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Constrained welfare egalitarianism in surplus-sharing problems

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ABSTRACT

The constrained equal welfare rule, f^{CE}, distributes the surplus according to the uniform gains method and, hence, equalizes the welfare of the agents subsequent to the allocation process, subject to making nobody worse off. We show that f^{CE} is the unique rule on the domain of surplus-sharing problems that satisfies efficiency, welfare monotonicity, path independence, and weak less first imposing an egalitarian bound for allowing positive payoffs to particular players. We provide an additional axiomatization employing consistency, a classical invariance property with respect to changes of the population. Finally, we show that the set of efficient solutions for cooperative TU games that support constrained welfare egalitarianism, i.e., distribute increments in the worth of the grand coalition according to f^{CE}, is characterized by aggregate monotonicity and bounded pairwise fairness requiring that a player can only gain if his initial payoff does not exceed the initial payoff of any other player by the amount to be divided.

1. Introduction

The notion of equity has a significant position in surplus-sharing problems, where a quantity of a divisible resource (e.g., money) is divided among a set of agents who regard resource or welfare egalitarianism as a social value. *Resource egalitarianism* is reached by distributing the available total resource equally among the agents, whereas *welfare egalitarianism* prioritizes to equalize the welfare of the agents after the allocation process.[1] Nevertheless, if the amount of resource that has to be distributed is small, it may happen that a rich agent has to transfer some of her money to poorer agents in order to reach welfare egalitarianism. The *constrained equal welfare* rule, f^{CE}, makes the approach to welfare egalitarianism compatible with individual self-interest. Imagine a situation in which a resource has to be divided among a set of agents that are ranked with respect to (w.r.t.) a *reference point*, representing some objective and measurable feature (sometimes called *status quo* or *welfare*). First, the lowest ranked agents receive equal shares of the resource until they become equal to the second lowest ranked agents, and so forth until the resource is exhausted. Many real-life allocation methods promote constrained welfare egalitarianism. For instance, in the distribution of grants or subsidies by public institutions, families with lower incomes often receive larger scholarships and, subsequent to a natural catastrophe, it is often decided that the more individuals suffer, the more financial support they get. As another interesting application we mention that, if the CO_2 emissions rights are distributed according to f^{CE}, then regions with higher past accumulated per-capita emissions receive less so that regions with lower past emissions are favored. In a discrete setting, it would be reasonable to allocate refugees to different countries according to f^{CE}, where the status quo captures the current numbers of refugees in the countries.

Distributing according to f^{CE} can be seen as a way of obtaining end-state fairness.[2] Several surplus-sharing rules have been established and characterized (see, e.g., Moulin, 1987; Young, 1988; Chun, 1989; Herrero et al., 1999; Pfingsten, 1991, 1998) that do not aim to diminish the inequalities of the ex-post allocations that arise when the rule has been applied. Moulin's (2002) *uniform gains method*, *UG*, however, does, but is, formally, defined in a different setting. In the current paper, we adopt Moulin's (1987) notion of a rule, which assigns to each surplus-sharing problem an allocation of the surplus among the agents so that f^{CE} is the rule which results from *UG*.[3] Moulin (2002) also studies the connections between *UG* and its counterpart for deficit-sharing

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1 Moreno-Ternero and Roemer (2012) provide a concise exposition of these two concepts of distributive justice.

2 See Ju and Moreno-Ternero (2017, 2018) for a discussion of different levels of fairness for the allocation of goods.

3 For details, see Remark 1 in Section 3 and Remark 5 in Section 4.

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problems, also known as bankruptcy problems introduced by O'Neill (1962)–see the surveys of Thomson (2003, 2015).

In this paper, we provide two characterizations of f^{CE} for surplus-sharing problems, which yield, when translating the employed simple and intuitive properties to Moulin's (2002) setting, also characterizations of *UG* that, as far as we know, has not yet been characterized. To this end, we introduce properties that prioritize agents with a lower status quo. *Less first* requires that if the relative welfare difference at the status quo between two agents exceeds the total amount to be divided, then the agent with higher welfare does not receive a positive amount. The weaker *weak less first* and *restricted less first* still impose certain egalitarian bounds for allowing positive payoffs for some agents with a significant level of welfare. Similar protective properties for those agents with small "initial starting point" have been used in other models. Examples are *no domination* (Moreno-Ternero and Roemer, 2012), in a model of resource allocation where agents are capable to transform wealth into non-transferable outcomes, or *ex-ante fairness* (Timoner and Izquierdo, 2016), in a context of rationing problems with ex-ante conditions (Hougaard et al., 2012, 2013). We first show that f^{CE} is characterized by *efficiency* or budget balance (requiring to exhaust the resource), *welfare monotonicity*, imposing that no agent is worse off after the application of the rule, *path independence* (Moulin, 1987), requiring that the assigned payoffs remain unchanged when applying the rule consecutively to an arbitrary partition of the resource, and *weak less first*. If we replace weak less first by *restricted less first* and add a weak version of *consistency*, a classical invariance property requiring that the share of the surplus of an agent remains unchanged if some other agents take their share and leave, we obtain an additional axiomatization. These characterizations provide significant insights in the differences of the constrained equal welfare rule f^{CE} compared to the *equal surplus rule* and the *equal welfare rule*, representing resource and welfare egalitarianism, respectively.

According to Megiddo (1974) a solution on a domain of *transferable utility* games is said to be *aggregate monotonic* if no player suffers when only the grand coalition becomes richer. If this increment in the worth of the grand coalition is distributed according to f^{CE} applied to the allocation initially proposed by the solution, then the solution *supports constrained welfare egalitarianism*. We show that an efficient solution supports constrained welfare egalitarianism if and only if it is aggregate monotonic and satisfies *bounded pairwise fairness* requiring that a player can only gain if his initial payoff does not exceed the initial payoff of another player by the amount to be divided. It turns out that Dutta–Ray's (1989) *egalitarian* solution and the *lexmax* solution (Arin et al., 2003) support constrained welfare egalitarianism on the domains of convex games and of games with large cores, respectively.

The remainder of the paper is organized as follows. Section 2 contains some general preliminaries. In Section 3 we introduce f^{CE} and compare it with the *equal surplus rule* and the *equal welfare rule*. Section 4 presents the axiomatic analysis of f^{CE}. In Section 5 we characterize the set of efficient single-valued solutions that support f^{CE}. The Appendix shows that each property in each of the characterization results is logically independent of the remaining properties.

2. Preliminaries

Let U be a set (the universe of potential agents) and $\mathcal{N}$ be the set of coalitions in U (a *coalition* is a nonempty finite subset of U). Given $S, T \in \mathcal{N}$, we use $S \subset T$ to indicate strict inclusion, that is, $S \subseteq T$ and $S \neq T$. By $|S|$ we denote the cardinality of the coalition $S \in \mathcal{N}$. Given $N \in \mathcal{N}$, $\mathbb{R}^N$ stands for the set of all real functions on N. An element $x \in \mathbb{R}^N$, $x = (x_i)_{i\in N}$, is a *payoff vector* for N. For each $x \in \mathbb{R}^N$, $x(S) = \sum_{i\in S} x_i$ with the convention $x(\emptyset) = 0$, and x_S denotes the restriction of x to S, i.e., $x_S = (x_i)_{i\in S} \in \mathbb{R}^S$. Given $y \in \mathbb{R}^N$, we write $x \geq y$ if $x_i \geq y_i$ for all $i \in N$. For all $\alpha \in \mathbb{R}$, we denote $\alpha_+ = \max\{0, \alpha\}$. If $y(N) = x(N)$, we say that y *weakly Lorenz dominates* x, denoted by $y \succeq_{\mathcal{L}} x$, if $\min\{y(S) \mid S \subseteq N,\ |S| = k\} \geq \min\{x(S) \mid S \subseteq N,\ |S| = k\}$, for all $k = 1, 2, \ldots, n-1$. We say that y *Lorenz dominates* x, denoted by $y \succ_{\mathcal{L}} x$, if at least one of the above inequalities is strict. Moreover, let $\mathcal{P}(x) = (N_1, N_2, \ldots, N_k)$ denote the ordered partition of N that is determined by $N_1 = \{i \in N \mid x_i \leq x_j\ \forall j \in N\}$ and $N_m = \{i \in N \setminus \bigcup_{j=1}^{m-1} N_j \mid x_i \leq x_j\ \forall j \in N \setminus \bigcup_{j=1}^{m-1} N_j\}$ for all $m = 2, \ldots, k$.

