \Gamma$. Now, suppose that σ satisfies AM. Then $\sigma'(N, v_{-t}) = -\sigma(N, w^t) \leq -\sigma(N, w) = \sigma'(N, v)$, which shows that σ' satisfies AAM. Conversely, if σ' satisfies AAM, $(N, w) \in \Gamma$, $t > 0$, and $(N, w^t) \in \Gamma$, then, with $v = w^{ad}$, $(w^t)^{ad} = v_{-t}$. Hence, $\sigma(N, w^t) = -\sigma'(N, v_{-t}) \geq -\sigma'(N, v) = \sigma(N, w)$, which proves that σ satisfies AM. □

It is not difficult to check that CS is self anti-dual. This fact, together with Proposition 1 and Theorem 1, lead to the following characterization result.

Theorem 2. *Let $N \in \mathcal{N}$. The egalitarian solution E on Γ^N_{conv} is the unique solution that satisfies* CS, AAM, *and* BMP.

Remark 2. The following examples show that each of the properties employed in Theorems 1 and 2 is logically independent of the remaining properties, provided $|N| \geq 3$. If $|N| \leq 2$, BR (BMP) and CS imply CE and, consequently, AM (AAM).

[14] A player $i \in N$ is a null player in a game (N, v) if $v(S \cup i) = v(S)$ for all $S \subseteq N \setminus \{i\}$.

(i) The equal split solution ED satisfies AM, AAM, BR, and BMP but not CS.
(ii) Let i, j be two distinct elements of N and define the game (N, u) by $u(S \cup \{i, j\}) = 1$ for all $S \subseteq N \setminus \{i, j\}$ and $u(S) = 0$, otherwise. Now define the solution σ as follows: $\sigma(N, v) = E(N, v)$ for all $(N, v) \in \Gamma^N_{conv}$ with $v \neq u$ and $\sigma_i(N, u) = 2/3$, $\sigma_j(N, u) = 1/3$, $\sigma_k(N, u) = 0$ for all $k \in N \setminus \{i, j\}$. Then, σ satisfies CS and BR (because $|N| > 2$), but not AM. The anti-dual solution of σ satisfies CS and BMP, but not AAM.
(iii) Let $\prec$ be a strict total order on N and $\preceq$ its reflexive cover. For all $(N, v) \in \Gamma^N_{conv}$ and all $i \in N$ define the *marginal contribution solution relative to* $\prec$ as follows: $mc^{\prec}_i(N, v) = v(\{j \in N \mid j \preceq i\}) - v(\{j \in N \mid j \prec i\})$. Then, $mc^{\prec}$ satisfies CS and AM but not BR. The anti-dual solution of $mc^{\prec}$ satisfies CS and AAM, but not BMP.

We now strengthen BR into SBR in Theorem 1, and it turns out that CS and SBR imply AM. Hence, these two properties alone characterize the egalitarian solution E.

Theorem 3. *Let $N \in \mathcal{N}$. The egalitarian solution E on Γ^N_{conv} is the unique solution that satisfies* CS *and* SBR*.*

Proof. Indeed, E satisfies CS and SBR by Remark 1. To show uniqueness, let σ be a solution satisfying CS and SBR. Let $(N, v) \in \Gamma^N_{conv}$. Denote $x = \sigma(N, v)$ and let $x^* = E(N, v)$ and $(S_1, \ldots, S_m)$ be the ordered partition of N such that $v(S_1 \cup \cdots \cup S_k) = x^*(S_1 \cup \cdots \cup S_k)$ for all $k = 1, \ldots, m$ (see Remark 1 and the preceding paragraph). It remains to show that $x = x^*$. Assume, on the contrary, $x \neq x^*$. Let m be minimal such that there exists $i \in S_m$ with $x_i < x^*_i =: \alpha$. Let $S = \{j \in N \mid x_j < \alpha\}$ and $T = N \setminus S$. Hence $T \supseteq \bigcup_{k=1}^{m-1} S_k$ and $x_j \geq x^*_j$ for all $j \in T$. By SBR and CS, $x(T) = v(T) = x^*(T)$, hence $x_j = x^*_j \geq \alpha$ for all $j \in T$. As $i \in S_m \setminus T$, $T \subset \bigcup_{k=1}^m S_k$, hence $x\left(\bigcup_{k=1}^m S_k\right) < x^*\left(\bigcup_{k=1}^m S_k\right) = v\left(\bigcup_{k=1}^m S_k\right)$, and a contradiction to CS is obtained. □

Remark 3. The following examples show that each of the properties employed in Theorem 3 is logically independent of the remaining properties, provided $|N| \geq 2$.