3. The constrained equal welfare rule

A *surplus-sharing* problem is a triple (N, x, t) where $N \in \mathcal{N}$ is the set of agents, $x \in \mathbb{R}^N$ is the *status quo* or *reference point*, and $t \geq 0$ the *surplus* in terms of money.[4] A *surplus-sharing* rule distributes the amount t among the members of N given $x \in \mathbb{R}^N$ which, depending on the situation, can denote the vector of individual opportunity costs or endowments of the agents or other objective references. Formally, it is a function f that assigns to each surplus-sharing problem (N, x, t) a vector $f(N, x, t) \in \mathbb{R}^N$ satisfying $\sum_{i\in N} f_i(N, x, t) \leq t$ (*feasibility*).[5] Let $\mathcal{F}$ denote the set of all surplus-sharing rules with a finite set of agents in $\mathcal{N}$. Note that $x + f(N, x, t)$ represents the "ex-post" welfare levels of the agents. We say that $f \in \mathcal{F}$ is *efficient* ($\mathbb{EF}$) if, for each surplus-sharing problem (N, x, t), $\sum_{i\in N} f_i(N, x, t) = t$. Moreover, $f \in \mathcal{F}$ satisfies *welfare monotonicity* ($\mathbb{WM}$) if $f_i(N, x, t) \geq 0$ for all $i \in N$ and for any surplus-sharing problem (N, x, t). $\mathbb{WM}$ requires that no agent transfers part of her status quo to others.

The *equal surplus rule*, f^{EQ}, defined by

$$f_i^{EQ}(N, x, t) = \frac{t}{|N|} \text{ for all surplus-sharing problems } (N, x, t) \text{ and all } i \in N, \tag{1}$$

distributes the available resource equally and, hence, implements resource egalitarianism. Clearly, f^{EQ} weakly Lorenz dominates every other efficient rule $f \in \mathcal{F}$, i.e., $f^{EQ}(N, x, t) \succeq_{\mathcal{L}} f(N, x, t)$ for each surplus-sharing problem (N, x, t). However, it is easy to find instances (see Example 1) of surplus-sharing problems and efficient rules $f \in \mathcal{F}$ where $x + f^{EQ}(N, x, t)$ is Lorenz dominated by $x + f(N, x, t)$.

The *equal welfare rule*, f^E, defined by

$$f_i^E(N, x, t) = \frac{x(N) + t}{|N|} - x_i \text{ for all surplus-sharing problems } (N, x, t) \text{ and all } i \in N, \tag{2}$$

equalizes the welfare of the agents ex-post[6] and, hence, implements welfare egalitarianism. Note that $x + f^E(N, x, t) \succeq_{\mathcal{L}} x + f(N, x, t)$ for each efficient rule $f \in \mathcal{F}$ and each surplus-sharing problem (N, x, t). However, f^E may require transfers between agents, i.e., it does not satisfy $\mathbb{WM}$. Hence, for small t some agents may lose when f^E is applied so that they prefer not to collaborate.

The *constrained equal welfare rule*, f^{CE}, defined by

$$f_i^{CE}(N, x, t) = (\lambda - x_i)_+ \text{ for all surplus-sharing problems } (N, x, t) \text{ and all } i \in N, \tag{3}$$

[4] Usually, in the definition of a surplus-sharing problem the condition $x \in \mathbb{R}^N_+$ is imposed. Here, we consider a more general domain of problems in which no restriction on x is required.

[5] Other models incorporate additional requirements in defining a surplus-sharing rule (see, for instance, (Moulin, 1987)).

[6] The counterpart of f^E in the setting of loss-sharing problems is a particular case of a new class of rules introduced by Gaertner and Xu (2020).

where $\lambda \in \mathbb{R}$ is determined by $\sum_{k \in N}(\lambda - x_k)_+ = t$, reconciles welfare egalitarianism with individual self-interest, i.e., satisfies $\mathbb{WM}$. Note that the ex-post allocation $x + f^{CE}(N, x, t)$ weakly Lorenz dominates the final outcome $x+f(N, x, t)$ for each efficient $f \in \mathcal{F}$ satisfying $\mathbb{WM}$, i.e.,[7]

$$x + f^{CE}(N, x, t) \succeq_{\mathcal{L}} x + f(N, x, t) \text{ for all surplus-sharing problems } (N, x, t). \tag{4}$$

Thus, f^{CE} treats equals (w.r.t. the status quo) equally and makes unequal agents as equal as possible. That is, it distributes the surplus to the poorer agents so that their payoffs become equal but not larger than the remaining agents' status quo payoffs. Note that f^{CE} generates constrained egalitarianism requiring that each agent can preserve her initial status quo.

The following remark shows the close relationship between f^{CE} and the *uniform gains method*, defined in a slightly different setting.

Remark 1. Let (N, x, t) be a surplus-sharing problem. Now, interpreting the status quo payoffs x_i, $i \in N$, as "claims" (provided they are nonnegative), a surplus-sharing *method* f^* distributes the total money $t + x(N)$ to the agents in such a way that each agent receives at least her claim. Formally, $\sum_{i \in N} f_i^*(N, x, x(N) + t) = x(N)+t$ and $f_i^*(N, x, x(N)+t) \geq x_i$ for all $i \in N$. The *uniform gains* method (Moulin, 2002), denoted by UG, is given by

$$UG(N, x, x(N) + t) = f^{CE}(N, x, t) + x. \tag{5}$$

Note that $f^E(N, x, t) = f^{CE}(N, x, t)$ if and only if $f^E(N, x, t) \geq 0$. Therefore, $f^E(N, x, t) = f^{CE}(N, x, t + t') - f^{EQ}(N, x, t')$ for each $t' \geq 0$ such that $f^E(N, x, t + t') \geq 0$. The following remark, useful in some proofs, explains how to explicitly calculate f^{CE}.

Remark 2. Let $N \in \mathcal{N}$, $x \in \mathbb{R}^N$, $t > 0$, and λ be such that $f_i^{CE}(N, x, t) = (\lambda - x_i)_+$ for all $i \in N$. Choose $i_1, \ldots, i_n$, where $n = |N|$, such that $\{i_1, \ldots, i_n\} = N$ and $x_{i_1} \leq \cdots \leq x_{i_n}$. For $k \in \{1, \ldots, n\}$ define $\alpha_k(t) = \alpha_k = x(\{i_1, \ldots, i_k\}) - kx_{i_k} + t$ and observe that $\alpha_1 = t > 0$ and, for $k < n$, $\alpha_k - \alpha_{k+1} = k(x_{i_{k+1}} - x_{i_k})$, hence $\alpha_1 \geq \cdots \geq \alpha_n$. Now, with $k_0 = \max\{k \in \{1, \ldots, n\} \mid \alpha_k > 0\}$, we get

$$\lambda = \frac{\alpha_{k_0}}{k_0} + x_{i_{k_0}} = \frac{x(\{i_1, \ldots, i_{k_0}\}) + t}{k_0}.$$

Hence, $\lambda = x_{i_k} + f_{i_k}^{CE}(N, x, t) < x_{i_{k'}} = x_{i_{k'}} + f_{i_{k'}}^{CE}(N, x, t)$, for all $k = 1, \ldots, k_0$ and all $k' = k_0 + 1, \ldots, n$.

The following example illustrates differences between the aforementioned surplus-sharing rules.

Example 1. Let $N = \{1, 2, 3, 4\}$, $x = (1, 3, 8, 0)$, and $t = 8$. Then

$$f^{EQ}(N, x, 8) = (2, 2, 2, 2) \text{ and } f^E(N, x, 8) = (4, 2, -3, 5).$$

Using the notation of Remark 2, $i_1 = 4$, $i_j = j - 1$ for $j = 2, 3, 4$, and

$$\begin{array}{lcllcl} \alpha_1 & = & x_{i_1} - 1x_{i_1} + 8 & & = & 8, \\ \alpha_2 & = & x_{i_1} + x_{i_2} - 2x_{i_2} + 8 & & = & 7, \\ \alpha_3 & = & x_{i_1} + x_{i_2} + x_{i_3} - 3x_{i_3} + 8 & & = & 3, \\ \alpha_4 & = & x_{i_1} + x_{i_2} + x_{i_3} + x_{i_4} - 4x_{i_4} + 8 & & = & -12. \end{array} \tag{6}$$

Thus, $k_0 = \max\{k \in \{1, 2, 3, 4\} \mid \alpha_k > 0\} = 3$, $\lambda = \frac{\alpha_3}{3} + x_{i_3} = 4$, and

$$\begin{array}{l} f_1^{CE}(N, (1, 3, 8, 0), 8) = (4 - 1)_+ = 3, \\ f_2^{CE}(N, (1, 3, 8, 0), 8) = (4 - 3)_+ = 1, \\ f_3^{CE}(N, (1, 3, 8, 0), 8) = (4 - 8)_+ = 0, \\ f_4^{CE}(N, (1, 3, 8, 0), 8) = (4 - 0)_+ = 4. \end{array} \tag{7}$$

Hence, $f^{CE}(N, x, 8) = (3, 1, 0, 4)$ and

$$f^{EQ}(N, x, 8) \succ_{\mathcal{L}} f^E(N, x, 8) \text{ and } f^{EQ}(N, x, 8) \succ_{\mathcal{L}} f^{CE}(N, x, 8).$$

Observe, however, that

$$x + f^{CE}(N, x, 8) = (4, 4, 8, 4) \succ_{\mathcal{L}} (3, 5, 10, 2) = x + f^{EQ}(N, x, 8),$$

and $x + f^E(N, x, 8) = (5, 5, 5, 5)$ Lorenz dominates both distributions. Nevertheless, under f^E, agent 3 has no incentive to cooperate because $f_3^E(N, x, 8) = -3$.

4. Axiomatic analysis of f^{CE}

In this section, we provide several axiomatizations of f^{CE} either for fixed or variable sets of agents. Although the properties are stated for variable sets of agents (i.e., for surplus-sharing problems (N, x, t) such that $N \in \mathcal{N}$), except for consistency, the remaining properties may be formulated for a fixed society $N \in \mathcal{N}$ of agents.

4.1. Properties

Together with $\mathbb{WM}$ and $\mathbb{EF}$, already defined in Section 3, we will use the following additional properties. A surplus-sharing rule $f \in \mathcal{F}$ satisfies

- *Equal treatment of equals* ($\mathbb{ET}$) if for all $N \in \mathcal{N}$, all $x \in \mathbb{R}^N$, all $t \in \mathbb{R}_+$, and all $i, j \in N$, if $x_i = x_j$ then $f_i(N, x, t) = f_j(N, x, t)$;
- *Resource monotonicity* ($\mathbb{RM}$) if for all $N \in \mathcal{N}$, all $x \in \mathbb{R}^N$, and all $t, t' \in \mathbb{R}_+$ with $t' > t$, $f(N, x, t') \geq f(N, x, t)$;
- *Path independence* ($\mathbb{PI}$) if for all $N \in \mathcal{N}$, all $x \in \mathbb{R}^N$, and all $t, t' \geq 0$, $f(N, x, t + t') = f(N, x, t) + f(N, x + f(N, x, t), t')$.

$\mathbb{ET}$ is a simple equity requirement which imposes that equal agents (w.r.t. the status quo) should receive the same amount of the resource. $\mathbb{RM}$ is a solidarity condition requiring that nobody is worse off when there is more to be divided. Moulin (1987) introduces $\mathbb{PI}$, which requires that, regardless of the partition of the total amount of resource to be allocated, its distribution may be dynamically obtained step-by-step by applying the surplus-sharing rule consecutively to the given elements of the partition, and taking into consideration the new status quo that emerges after the allocation process in the previous step.

Remark 3. Note that $\mathbb{PI}$ and $\mathbb{WM}$ imply $\mathbb{RM}$. Moreover, if $f \in \mathcal{F}$ satisfies $\mathbb{RM}$ and $\mathbb{EF}$, then, for all $N \in \mathcal{N}$ and all $x \in \mathbb{R}^N$, $f(N, x, \cdot) : \mathbb{R}_+ \to \mathbb{R}_+^N$ is a *continuous mapping*.

We now present three properties that require to prioritize agents with a lower status quo. They can be interpreted as solidarity requirements that establish certain egalitarian bounds for allowing positive payoffs to particular players. A surplus-sharing rule $f \in \mathcal{F}$ satisfies

- *Less first* ($\mathbb{LF}$) if for all $N \in \mathcal{N}$, all $x \in \mathbb{R}^N$, all $t \in \mathbb{R}_+$, and all $i, j \in N$, $i \neq j$, $f_i(N, x, t) > 0$ implies $x_i - x_j < t$;
- *Weak less first* ($\mathbb{WLF}$) if for all $N \in \mathcal{N}$, all $x \in \mathbb{R}^N$, all $t \in \mathbb{R}_+$, and all $i, j \in N$, $i \neq j$, $f_i(N, x, t) > 0$ and $x_i - x_j \geq t$ imply $f_j(N, x, t) \geq t$;
- *Restricted less first* ($\mathbb{RLF}$) if for all $N \in \mathcal{N}$, all $x \in \mathbb{R}^N$, all $t \in \mathbb{R}_+$, and all $i, j \in N$, $i \neq j$, with $x_i \geq x_k$ for all $k \in N$, $f_i(N, x, t) > 0$ implies $x_i - x_j < t$.

$\mathbb{LF}$ applies to any pair of agents, and it requires that an agent does not gain if her status quo exceeds the status quo of another agent by the surplus, while $\mathbb{WLF}$ imposes that the richest agent in the pair can only gain if the poorest agent in the pair receives at least the total surplus. Under $\mathbb{WM}$, $\mathbb{WLF}$ is equivalent to $\mathbb{LF}$. $\mathbb{RLF}$ is substantially weaker and imposes $\mathbb{LF}$ only to pairs of agents

[7] Indeed, for all $N \in \mathcal{N}$, all $x \in \mathbb{R}^N$, and all $t \in \mathbb{R}_+$, it is easily deduced that $x + f^{CE}(N, x, t) \succ_{\mathcal{L}} x + z$ for all $z \in \mathbb{R}_+^N \setminus \{f^{CE}(N, x, t)\}$ with $z(N) = t$.

Table 1
Rules and properties.

	$\mathbb{EF}$	$\mathbb{WM}$	$\mathbb{ET}$	$\mathbb{RM}$	$\mathbb{PI}$	$\mathbb{LF}$	$\mathbb{WLF}$	$\mathbb{RLF}$	$\mathbb{CCO}$	2-$\mathbb{CCO}$
f^{EQ}	✓	✓	✓	✓	✓	×	×	×	✓	✓
f^{E}	✓	×	✓	✓	✓	×	✓	✓	✓	✓
f^{CE}	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓

containing an agent with the highest status quo. We remark that $\mathbb{WLF}$ and $\mathbb{RLF}$, both implied by $\mathbb{LF}$, highlight the normative differences between f^{EQ}, f^{E}, and f^{CE}. Notice that a rule satisfying any of these properties may allow that some agents transfer part of their welfare to others, that is, none of the properties implies $\mathbb{WM}$.