(i) The equal split solution ED satisfies SBR but not CS.
(ii) The marginal contribution solution $mc^{\prec}$ satisfies CS but not SBR.

Arin and Iñarra (2001) characterize E on Γ^N_{conv} by CS and RS.[15] Hence, under CS, RS and SBR are equivalent on the domain of convex games. The following two examples show that, even under EF, SBR does not imply and is not implied by RS (provided that $|U| \geq 4$).

Example 2. Let $|N| = 3$, say $N = \{1, 2, 3\}$, (N, v) the inessential game generated by $(0, 1, 4)$, and $x = (0, 2, 3)$. Then $x \in X(N, v)$, $x_2 + x_3 = v(\{2, 3\})$, and $x_3 \leq v(\{3\})$. Hence, the solution that assigns x to (N, v) and coincides with E for all other games satisfies EF and SBR. However, the inequality $s_{12}(x, v) = v(\{1, 3\}) - x_1 - x_3 = 1 > 0 = s_{21}(x, v)$ shows that it does not satisfy RS.

Example 3. Let $|N| = 4$, say $N = \{1, \ldots, 4\}$, and (N, v) be the convex game defined by $v(\{1\}) = 5, v(\{2\}) = v(\{3\}) = 4, v(\{1, 2\}) = v(\{1, 3\}) = 13, v(\{2, 3\}) = 12, v(\{1, 2, 3\}) = 21$, and $v(S \cup \{4\}) = v(S)$ for all $S \subseteq N$.[16] Now, observe that $x = (6, 5, 5, 5) \in X(N, v)$, $s_{1j}(x, v) = s_{j1}(x, v) = 2$ for $j = 2, 3$, $s_{14}(x, v) = 5$, and $s_{4,1}(N, v) < 0$. Hence, the solution that assigns x to (N, v) and coincides with E for all other games satisfies EF and RS. As $x_1 > v(\{1\})$, it does neither satisfy BR nor SBR nor BMP.

To conclude this subsection, we show that EDS (and thus IR) does not replace CS in Theorems 1, 2, and 3. To this end, we introduce a modification $\widetilde{\sigma}$ of the solution σ in Example 1 that satisfies EDS (and thus IR), AM, AAM, and SBR (and thus BR and BMP). The definition of $\widetilde{\sigma}(N, v)$ differs from the definition of $\sigma(N, v)$ only if $n = |N| \geq 2$ and $v_r(N) < v(N) < v_r(N) + t^*$, in which case we define $\widetilde{\sigma}(N, v) = E(N, v_r)$. Note that $\widetilde{\sigma} \neq E$ since it does not satisfy EF, and thus neither CS. SBR and thus BR and BMP come directly from the fact that E satisfies SBR and $v(S) = v_r(S)$ for all $S \subset N$ while $v(N) \geq v_r(N)$. IR and AM are straightforward since $E(N, v_r) \in C(N, v_r)$ and nobody is worse off when increasing the value of the grand coalition. To show EDS, we may restrict our attention to the case $v_r(N) < v(N) < v_r(N) + t^*$. In this case, $ED(N, v) \notin C(N, v)$, and thus there exist $S \subset N$ and $i \in S$ such that $E_i(N, v_r) \geq v_r(S)/|S| = v(S)/|S| > v(N)/|N|$, which proves EDS. Finally, it can be shown that the solution $\widetilde{\sigma}$ is self anti-dual and, hence, it also satisfies AAM, which invalidates to weaken CS in Theorem 2.

4.2. Characterizations of the egalitarian solution with consistency

In this subsection, and making use of some lemmas that point out logical relations among consistency, stability and monotonicity properties, we obtain new axiomatizations of E. The first lemma shows that 2-DMC implies CS on the domain of convex games with at least two players, denoted by $\Gamma^{\geq 2}_{conv}$.