We now show that $\mathbb{ET}$ is a consequence of $\mathbb{EF}$, $\mathbb{WM}$, $\mathbb{PI}$, and $\mathbb{LF}$.

Proposition 1. *If a surplus-sharing rule satisfies $\mathbb{EF}$, $\mathbb{WM}$, $\mathbb{PI}$, and $\mathbb{LF}$ then also $\mathbb{ET}$.*

Proof. Let $N \in \mathcal{N}$, $x \in \mathbb{R}^N$, $t \in \mathbb{R}_+$, and $i, j \in N$, $i \neq j$, such that $x_i = x_j$. Let $f \in \mathcal{F}$ satisfy $\mathbb{EF}$, $\mathbb{WM}$, $\mathbb{PI}$ and $\mathbb{LF}$.

If $t = 0$, then by $\mathbb{EF}$ and $\mathbb{WM}$, $f_i(N, x, t) = f_j(N, x, t) = 0$.

If $t > 0$ suppose, w.l.o.g., $f_i(N, x, t) < f_j(N, x, t)$. Note that by $\mathbb{WM}$, $f_j(N, x, t) > 0$. Moreover, since $\mathbb{WM}$ and $\mathbb{PI}$ imply $\mathbb{RM}$, for all $0 \leq t' \leq t$ we have that $f(N, x, t') \leq f(N, x, t)$. By continuity and $\mathbb{RM}$ of f (see Remark 3), $t^* = \min\{\tau \in \mathbb{R}_+ \mid f_j(N, x, \tau) = f_j(N, x, t)\}$ exists and, as $f_j(N, x, t^*) > f_i(N, x, t^*)$, for each $0 < \hat{t} < t^*$ close enough to t^*, $f_j(N, x, \hat{t}) - f_i(N, x, \hat{t}) > t^* - \hat{t}$. As $x_i = x_j$, we obtain $t^* - \hat{t} < x_j + f_j(N, x, \hat{t}) - (x_i + f_i(N, x, \hat{t}))$. Hence, by $\mathbb{LF}$ and $\mathbb{WM}$, $f_j(N, x + f(N, x, \hat{t}), t^* - \hat{t}) = 0$. But then, by $\mathbb{PI}$, $f_j(N, x, t^*) = f_j(N, x, \hat{t})$ which means that $f_j(N, x, t) = f_j(N, x, \hat{t})$, contradicting the minimality of t^*. □

Remark 4. Let us stress that it is not possible to exclusively replace $\mathbb{PI}$ by $\mathbb{RM}$ in Proposition 1. Indeed, select $i \in U$ and define $f \in \mathcal{F}$ as follows. Let $N \in \mathcal{N}$, $x \in \mathbb{R}^N$, and $t \geq 0$. If $i \notin N$ or $i \in N$ and $x_i > x_j$ for some $j \in N \setminus \{i\}$, define $f(N, x, t) = f^{CE}(N, x, t)$. If $i \in N$ and $x_i \leq x_j$ for all $j \in N$, define $f_i(N, x, t) = t$ and $f_j(N, x, t) = 0$ for all $j \in N \setminus \{i\}$. Then, f satisfies $\mathbb{EF}$, $\mathbb{WM}$, $\mathbb{RM}$, and $\mathbb{LF}$ but not $\mathbb{ET}$.

Finally, we introduce *conditional consistency*, a weak version of the classical consistency property which forces the solution to coincide in both the original and the reduced surplus-sharing problem that results when some agents leave. Conditional consistency applies only if what is left to share among the agents in the reduced problem is nonnegative. A surplus-sharing rule $f \in \mathcal{F}$ satisfies

- *Conditional consistency* ($\mathbb{CCO}$) if for all $N \in \mathcal{N}$, all $x \in \mathbb{R}^N$, all $t \in \mathbb{R}_+$, and all $\emptyset \neq S \subset N$, the following condition holds: if $t - \sum_{i \in N \setminus S} f_i(N, x, t) \geq 0$, then $f_S(N, x, t) = f\left(S, x_S, t - \sum_{i \in N \setminus S} f_i(N, x, t)\right)$.

Bilateral conditional consistency (2-$\mathbb{CCO}$) requires $\mathbb{CCO}$ for reduced surplus-sharing problems with two agents, i.e., $|S| = 2$.

Which of the properties are satisfied by f^{EQ}, f^{E}, and f^{CE} is shown in Table 1, which also indicates that the solidarity requirements $\mathbb{LF}$, $\mathbb{WLF}$ and $\mathbb{RLF}$, together with $\mathbb{WM}$, are crucial in our axiomatizations in order to distinguish f^{CE} from f^{EQ} and f^{E}.

In the following section we prove that f^{CE} satisfies $\mathbb{PI}$ and $\mathbb{LF}$, and in the proof of Theorem 2 we show that it satisfies $\mathbb{CCO}$. Note that it is straightforward or similar to check the correctness of the remaining entries of the table.

4.2. Characterizations with and without consistency

First, we deal with a fixed agent set $N \in \mathcal{N}$. By definition, f^{CE} satisfies $\mathbb{EF}$ and $\mathbb{WM}$.

Proposition 2. *The surplus-sharing rule f^{CE} satisfies $\mathbb{PI}$ and $\mathbb{LF}$.*

Proof. Let $x \in \mathbb{R}^N$ and $t \geq 0$.

To show $\mathbb{PI}$, let $i_1, \ldots, i_n$ be defined as in Remark 2, $t = t_1 + t_2$, $t_1, t_2 > 0$,

$$k_0^1 = \max\{k \in \{1, \ldots, n\} \mid x(\{i_1, \ldots, i_k\}) + t_1 > k x_{i_k}\}$$

and

$$k_0 = \max\{k \in \{1, \ldots, n\} \mid x(\{i_1, \ldots, i_k\}) + t > k x_{i_k}\}.$$

That is, with

$$\lambda_1 = \frac{x(\{i_1, \ldots, i_{k_0^1}\}) + t_1}{k_0^1} \text{ and } \lambda = \frac{x(\{i_1, \ldots, i_{k_0}\}) + t}{k_0},$$

we have $f_i^{CE}(N, x, t_1) = (\lambda_1 - x_i)_+$ and $f_i^{CE}(N, x, t) = (\lambda - x_i)_+$, for all $i \in N$. Let $y = x + f^{CE}(N, x, t_1)$. By Remark 2, $y_{i_1} = \cdots = y_{i_{k_0^1}} < y_{i_{k_0^1+1}} \leq \cdots \leq y_{i_n}$ and $k_0^1 \leq k_0$. As $k_0\lambda - x(\{i_1, \ldots, i_{k_0}\}) = t$ and $k_0^1\lambda_1 - x(\{i_1, \ldots, i_{k_0^1}\}) = t_1$, we conclude that

$$\begin{aligned} & k_0\lambda - y(\{i_1, \ldots, i_{k_0}\}) \\ & = k_0(\lambda - \lambda_1) - x(\{i_{k_0^1+1}, \ldots, i_{k_0}\}) \\ & = k_0\lambda - x(\{i_1, \ldots, i_{k_0}\}) + x(\{i_1, \ldots, i_{k_0^1}\}) - k_0^1\lambda_1 \\ & = t - t_1 = t_2 \end{aligned}$$

so that $\mathbb{PI}$ is shown.

To show $\mathbb{LF}$, suppose there are $i, j \in N$, $i \neq j$, with $x_i - x_j \geq t$ and $f_i^{CE}(N, x, t) > 0$. Since $x_i \geq x_j$, $f_i^{CE}(N, x, t) \leq f_j^{CE}(N, x, t)$ and thus $f_j^{CE}(N, x, t) > 0$. This means that $x_i + f_i^{CE}(N, x, t) = x_j + f_j^{CE}(N, x, t)$ (see Remark 2), which implies $x_i - x_j = f_j^{CE}(N, x, t) - f_i^{CE}(N, x, t) \geq t$. But then $f_j^{CE}(N, x, t) > t$, contradicting $\mathbb{EF}$. Hence, $f_i^{CE}(N, x, t) = 0$. □

Our first characterization result imposes $\mathbb{EF}$, $\mathbb{WM}$, $\mathbb{PI}$, and $\mathbb{WLF}$.

Theorem 1. *The unique surplus-sharing rule that satisfies $\mathbb{EF}$, $\mathbb{WM}$, $\mathbb{PI}$, and $\mathbb{WLF}$ is f^{CE}.*

Proof. f^{CE} satisfies $\mathbb{EF}$ and $\mathbb{WM}$ and, by Proposition 2, $\mathbb{PI}$ and $\mathbb{LF}$, hence $\mathbb{WLF}$.

For the uniqueness part, let f be a surplus-sharing rule that satisfies the desired axioms, hence $\mathbb{LF}$ and, by Proposition 1, also $\mathbb{ET}$. Let (N, x, t) be a surplus-sharing problem. It remains to show that $f(N, x, t) = f^{CE}(N, x, t)$. We proceed by induction on $m(x) = |\{x_i \mid i \in N\}|$. If $m(x) = 1$, then the proof is finished by $\mathbb{ET}$ and $\mathbb{EF}$. Our inductive hypothesis is that $f(N, x, t) = f^{CE}(N, x, t)$ whenever $m(x) < k$ for some $k \in \mathbb{N}$, $k > 1$. Now, assume that $m(x) = k$. Let $S(x) = S = \{i \in N \mid x_i \leq x_j \text{ for all } j \in N\}$, $\alpha(x) = \alpha = \min_{i \in N} x_i$, and $\beta(x) = \beta = \min_{i \in N \setminus S} x_i$. Let $|S| = s$. By $\mathbb{ET}$, $f_i(N, x, t) = f_j(N, x, t)$ for all $i \in S$. We distinguish two cases:

Case 1: $t \leq s(\beta - \alpha)$. By $\mathbb{EF}$ and $\mathbb{WM}$ it remains to show that $f_i(N, x, t) = t/s$ for all $i \in S$. Assume the contrary. As $f_i(N, x, 0) = f_i^{CE}(N, x, 0) = 0$ for all $i \in N$, by $\mathbb{WM}$, continuity (see Remark 2) of $f(N, x, \cdot)$ implies that, for all $i \in S$,

$$t' = \max\{\tilde{t} \in \mathbb{R} \mid 0 \leq \tilde{t} \leq t, f_i(N, x, \tilde{t}) = f_i^{CE}(N, x, \tilde{t})\}$$

exists and, by our assumption, $t' < t$. Let $x' = x + f(N, x, t')$. Note that $S(x') = S$, $\beta(x') = \beta$, and $\alpha(x') = \alpha(x) + t'/s$. Now, for any $0 < t'' < (\beta - \alpha(x'))/s$, $f_j(N, x', t'') = 0$ for all $j \in N \setminus S$ by $\mathbb{LF}$ and $\mathbb{WM}$ so that $f(N, x', t'') = f^{CE}(N, x', t'')$ by $\mathbb{ET}$ and $\mathbb{EF}$. Therefore, by $\mathbb{PI}$ of f and f^{CE}, $f(N,x,t'+t'') = f(N,x,t') + f(N,x',t'') = f^{CE}(N,x,t') + f^{CE}(N,x',t'') = f^{CE}(N,x,t'+t'')$, which contradicts the maximality of t'.

Case 2: $t > s(\beta - \alpha) = t'$. By Case 1, $f(N,x,t') = f^{CE}(N,x,t')$. Let $x' = x + f(N,x,t')$. Then $m(x') = m(x) - 1$ so that, by the inductive hypothesis, $f(N,x',t-t') = f^{CE}(N,x',t-t')$. Finally, by $\mathbb{PI}$ we receive $f(N,x,t) = f(N,x,t') + f(N,x',t-t') = f^{CE}(N,x,t') + f^{CE}(N,x',t-t') = f^{CE}(N,x,t)$. □

Hence, as f^E satisfies all properties in Theorem 1 except $\mathbb{WM}$, this property is essential for distinguishing f^{CE} from f^E. Note that neither f^{EQ} satisfies $\mathbb{WLF}$ nor f^E satisfies $\mathbb{LF}$ (see Table 1).

Now, we consider a variable society of agents. Our second characterization replaces $\mathbb{WLF}$ in Theorem 1 by $\mathbb{RLF}$ and 2-$\mathbb{CCO}$. This axiomatization permits a direct normative comparison of the three rules.

Theorem 2. *The unique surplus-sharing rule that satisfies* $\mathbb{EF}$, $\mathbb{WM}$, $\mathbb{PI}$, $\mathbb{RLF}$, *and* 2-$\mathbb{CCO}$ *is* f^{CE}.

Proof. f^{CE} satisfies $\mathbb{EF}$, $\mathbb{WM}$, $\mathbb{PI}$, and $\mathbb{RLF}$. To show 2-$\mathbb{CCO}$ (in fact $\mathbb{CCO}$), let $N \in \mathcal{N}$, $x \in \mathbb{R}^N$, $t \geq 0$, and $\emptyset \neq S \subset N$, then $t' = t - \sum_{i \in N\setminus S} f_i(N,x,t) = \sum_{i\in S} f_i(N,x,t) \geq 0$ by $\mathbb{EF}$ and $\mathbb{WM}$. Let $y = f^{CE}(N,x,t)$ and $z = \left(f^{CE}(S,x_S,t'), y_{N\setminus S}\right)$. By (4), $x+y \succeq_{\mathcal{L}} x+z$. As $y_{N\setminus S} = z_{N\setminus S}$, the definition of Lorenz domination yields $x_S + y_S \succeq_{\mathcal{L}} x_S + z_S = x_S + f^{CE}(S,x_S,t')$. Again by (4), $x_S + f^{CE}(S,x_S,t') \succeq_{\mathcal{L}} x_S + y_S$, so that we obtain $y_S = f^{CE}(S,x_S,t')$. Hence, f^{CE} satisfies $\mathbb{CCO}$ and consequently also 2-$\mathbb{CCO}$.

For the uniqueness part, let $N \in \mathcal{N}$, $x \in \mathbb{R}^N$, $t \geq 0$, and f be a surplus-sharing rule that satisfies the desired properties. By Theorem 1, it suffices to show that f satisfies $\mathbb{WLF}$. To this end, let $N \in \mathcal{N}$ with $|N| \geq 2$, $x \in \mathbb{R}^N$, and $t \geq 0$. If $i,j \in N$ such that $i \neq j$ and $f_i(N,x,t) > 0$, we have, with $S = \{i,j\}$, by $\mathbb{EF}$ and $\mathbb{WM}$, $t' = t - \sum_{k\in N\setminus S} f_k(N,x,t) \geq 0$ so that, by 2-$\mathbb{CCO}$, $f(S,x_S,t') = f_S(N,x,t)$. Hence, by $\mathbb{RLF}$ applied to (S,x_S,t'), $x_i - x_j < t' \leq t$ and thus $\mathbb{LF}$ is shown, and hence, $\mathbb{WLF}$. □

The solidarity requirement $\mathbb{RLF}$ is needed to distinguish f^{EQ} from f^{CE} because f^{EQ} satisfies all other properties of Theorem 2. On the other hand, f^E meets both 2-$\mathbb{CCO}$ and $\mathbb{RLF}$, which shows that $\mathbb{WM}$ is crucial to compare f^{CE} with f^E from a normative point of view (see Table 1).

Remark 5. Characterizations of the uniform gains method *UG* (originally defined on the domain of all surplus sharing problems (N,x,t) with $x \geq 0$ and $t \geqslant x(N)$) emerge from Theorems 1 and 2 using (5). Indeed, the definition of 2-$\mathbb{CCO}$ remains unchanged, $\mathbb{PI}$ is the analogue of *composition* (Moulin, 2002), $\mathbb{WLF}$ and $\mathbb{RLF}$ can be easily adapted replacing, in the corresponding definitions, the surplus t by $t - x(N)$, and $\mathbb{EF}$ and $\mathbb{WM}$ are parts of the definition of a surplus-sharing method (see Remark 1).