Lemma 2. *If the solution σ on $\Gamma^{\geq 2}_{conv}$ satisfies* 2-DMC*, then it satisfies* CS *as well.*

Proof. Let $(N, v) \in \Gamma^{\geq 2}_{conv}$. We consider two cases:

(i) $|N| = 2$. By the assumption $|U| \geq 3$ there exists $k \in U \setminus N$. Let $M = N \cup \{k\}$ and (M, w) be the game that arises from (N, v) by adding the null player k, i.e., w is given by $w(S) = v(S \cap N)$ for all $S \subseteq M$. Note that (M, w) is still convex.
Claim: If (N, v) is inessential, then $\sigma(N, v)$ is the unique element of $C(N, v)$.
In order to show the claim, note that (M, w) is inessential. Let $y \in \mathbb{R}^M$ be defined by $y_i = w(\{i\})$ for all $i \in M$, hence $y(S) = w(S)$ for all $S \subseteq M$. Moreover, let $x = \sigma(M, w)$. For any $i \in M$, by the definition of the Davis–Maschler reduced game, $w_{M\setminus\{i\},x}(\{j\}) \geq w(\{i, j\}) - x_i = y(\{i, j\}) - x_i$ for both $j \in M \setminus \{i\}$ and $w_{M\setminus\{i\},x}(M \setminus \{i\}) = w(M) - x_i = y(M) - x_i$. By 2-DMC, $(M \setminus \{i\}, w_{M\setminus\{i\},x})$ is convex so that $\sum_{j\in M\setminus\{i\}} w_{M\setminus\{i\},x}(\{j\}) \leq w_{M\setminus\{i\},x}(M \setminus \{i\})$. We conclude that
$$\sum_{j\in M\setminus\{i\}} [y(\{i, j\}) - x_i] = y(M) + y_i - 2x_i \leq y(M) - x_i,$$
hence $x_i \geq y_i$ for all $i \in M$. Now, as $x(M) \leqslant w(M) = y(M)$, we have $x = y$. Finally, since $(N, w_{N,y}) = (N, v)$, by 2-DMC, $x_N = y_N = \sigma(N, v)$ with $C(N, v) = \{x_N\}$.
Now let $x = \sigma(M, w)$, $i \in N$, and $N = \{i, j\}$. By 2-DMC, $(M \setminus \{i\}, w_{M\setminus\{i\},x})$ is convex and $x_{M\setminus\{i\}} = \sigma(M \setminus \{i\}, w_{M\setminus\{i\},x})$. By definition of the Davis–Maschler reduced game,
$$w_{M\setminus\{i\},x}(\{j\}) = \max\{w(\{j\}), w(\{i, j\}) - x_i\} = \max\{v(\{j\}), v(N) - x_i\},$$
$$w_{M\setminus\{i\},x}(\{k\}) = \max\{w(\{k\}), w(\{i, k\}) - x_i\} = \max\{0, v(\{i\}) - x_i\},$$
and
$$w_{M\setminus\{i\},x}(M \setminus \{i\}) = w(M) - x_i = v(N) - x_i$$

[15] In fact, they show that the *egalitarian core*, defined to be the core elements of a balanced game satisfying RS, coincides with E for convex games.

[16] The game (N, v) arises adding the null player 4 to the 3-person convex game $(\{1, 2, 3\}, w)$ defined by: $w(\{1\}) = 5, w(\{2\}) = w(\{3\}) = 4, w(\{1, 2\}) = w(\{1, 3\}) = 13, w(\{2, 3\}) = 12$, and $w(\{1, 2, 3\}) = 21$. Hence, (N, v) is convex.

so that 2-DMC implies $v(N) - x_i \geq \max\{v(\{j\}), v(N) - x_i\} + \max\{0, v(\{i\}) - x_i\}$. Hence, $x_i \geq v(\{i\})$ and $v(N) - x_i \geq v(\{j\})$. We conclude that $(M \setminus \{i\}, w_{N\setminus\{i\},x})$ is inessential and thus, by 2-DMC and our claim, $x_j = v(N) - x_i$ and $x_k = 0$. Therefore, $x_N \in C(N, v)$ and the proof is finished by 2-DMC.

(ii) $|N| \geq 3$. Let $x = \sigma(N, v)$ and assume that $x \notin C(N, v)$. If $x(N) < v(N)$ select any $S \subseteq N$ with $|S| = 2$. By 2-DMC, $(S, v_{S,x}) \in \Gamma^{\geq 2}_{conv}$ and $x_S = \sigma(S, v_{S,x})$. Now $v_{S,x}(S) = v(N) - x(N \setminus S) > x(S)$ so that $x_S \notin C(S, v_{S,x})$ which contradicts case (i). Therefore, we may assume that $x(N) = v(N)$ and $x(T) < v(T)$ for some $\emptyset \neq T \subsetneq N$ so that there exist $i \in T$ and $j \in N \setminus T$. Let $S = \{i, j\}$ and observe that $v_{S,x}(\{i\}) \geq v(T) - x(T \setminus \{i\}) > x_i$ by definition of the Davis–Maschler reduced game. Therefore x_S is not individually rational for $(S, v_{S,x})$ and the desired contradiction is obtained by 2-DMC and case (i). □

Remark 4. Lemma 2 does not hold on the domain of all convex games, including all 1-person games. Indeed, let $(N, v) \in \Gamma_{conv}$ and $\varepsilon > 0$. Define the single-valued solution ρ as follows: $\rho(N, v) = E(N, v)$ if $|N| \geq 2$, and $\rho(N, v) = v(N) - \varepsilon$ otherwise. Then, ρ satisfies 2-DMC but not CS.

Since, on $\Gamma^{\geq 2}_{conv}$, CS implies EF, EDS, and IR, an immediate consequence of Lemma 2 is the following corollary.

Corollary 1. *If the solution σ on $\Gamma^{\geq 2}_{conv}$ satisfies* 2-DMC*, then it satisfies* EF, EDS*, and* IR *as well.*

Next, we show that IR combined with the strong aggregate monotonicity property of CWE imply CE.

Lemma 3. *If the solution σ on Γ_{conv} satisfies* IR *and* CWE*, then it satisfies* CE *as well.*

Proof. Let (N, v) be a two person convex game with $N = \{i, j\}$, $i \neq j$ and $v(\{i\}) \leq v(\{j\})$. Let $t = v(N) - v(\{i\}) - v(\{j\}) \geq 0$. By IR and CWE, for all $k \in N$, $\sigma_k(N, v) = \max\{\lambda, v(\{k\})\}$, where $\lambda \in \mathbb{R}$ is determined by $\sum_{k\in N}(\lambda - v(\{k\}))_+ = t$. If $t > v(\{j\}) - v(\{i\})$, then $\lambda = \frac{v(N)}{2} > v(\{j\}) \geq v(\{i\})$ and hence $\sigma_j(N, v) = \sigma_i(N, v) = \frac{v(N)}{2}$. If $t \leq v(\{j\}) - v(\{i\})$, then $\lambda = v(N) - v(\{j\}) < v(\{j\})$ and thus $\sigma_j(N, v) = v(\{j\})$ and $\sigma_i(N, v) = v(N) - v(\{j\})$. In both cases, $\sigma(N, v) = CE(N, v)$. □

Lemma 3.2 of Klijn et al. (2000) shows that EF, EDS, and BMP imply CE. It is straightforward and left to the reader to check that EF, IR, and BR (or RS) together also imply CE. Theorem 5.3 of Dutta (1990) characterizes the egalitarian solution E by means of CE and DMC. In fact, in the uniqueness part of his proof, Dutta only used 2-DMC rather than DMC.[17] Moreover, Calleja et al. (2021) showed that the egalitarian solution E satisfies CWE. Combining these results with Corollary 1 and Lemma 3 we obtain the following new characterizations.

Theorem 4. *On the domain $\Gamma^{\geq 2}_{conv}$,*

(i) the egalitarian solution E is the unique solution that satisfies 2-DMC *and* BR*.*
(ii) the egalitarian solution E is the unique solution that satisfies 2-DMC *and* BMP*.*
(iii) the egalitarian solution E is the unique solution that satisfies 2-DMC *and* CWE*.*

In view of Lemma 2, it is worth to point out that, on $\Gamma^{\geq 2}_{conv}$, imposing 2-DMC allows to drop the monotonicity property (AM or AAM) in Theorems 1 and 2, respectively. Obviously, Theorem 3 and the characterization in Arin and Iñarra (2001) can be rewritten imposing 2-DMC, which is stronger than CS. Moreover, because of Lemma 3, under 2-DMC, the properties imposing egalitarian bounds can be replaced by CWE.

Remark 5. Each of the properties in Theorem 4 is logically independent of the remaining properties.

(i) The equal split solution *ED* satisfies BMP, BR, and CWE but not 2-DMC.
(ii) Schmeidler's (1969) nucleolus, ν,[18] satisfies 2-DMC but neither BMP nor BR nor CWE.

Note that, in view of Remark 4, none of the characterizations of E presented in Theorem 4 hold when expanding the domain of convex games with at least 2 players to the domain of all convex games, including all 1-person games. A way to extend the results to the entire domain Γ_{conv} is to impose, additionally and only for 1-person games, IR (or EF).

The remainder of this section is devoted to the question to what extent the characterizations in Theorem 4 still hold if we replace 2-DMC by 2-HMC. We do not know if, on the domain of convex games with at least two players, 2-HMC implies CS. However, on the full domain of convex games, if we additionally impose IR, then we can show that EF is also satisfied. We finally deduce that we can replace 2-DMC by 2-HMC in the modified version of Theorem 4 that works on the domain of all convex games when employing IR in addition.