5. Game theoretical support of f^{CE}

In this section, we study solutions for TU games, which distribute possible increments of the grand coalition according to f^{CE}. A *TU game*, for short *game*, is a pair (N,v) where $N \in \mathcal{N}$ and v is a function that associates a real number $v(S)$ with each $S \subseteq N$. We assume that $v(\emptyset) = 0$. For $t \in \mathbb{R}$ and a game (N,v), denote by (N,v^t) the game that differs from (N,v) at most inasmuch as $v^t(N) = v(N) + t$. By Γ we denote the set of all games.

We often consider a domain of games that allow to increase the worth of the grand coalition. Thus, we say that $\Gamma' \subseteq \Gamma$ is *closed under increments* if for all $(N,v) \in \Gamma'$ and all $t > 0$, $(N,v^t) \in \Gamma'$. A game (N,v) is *convex* (Shapley, 1971) if and only if $v(S) + v(T) \leq v(S\cup T) + v(S\cap T)$, for every $S,T \subseteq N$. The set of convex games is denoted by Γ_{vex}. A game (N,v) is *balanced* if and only if it has a nonempty core (Bondareva, 1963; Shapley, 1967). By Γ_{bal} we denote the set of balanced games. Let (N,v) be an arbitrary game. The core is *large* (Sharkey, 1982) if, for all $y \in \mathbb{R}^N$ such that $y(S) \geq v(S)$ for all $S \subseteq N$, there exists $x \in C(N,v)$ such that $x \leq y$. By Γ_{lc} we denote the set of games with large core. Note that $\Gamma_{vex} \subset \Gamma_{lc} \subset \Gamma_{bal}$ (if $|U| \geq 3$) and that Γ_{vex}, Γ_{bal}, and Γ_{lc} are closed under increments.

The set of *feasible payoff vectors* of (N,v) is defined by $X^*(N,v) = \{x \in \mathbb{R}^N \mid x(N) \leq v(N)\}$, the set of *efficient* payoff vectors by $X(N,v) = \{x \in \mathbb{R}^N \mid x(N) = v(N)\}$, and the *core* by $C(N,v) = \{x \in X(N,v) \mid x(S) \geq v(S) \text{ for all } S \subseteq N\}$. A *(single-valued) solution* on a domain $\Gamma' \subseteq \Gamma$ is a function σ that associates with each $(N,v) \in \Gamma'$ a unique element $\sigma(N,v)$ of $X^*(N,v)$. A solution σ on Γ' satisfies

- *Efficiency* (EF) if for all $(N,v) \in \Gamma'$, $\sigma(N,v) \in X(N,v)$;
- *Aggregate monotonicity* (Megiddo, 1974) (AM) if for all $(N,v) \in \Gamma'$ and all $t > 0$ such that $(N,v^t) \in \Gamma'$, $\sigma(N,v^t) \geq \sigma(N,v)$.

EF requires to exhaust the entire worth of the grand coalition. AM requires that no player suffers when the grand coalition becomes richer. We say that, in the special case of AM in which the increment in the worth of the grand coalition is distributed according to f^{CE}, the solution supports constrained welfare egalitarianism.

Definition 1. A solution σ on Γ' is said to *support constrained welfare egalitarianism* if for all $(N,v) \in \Gamma'$ and all $t > 0$, whenever $(N,v^t) \in \Gamma'$ it holds that

$$\sigma(N,v^t) = \sigma(N,v) + f^{CE}(N,\sigma(N,v),t). \tag{8}$$

Obviously, the equal division solution, defined by $ED_i(N,v) = \frac{v(N)}{|N|}$ for all $(N,v) \in \Gamma$ and all $i \in N$, supports constrained welfare egalitarianism on the set of all games. We show that the egalitarian solution of Dutta and Ray (1989), denoted by E, supports it as well on the domain of convex games. It is well known that E selects the unique core element that Lorenz dominates every other core point.[8] That is, given $(N,v) \in \Gamma_{vex}$,

$$E(N,v) \in C(N,v) \text{ and } E(N,v) \succ_{\mathcal{L}} y \text{ for all } y \in C(N,v)\setminus\{E(N,v)\}.$$

Outside the domain of convex games, the existence of E is not guaranteed. An alternative way to harmonize egalitarian considerations and the core, already proposed by Dutta and Ray (1989) and latter adopted by Arin and Iñarra (2001) and Arin et al. (2003), is to focus on a Lorenz maximal allocation within the core. Indeed, a solution σ on a domain $\Gamma' \subseteq \Gamma_{bal}$ is called *egalitarian* if for all $(N,v) \in \Gamma'$, $\sigma(N,v) \in C(N,v)$ and there is no $y \in C(N,v)$ such that $y \succ_{\mathcal{L}} \sigma(N,v)$.

According to Arin et al. (2003), the *lexmax* solution *Lmax*, which is an egalitarian solution, is defined, for $(N,v) \in \Gamma_{bal}$, by

$$Lmax(N,v) = \left\{x \in C(N,v) \mid \hat{x} \preceq_{lex} \hat{y} \text{ for all } y \in C(N,v)\right\},$$

where $\hat{x}$ and $\hat{y}$ are obtained from x and y, respectively, by ordering their coordinates in a non-increasing way.[9] Then *Lmax* is a singleton and it is Lorenz undominated within the core.

We now show that an egalitarian solution supports constrained welfare egalitarianism if it satisfies AM.

Theorem 3. *Let* σ *be an egalitarian solution on* $\Gamma' \subseteq \Gamma_{bal}$ *closed under increments. Then* σ *supports constrained welfare egalitarianism on* Γ' *if and only if it satisfies* AM.

8 This is not the original definition, but is shown in Dutta and Ray (1989) to coincide with their egalitarian solution for convex games.

9 Recall that for two vectors $x,y \in \mathbb{R}^N$, we say that $x \preceq_{lex} y$ if $x = y$ or there exists $k \in \{1,\ldots,|N|\}$ such that $x_i = y_i$ for $1 \leq i \leq k-1$ and $x_k < y_k$.

Proof. Only the if-part remains. Suppose there is $(N, v) \in \Gamma'$ and $t > 0$ such that $\sigma(N, v^t) \neq x^* + f^{CE}(N, x^*, t)$, where $x^* = \sigma(N, v)$. By AM, there exists $z \in \mathbb{R}^N_+$ with $z(N) = t$ such that $\sigma(N, v^t) = x^* + z$. By (4), $x^* + f^{CE}(N, x^*, t) \succ_{\mathcal{L}} x^* + z$. But $x^* + f^{CE}(N, x^*, t) \in C(N, v^t)$, which leads to a contradiction. □

Since E and $Lmax$ are egalitarian solutions satisfying AM on Γ_{vex} and Γ_{lc}, respectively (see, for instance, (Hokari and van Gellekom, 2002; Arin et al., 2003)), Theorem 3 implies the following corollary.

Corollary 1.

(i) On Γ_{vex}, E supports constrained welfare egalitarianism.
(ii) On Γ_{lc}, $Lmax$ supports constrained welfare egalitarianism.

The following example shows that, on the domain of all balanced games, there is no egalitarian solution supporting constrained welfare egalitarianism, provided $|U| \geq 3$.

Example 2. Let $N = \{1, 2, 3\}$, $v(\{i\}) = v(\{2, 3\}) = 0$, for all $i \in N$, and $v(\{1, 2\}) = v(\{1, 3\}) = v(\{1, 2, 3\}) = 1$. Then, $C(N, v) = \{(1, 0, 0)\}$ and $\left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right) \in C(N, v^1)$, so that, for an egalitarian solution σ, we obtain $\sigma(N, v) = (1, 0, 0)$ and $\sigma(N, v^1) = \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$. Hence, σ violates AM.