Lemma 4. *If the solution σ on Γ_{conv} satisfies* IR *and* 2-HMC*, then it satisfies* EF *as well.*

Proof. Let $(N, v) \in \Gamma_{conv}$. If $|N| = 1$, the proof is finished by IR (and feasibility).

If $|N| = 2$, by the assumption $|U| \geq 3$ there exists $k \in U \setminus N$. Let $M = N \cup \{k\}$ and (M, w) be the game that arises from (N, v) by adding the null player k, i.e., w is given by $w(S) = v(S \cap N)$ for all $S \subseteq M$. Note that (M, w) is still convex. Recall that, if (N, v) is inessential, then $\sigma(N, v)$ is the unique element of $C(N, v)$ by IR (and feasibility). Let $x = \sigma(M, w)$, $i \in N$, and $N = \{i, j\}$. Then $w_{M\setminus\{i\},\sigma}(\{j\}) = v(N) - \sigma_i(N, v)$ and $w_{M\setminus\{i\},\sigma}(\{k\}) = v(\{i\}) - \sigma_i(\{i, k\}, w) = 0$, where the last equation follows because $(\{i, k\}, w)$ is inessential. By IR and 2-HMC, $x_j \geq v(N) - \sigma_i(N, v)$ and $x_k \geq 0$. Let $y = \sigma(N, v)$. As $y(N) \leq v(N)$, $x_j \geq v(N) - y_i$ and, analogously, $x_i \geq v(N) - y_j$, we have $v(N) \geq x(M) \geq 2v(N) - y(N) + x_k \geq v(N) + x_k \geq v(N)$ so that all inequalities must be equations, i.e., $x_i + x_j = v(N)$ and $x_k = 0$. Since $(\{i, k\}, w)$ and $(\{j, k\}, w)$ are inessential, by IR (and feasibility) $\sigma_k(\{i, k\}, w) = \sigma_k(\{j, k\}, w) = 0$ and thus $(N, w_{N,\sigma}) = (N, v)$. Finally, by 2-HMC we conclude that $x_N = \sigma(N, w_{N,\sigma}) = \sigma(N, v)$ is efficient.

If $|N| \geq 3$, assume that $x = \sigma(N, v)$ satisfies $x(N) < v(N)$. Then, for any $S \subseteq N$ with $|S| = 2$, $x(S) < v_{S,\sigma}(S) = v(N) - x(N \setminus S)$, a contradiction. □

As we have seen before, EF, IR, and either BMP or BR or RS or CWE imply CE. Theorem 5.4 of Dutta (1990) stating that CE and HMC characterize the egalitarian solution E is not entirely correct as shown by Hokari (2002). In fact, E does not satisfy HMC because a HM-reduced game of a convex game is not necessarily

[17] However, since the set of convex games is closed under DM reduction, Theorem 5.3 of Dutta (1990) holds in the full domain of convex games, including 1-person games.

[18] That is, the unique feasible payoff vector that lexicographically minimizes the non-increasingly ordered vector of excesses $(v(S) - x(S))_{S\subseteq N}$ over the set of feasible payoff vectors.

convex. However, the following mild modification of the foregoing theorem holds. A careful inspection of Dutta's proof shows that HM-reduced games of convex games have a non-empty core. Hence, two-person HM-reduced games are convex. Therefore, E satisfies 2-HMC on the domain of convex games. Also, as in the case of Dutta's characterization of E with DMC, the uniqueness proof only uses 2-HMC rather than HMC.[19] Hence, on $\Gamma_{conv}^{\geq 2}$, E is characterized by CE and 2-HMC. These observations, together with Lemma 4, lead to the following characterizations.

Theorem 5. *On the domain Γ_{conv},*

(i) the egalitarian solution E is the unique solution that satisfies 2-HMC, IR, *and* BR*;*
(ii) the egalitarian solution E is the unique solution that satisfies 2-HMC, IR, *and* BMP*;*
(iii) the egalitarian solution E is the unique solution that satisfies 2-HMC, IR, *and* RS*;*
(iv) the egalitarian solution E is the unique solution that satisfies 2-HMC, IR, *and* CWE*.*

Indeed, by Lemma 4, CS and AM (or AAM) can be replaced by 2-HMC and IR in Theorem 1 (Theorem 2). Clearly, Theorem 5 (i) still holds when replacing BR by the stronger SBR. Moreover, the egalitarian bounds properties can be replaced by CWE.

Remark 6. Each of the properties in Theorem 5 is logically independent of the remaining properties.

(i) The single-valued solution ρ as defined in Remark 4 satisfies 2-HMC, BR, BMP, RS, and CWE but not IR.
(ii) For all $(N, v) \in \Gamma_{conv}$ define the solution σ as follows: $\sigma(N, v) = E(N, v_r)$. Then, σ satisfies IR, BR, BMP, and RS, but not 2-HMC.
(iii) For all $(N, v) \in \Gamma_{conv}$ define the solution σ as follows: $\sigma(N, v) = v(N, v)$ if $(N, v) = (N, v_r)$, and $\sigma_i(N, v) = v_i(N, v_r) + (\lambda - v_i(N, v_r))_+$ otherwise, where $\lambda \in \mathbb{R}$ is determined by $\sum_{i \in N}(\lambda - v_i(N, v_r))_+ = v(N) - v_r(N)$. Then, σ satisfies IR, CWE but not 2-HMC.
(iv) Let $\prec$ be a strict total order on U. The marginal contribution solution $mc^{\prec}$ satisfies 2-HMC and IR, but neither BMP nor BR nor RS nor CWE.

5. (Im)possibilities on larger domains

In this section, we investigate larger domains of games. It is well known that, on balanced games, the existence of the egalitarian solution E is not guaranteed. An alternative way to combine core stability and egalitarianism, already proposed by Dutta and Ray (1989) and latter adopted simultaneously by Arin and Iñarra (2001) and Hougaard et al. (2001), is to focus on the set of Lorenz maximal allocations within the core. According to Arin and Iñarra (2001) and Arin et al. (2003), a solution σ is said to be *core egalitarian*[20] if, for all $(N, v) \in \Gamma_{bal}$, $\sigma(N, v) \in C(N, v)$ and there is no $y \in C(N, v)$ such that $y \succ_{\mathcal{L}} \sigma(N, v)$. Examples of core egalitarian solutions are the *lexmin solution* (Yanovskaya, 1995; Arin and Iñarra, 2001) and its anti-dual, the *lexmax solution* (Arin et al., 2003). A proper subset of balanced games including convex games for which the egalitarian solution exists is the set of exact partition games (Llerena and Mauri, 2017), denoted by Γ_{expa}. A game (N, v) is an *exact partition* game if there exists a core element x such that, for the ordered partition $(S_1, \ldots, S_m)$ induced by x (see Remark 1 for the definition of "induced"), $x(S_1 \cup \cdots \cup S_k) = v(S_1 \cup \cdots \cup S_k)$ for all $k = 1, \ldots, m$. For an exact partition game (N, v), $E(N, v)$ is the unique singleton $\{x\}$ with the foregoing property. By Γ_{expa}^N we denote the subset of games in Γ_{expa} with $N \in \mathcal{N}$ as player set. Notice that exact partition games are closed under increments of the worth of the grand coalition, i.e., for all $t > 0$, $(N, v^t) \in \Gamma_{expa}$ if $(N, v) \in \Gamma_{expa}$. Moreover, Dietzenbacher and Yanovskaya (2020a) prove that this domain is closed under the anti-duality operation, i.e., $(N, v) \in \Gamma_{expa}$ if and only if $(N, v^{ad}) \in \Gamma_{expa}$.[21]

We study whether or not the results in Section 4 can be extended either to Γ_{expa} or Γ_{bal}. Following the arguments in the proofs of Theorems 1 and 3 it can be checked that, for $N \in \mathcal{N}$, both characterizations remain valid on Γ_{expa}^N. Furthermore, since the egalitarian solution E is self anti-dual on the domain of exact partition games (see Dietzenbacher and Yanovskaya, 2020b), Theorem 2 also holds on Γ_{expa}^N. Unfortunately, as shown below, none of the three theorems can be extended to the domain of balanced games, since the properties listed in these theorems are incompatible on Γ_{bal}^N. We first show that the maximal domain where CS and SBR are compatible and, thus, the characterization in Theorem 3 holds, is the domain of exact partition games.