To conclude, with the help of one further property called bounded pairwise fairness, we characterize the set of solutions that support constrained welfare egalitarianism on a domain Γ' closed under increments. A solution σ on Γ' satisfies

- *Bounded pairwise fairness* (BPF) if for all $(N, v) \in \Gamma'$, all $t > 0$ such that $(N, v^t) \in \Gamma'$, and all $i, j \in N$, $\sigma_i(N, v^t) - \sigma_i(N, v) > 0$ implies $\sigma_i(N, v) - \sigma_j(N, v) < t$.

Note that BPF is a priority requirement imposing that, if the difference in payoffs between two players in the initial game (N, v) exceeds the total additional amount t to be divided, then in the game (N, v^t) the originally richer player cannot become better off.

Theorem 4. *Let $\Gamma' \subseteq \Gamma$ be closed under increments and σ on Γ' be an efficient solution. Then, σ on Γ' supports constrained welfare egalitarianism if and only if it satisfies* AM *and* BPF.

Proof. Let σ be an efficient solution on Γ' supporting constrained welfare egalitarianism, hence AM. To check BPF, let $(N, v) \in \Gamma'$, $t > 0$, and $i, j \in N$ such that $\sigma_i(N, v) - \sigma_j(N, v) \geq t$. By $\mathbb{LF}$ of f^{CE}, $f^{CE}_i(N, \sigma(N, v), t) = 0$, and thus $\sigma_i(N, v^t) = \sigma_i(N, v)$ which proves BPF. To prove the reverse implication, let σ be an efficient solution satisfying AM and BPF.

Claim: For all $(N, v) \in \Gamma'$, all $i, j \in N$, and all $t > 0$, if $\sigma_i(N, v) = \sigma_j(N, v)$, then $\sigma_i(N, v^t) = \sigma_j(N, v^t)$.

Suppose, on the contrary, there exist $i, j \in N$ and $t > 0$ such that $\sigma_i(N, v) = \sigma_j(N, v)$ but $\sigma_j(N, v^t) > \sigma_i(N, v^t)$. By EF and AM, σ satisfies some weak kind of continuity. Namely, it can easily be deduced that there exists a minimal $t^* \in (0, t]$ such that $\sigma_j(N, v^{t^*}) = \sigma_j(N, v^t)$. Hence,

$$\sigma_j(N, v^{t''}) < \sigma_j(N, v^{t^*}) = \sigma_j(N, v^t) \text{ for all } t'' \in [0, t^*). \tag{9}$$

Note that $\sigma_j(N, v) < \sigma_j(N, v^t)$ since, otherwise, $\sigma_i(N, v) = \sigma_j(N, v) = \sigma_j(N, v^t) > \sigma_i(N, v^t)$, contradicting AM. Let $\hat{t} \in (0, t^*)$ be such that $2 \cdot \left(v^{t^*}(N) - v^{\hat{t}}(N)\right) \leq \sigma_j(N, v^t) - \sigma_i(N, v^t)$. By EF and AM, we obtain

$$\begin{aligned}
2 \cdot (v^{t^*}(N) - v^{\hat{t}}(N))
&\leq \sigma_j(N, v^t) - \sigma_i(N, v^t) \\
&\leq \sigma_j(N, v^{t^*}) - \sigma_i(N, v^{t^*}) \\
&= \sigma_j(N, v^{\hat{t}}) - \sigma_i(N, v^{\hat{t}}) + \sigma_j(N, v^{t^*}) - \sigma_j(N, v^{\hat{t}}) \\
&\quad - (\sigma_i(N, v^{t^*}) - \sigma_i(N, v^{\hat{t}})) \\
&\leq \sigma_j(N, v^{\hat{t}}) - \sigma_i(N, v^{\hat{t}}) + \textstyle\sum_{j \in N} \left(\sigma_j(N, v^{t^*}) - \sigma_j(N, v^{\hat{t}})\right) \\
&\quad - (\sigma_i(N, v^{t^*}) - \sigma_i(N, v^{\hat{t}})) \\
&\leq \sigma_j(N, v^{\hat{t}}) - \sigma_i(N, v^{\hat{t}}) + v^{t^*}(N) - v^{\hat{t}}(N).
\end{aligned}$$

Hence, $v^{t^*}(N) - v^{\hat{t}}(N) \leq \sigma_j(N, v^{\hat{t}}) - \sigma_i(N, v^{\hat{t}})$. Now, by AM and BPF, $\sigma_j(N, v^{t^*}) = \sigma_j(N, v^{\hat{t}})$, contradicting (9), and our claim follows.

Now take $(N, v) \in \Gamma'$ and $t > 0$. Denote $\sigma(N, v) = x$ and $\sigma(N, v^t) = x^t$. Let $\mathcal{P}(x) = (N_1, N_2, \ldots, N_k)$ be the ordered partition of N as defined in Section 2. We proceed by induction on $|\mathcal{P}(x)|$.

If $k = 1$, by EF, $x_i = \frac{v(N)}{n}$ for all $i \in N$, where $|N| = n$. Hence, by our claim, $x^t_i = x^t_j$ for all $i, j \in N$, and by EF, for all $i \in N$,

$$x^t_i = \frac{v^t(N)}{n} = \frac{v(N)}{n} + \frac{t}{n} = x_i + f^{CE}_i(N, x, t),$$

where the last equality comes from $\mathbb{ET}$ of f^{CE}.

Our induction hypothesis is that $x^t = x + f^{CE}(N, x, t)$ whenever $k < \ell$ for some $\ell \in \mathbb{N}$, $\ell > 1$.

We now assume $k = \ell$. Take $i_1 \in N_1$, with $n_1 = |N_1|$ and $i_2 \in N_2$. We distinguish two cases:

Case 1: $x_{i_2} - x_{i_1} \geq \frac{t}{n_1}$. By our claim, for all $i, j \in N_1$, $x^t_i = x^t_j$, and AM together with BPF lead to $x^t_i = x_i$ for all $i \in N \setminus N_1$. Now, taking into account that f^{CE} satisfies $\mathbb{LF}$ and $\mathbb{ET}$, we have that $x^t = x + f^{CE}(N, x, t)$.

Case 2: $x_{i_2} - x_{i_1} < \frac{t}{n_1}$. Let $t' = n_1(x_{i_2} - x_{i_1})$ and $\sigma(N, v^{t'}) = x^{t'}$. Note that $t - t' > 0$. By BPF, $x^{t'}_i = x_i$ for all $i \in N \setminus N_1$. By our claim and EF, $x^{t'}_i = x_i + (x_{i_2} - x_{i_1}) = x_{i_2}$ for all $i \in N_1$. Since $|\mathcal{P}(x^{t'})| = \ell - 1$, by induction hypothesis $x^t = x^{t'} + f^{CE}(N, x^{t'}, t - t')$. Moreover, from $\mathbb{LF}$ and $\mathbb{ET}$ of f^{CE} we receive $x^{t'} = x + f^{CE}(N, x, t')$. Finally, from $\mathbb{PI}$ of f^{CE} we obtain $x^t = x^{t'} + f^{CE}(N, x^{t'}, t - t') = x + f^{CE}(N, x, t)$. Hence, σ supports constrained welfare egalitarianism on a domain Γ' of games that is closed under increments. □

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Appendix

We now provide examples that show that each property in each of the characterization results is logically independent of the remaining properties.