Proposition 2. *Let $N \in \mathcal{N}$ and σ be a solution on $\Gamma^N \subseteq \Gamma_{bal}^N$ satisfying* CS *and* SBR*. Then $\Gamma^N \subseteq \Gamma_{expa}^N$.*

Proof. Let $N \in \mathcal{N}$ and σ be a solution on $\Gamma^N \subseteq \Gamma_{bal}^N$ satisfying CS and SBR. Let $(N, v) \in \Gamma^N$, $x = \sigma(N, v)$ and $(S_1, \ldots, S_m)$ be the ordered partition of N induced by x. Let $k \in \{1, \ldots, m-1\}$, $i \in S_k$, and $\alpha = x_i$. Then, with $S = \{i \in N \mid x_i < \alpha\} = S_{k+1} \cup \cdots \cup S_m$, $N \setminus S = S_1 \cup \cdots \cup S_k$. By CS and SBR $x(S_1 \cup \cdots \cup S_k) = v(S_1 \cup \cdots \cup S_k)$, which proves that $(N, v) \in \Gamma_{expa}^N$. □

On the domain of convex games SBR and RS are equivalent under CS, but RS and CS remain compatible for the set of all balanced games. Indeed, the lexmin solution satisfies both properties (Arin and Iñarra, 2001). In the following proposition we show that CS and either BMP or BR are incompatible on the class of balanced games.

Proposition 3. *Let $N \in \mathcal{N}$ with $|N| \geq 3$. Then, there is no solution on Γ_{bal}^N satisfying* CS *and either* BMP *or* BR*.*

Proof. Let $N \in \mathcal{N}$ with $|N| \geq 3$ and σ be a solution on Γ_{bal}^N satisfying CS. Let i, j be two distinct elements of N and define the game $(N, v) \in \Gamma_{bal}^N$, with $|N| \geq 3$, by $v(\{i\}) = v(\{j\}) = 1/2$, $v(N) = 1$, and $v(S) = 0$ otherwise. By CS, $\sigma_i(N, v) = \sigma_j(N, v) = 1/2$ and $\sigma_k(N, v) = 0$ for all $k \in N \setminus \{i, j\}$, a contradiction to BMP and BR. □

Observe that in the proof of Proposition 3 CS can be replaced by IR. Hence, on balanced games, IR (and thus EDS) are also incompatible with any of the egalitarian bounds property. So, there is a trade off between stability and egalitarian bounds properties, the combination of which is extensively used to characterize E on the domains of convex or exact partition games.

[19] To be more precise, recall that E does not satisfy HMC (see Footnote 10), but 2-HMC because all 2-person HM-reduced games w.r.t. E are balanced and, hence, convex. Therefore, Dutta's proof shows that E is characterized by 2-HMC and CE on $\Gamma_{conv}^{\geq 2}$, and this result may be extended to the entire set Γ_{conv} by adding IR (or EF) only for 1-person games, or by replacing 2-HMC by weak HMC, requiring HMC for coalitions S with $|S| = 2$ and $|S| = 1$. Indeed, if a solution satisfies weak HMC and EF for convex 2-person games, then it satisfies EF in general.

[20] Although they use the term egalitarian, here, and to avoid confusion, we rename it by core egalitarian.

[21] Recently, Dietzenbacher and Yanovskaya (2020a,b) extend some of the axiomatizations of E on the domain of convex games to the larger domain of exact partition games.

Interestingly, it is not difficult to check that Lemmas 2–4 hold on the sets of exact partition games and balanced games. On the contrary, since 2-DMC implies CS, from Proposition 3 it follows that 2-DMC and either BMP or BR are not compatible on Γ_{bal}, and, hence, Theorem 4 (*i*) and (*ii*) do not longer hold. To see that Theorem 4 (*iii*) is also not valid on Γ_{bal} we show that 2-DMC and CWE are incompatible.

Proposition 4. *There is no solution on* Γ_{bal} *satisfying* 2-DMC *and* CWE.

Proof. Let σ be a solution on Γ_{bal} satisfying 2-DMC and CWE. Then, as Lemma 2 also holds for balanced rather than convex games (the proof may be literally copied), when restricting σ to games with at least two players, it satisfies CS as well. Let $(N, v) \in \Gamma_{bal}$ be the game with $|N| = 3$, say $N = \{1, 2, 3\}$, and characteristic function $v(\{h\}) = 0$ for all $h \in N$, $v(\{1, 2\}) = v(\{1, 3\}) = 1$, $v(\{2, 3\}) = 0$, and $v(N) = 1$. By CS, $\sigma(N, v) = (1, 0, 0)$. By CWE, $\sigma(N, v^1) = (1, 1/2, 1/2) := x$. Now, with $N' = \{1, 2\}$, $v^1_{N',x}(\{1\}) = 1/2$, $v^1_{N',x}(\{2\}) = 0$, and $v^1_{N',x}(N') = 3/2$. Therefore, $(v^1_{N',x})_r = (v^1_{N',x})^{-1}$, and, applying CS, $\sigma(N', (v^1_{N',x})_r) = (1/2, 0)$. By CWE, $\sigma(N', v^1_{N',x}) = (3/4, 3/4)$, which contradicts 2-DMC. □