(i) On the logical independence of the properties in Theorem 1 when $|U| \geq 2$:

– The equal surplus rule f^{EQ} satisfies $\mathbb{EF}$, $\mathbb{WM}$, $\mathbb{PI}$ but not $\mathbb{WLF}$.
– Let $N \in \mathcal{N}$, $x \in \mathbb{R}^N$, and $t \geq 0$. As in Section 2, we denote $N_1 = \{i \in N \mid x_i \leq x_j \, \forall j \in N\}$. Define

$$f_i^{\leq}(N, x, t) = \begin{cases} \frac{t}{|N_1|} & \text{if} \quad i \in N_1, \\ 0 & \text{if} \quad i \in N \setminus N_1. \end{cases}$$

Then, $f^{\leq}$ satisfies $\mathbb{EF}$, $\mathbb{WM}$, and $\mathbb{WLF}$ but not $\mathbb{PI}$.
– f^E satisfies $\mathbb{EF}$, $\mathbb{PI}$, and $\mathbb{WLF}$ but not $\mathbb{WM}$;
– Let $N \in \mathcal{N}$, $x \in \mathbb{R}^N$, and $t \geq 0$. Define

$$f^0 = (0, 0, \ldots, 0) \in \mathbb{R}^N.$$

Then, f^0 satisfies $\mathbb{WM}$, $\mathbb{PI}$, and $\mathbb{WLF}$ but not $\mathbb{EF}$.

(ii) On the logical independence of the properties in Theorem 2 **when** $|U| \geq 2$:

– f^{EQ} satisfies $\mathbb{EF}$, $\mathbb{WM}$, $\mathbb{PI}$, and 2-$\mathbb{CCO}$ but not $\mathbb{RLF}$;
– $f^{\leq}$ satisfies $\mathbb{EF}$, $\mathbb{WM}$, $\mathbb{RLF}$, and 2-$\mathbb{CCO}$ but not $\mathbb{PI}$;
– f^E satisfies $\mathbb{EF}$, $\mathbb{PI}$, $\mathbb{RLF}$, and 2-$\mathbb{CCO}$ but not $\mathbb{WM}$;
– f^0 satisfies $\mathbb{WM}$, $\mathbb{PI}$, $\mathbb{RLF}$, and 2-$\mathbb{CCO}$ but not $\mathbb{EF}$.
– Let $N \in \mathcal{N}$, $x \in \mathbb{R}^N$, and $t \geq 0$. Define $\hat{f}(N, x, t)$ as follows: for all $i \in N$,

$$\hat{f}_i(N, x, t) = \begin{cases} \frac{t}{n\gamma - x(N)}(\gamma - x_i) & \text{if} \quad t < n\gamma - x(N), \\ f_i^{CE}(N, x, t) & \text{if} \quad t \geq n\gamma - x(N) \end{cases}$$

where $n = |N|$ and $\gamma = \gamma(x) = \max_{i \in N} x_i$.
Then, $\hat{f}$ satisfies $\mathbb{EF}$, $\mathbb{WM}$, $\mathbb{RLF}$, and $\mathbb{PI}$ but not 2-$\mathbb{CCO}$.

(iii) On the logical independence of the properties in Theorem 4:

Clearly, for one person games, EF implies both AM and BPF. However, for a domain of games Γ' that is closed under increments and not contained in the domain of one player games, by means of examples we show that each of the two properties in Theorem 4 is logically independent of the remaining property. To this end choose an arbitrary game $(N_*, v_*) \in \Gamma'$ with $|N_*| \geq 2$. Let $k \in N_*$ and $x \in \mathbb{R}^{N_*}$ given by

$$x_k = \frac{v_*(N_*)}{|N_*|} - (|N_*| - 1) \text{ and } x_i = \frac{v_*(N_*)}{|N_*|} + 1 \text{ for all } i \in N_* \setminus \{k\}.$$

– Define σ^1 as follows. First, $\sigma^1(N_*, v_*) = x$. Now, for all $t \in \mathbb{R}$ such that $(N_*, v_*^t) \in \Gamma'$, put $\sigma^1(N_*, v_*^t) = x + \frac{t}{|N_*|} \cdot e^N$, where, for any $\emptyset \neq S \subseteq N_*$, $e^S \in \mathbb{R}^{N_*}$ denotes the *indicator* function of S defined by

$$e_i^S = \begin{cases} 1, \text{ if } i \in S, \\ 0, \text{ if } i \in N_* \setminus S. \end{cases}$$

For all other $(N, v) \in \Gamma'$, $\sigma^1(N, v) = \frac{v(N)}{|N|} e^N$. Then, σ^1 is an efficient solution that satisfies AM but violates BPF.

– For $(N, v) \in \Gamma'$, define

$$\sigma^2(N, v) = \begin{cases} x + te^{\{k\}} & \text{, if } (N, v) = (N_*, v_*^t) \text{ for some } t \leq 0, \\ \frac{v(N)}{|N|} e^N & \text{, for all other } (N, v) \in \Gamma'. \end{cases}$$

Then, σ^2 is an efficient solution that satisfies BPF but violates AM.

References

Arin, J., Iñarra, E., 2001. Egalitarian solutions in the core. Internat. J. Game Theory 30, 187–193.
Arin, J., Kuipers, J., Vermeulen, D., 2003. Some characterizations of egalitarian solutions on classes of TU-games. Math. Social Sci. 46, 327–345.
Bondareva, O.N., 1963. Some applications of linear programming methods to the theory of cooperative games. Probl. Kibernitiki 10, 119–139.
Chun, Y., 1989. A noncooperative justification for egalitarian surplus sharing. Math. Social Sci. 17, 245–261.
Dutta, B., Ray, D., 1989. A concept of egalitarianism under participation constraints. Econometrica 57, 615–635.
Gaertner, W., Xu, Y., 2020. Loss sharing: Characterizing a new class of rules. Math. Social Sci. 107, 37–40.
Herrero, C., Maschler, M., Villar, A., 1999. Individual rights and collective responsibility: the rights-egalitarian solution. Math. Social Sci. 37, 59–77.
Hokari, T., van Gellekom, 2002. Population monotonicity and consistency in convex games: Some logical relations. Internat. J. Game Theory 31, 593–607.
Hougaard, J.L., Moreno-Ternero, J., Osterdal, L.P., 2012. A unifying framework for the problem of adjudicating conflicting claims. J. Math. Econom. 48, 107–114.
Hougaard, J.L., Moreno-Ternero, J., Osterdal, L.P., 2013. Rationing in the presence of baselines. Soc. Choice Welf. 40, 1047–1066.
Ju, B.-G., Moreno-Ternero, J., 2017. Fair allocation of disputed properties. Internat. Econom. Rev. 58, 1279–1301.
Ju, B.-G., Moreno-Ternero, J., 2018. Entitlement theory of justice and end-state fairness in the allocation of goods. Econ. Philos. 34, 317–341.
Megiddo, N., 1974. On the monotonicity of the bargaining set, the kernel, and the nucleolus of a game. SIAM J. Appl. Math. 27, 355–358.
Moreno-Ternero, J., Roemer, J.E., 2012. A common ground for resource and welfare egalitarianism. Games Econom. Behav. 75, 832–841.
Moulin, H., 1987. Equal or proportional division of a surplus, and other methods. Internat. J. Game Theory 16, 161–186.
Moulin, H., 2002. Axiomatic cost and surplus-sharing. In: Arrow, K., Sen, A., Suzumura, K. (Eds.), Handbook of Social Choice and Welfare, vol. 1, pp. 289–357.
O'Neill, B., 1962. A problem of rights arbitration from the Talmud. Math. Social Sci. 2, 345–371.
Pfingsten, A., 1991. Surplus-sharing methods. Math. Social Sci. 21, 287–301.
Pfingsten, A., 1998. Cheating by groups and cheating over time in surplus sharing problems. Math. Social Sci. 36, 243–249.
Shapley, L.S., 1967. On balanced sets and cores. Nav. Res. Logist. Q. 14, 453–460.
Shapley, L.S., 1971. Cores of convex games. Internat. J. Game Theory 1, 11–26.
Sharkey, W., 1982. Cooperative games with large cores. Internat. J. Game Theory 11, 175–182.
Thomson, W., 2003. Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: a survey. Math. Social Sci. 45, 249–297.
Thomson, W., 2015. Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: An update. Math. Social Sci. 74, 41–59.
Timoner, P., Izquierdo, J.M., 2016. Rationing problems with ex-ante conditions. Math. Social Sci. 79, 46–52.
Young, H.P., 1988. Distributive justice in taxation. J. Econom. Theory 44, 321–335.