By definition, on the set balanced games any core egalitarian solution satisfies CS and, thus, IR and EDS. Moreover, as a consequence of Propositions 2 and 3, it does not satisfy BMP, BR (SBR), or RS. Example 2 of Calleja et al. (2021) shows that there is no core egalitarian solution satisfying AM, and, with the help of the anti-dual games of this example it is straightforward to verify that AAM is also violated. However, if a core egalitarian solution meets AM on some subdomain of balanced games closed under increments w.r.t. the worth of the grand coalition, then it satisfies CWE as well (see Theorem 4 in Calleja et al., 2021).

Clearly, on the set of exact partition games, all properties employed in Theorem 4 are still satisfied by E. However, as E does no longer satisfy converse DM-consistency[22] on this larger domain (see Example 1 of Llerena and Mauri, 2017), the question of whether E is the unique solution on Γ_{expa} that satisfies 2-DMC and CE remains open.

Concerning Theorem 5, the following proposition shows that 2-HMC and CE are not compatible on the domains Γ_{expa} and Γ_{bal}. As a consequence, none of the characterizations in Theorem 5 hold on these larger domains, and, moreover, the characterization in Theorem 5.4 of Dutta (1990) cannot be extended to Γ_{expa} nor to Γ_{bal}.

Proposition 5. *Neither on* Γ_{expa} *nor on* Γ_{bal} *there is a solution satisfying* 2-HMC *and* CE.

Proof. Note first that CE implies EF for 2-person balanced games and, as the proof of Lemma 4 can be easily modified for the sets of exact partition games or balanced games, a solution that satisfies 2-HMC and CE would also satisfy EF on each of the two sets of games. Now suppose, on the contrary, there is a solution σ on Γ_{bal} satisfying 2-HMC and CE. Let (N, v) be the game with $|N| = 3$, say $N = \{1, 2, 3\}$, and characteristic function as follows: $v(\{h\}) = 0$ for all $h \in N$, $v(\{1, 2\}) = 1$, $v(\{1, 3\}) = v(\{2, 3\}) = 1/2$, and $v(N) = 1$. Then $C(N, v) = \{(1/2, 1/2, 0)\}$, which shows that (N, v) is an exact partition game. Now consider the HM-reduced game of (N, v) w.r.t. $N' = \{1, 2\}$ at σ. By CE,

$$v_{N',\sigma}(\{1\}) = v(\{1, 3\}) - \sigma_3(\{1, 3\}, v) = 1/2 - 1/4 = 1/4,$$

$$v_{N',\sigma}(\{2\}) = v(\{2, 3\}) - \sigma_3(\{2, 3\}, v) = 1/2 - 1/4 = 1/4,$$

and

$$v_{N',\sigma}(N') = 1 - \sigma_3(N, v).$$

By 2-HMC, $(N', v_{N',\sigma})$ is balanced so that $1 - \sigma_3(N, v) \geq 1/4 + 1/4 = 1/2$, i.e., $\sigma_3(N, v) \leq 1/2$. Considering the HM-reduced games of (N, v) w.r.t. $N'' = \{1, 3\}$ and $N''' = \{2, 3\}$ at σ, similar arguments show that $\sigma_2(N, v) \leq 1/4$ and $\sigma_1(N, v) \leq 1/4$. By EF, $\sigma(N, v) = (1/4, 1/4, 1/2)$. Hence, $v_{N'',\sigma}(\{1\}) = 1/2$, $v_{N'',\sigma}(\{3\}) = 1/4$, and $v_{N'',\sigma}(N'') = 3/4$. By CE, $\sigma_1(N'', v_{N'',\sigma}) = 1/2 \neq 1/4 = \sigma_1(N, v)$, which contradicts 2-HMC. □

To conclude, note that any core egalitarian solution satisfies 2-DMC which follows from Theorem 2 in Arin and Iñarra (2001) and the fact that two-person balanced games are convex, and thus E is the unique core egalitarian allocation. So, in contrast to Proposition 5, 2-DMC and CE are compatible on Γ_{bal}.

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[22] See Peleg and Sudhölter (2007) for a formal definition of converse DM-consistency.

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