Discrete Applied Mathematics 341 (2023) 60–81

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Minimal balanced collections and their application to core stability and other topics of game theory☆

Dylan Laplace Mermoud [a,*], Michel Grabisch [b], Peter Sudhölter [c]

[a] *Centre d'Économie de la Sorbonne, Université Paris 1 Panthéon-Sorbonne, France*
[b] *Paris School of Economics, Université Paris 1 Panthéon-Sorbonne, France*
[c] *Department of Economics, University of Southern Denmark, Denmark*

A R T I C L E   I N F O

Article history:
Received 10 October 2022
Received in revised form 20 July 2023
Accepted 24 July 2023
Available online xxxx

Keywords:
Minimal balanced collection
Cooperative game
Core
Stable set
Balancedness
Hypergraph
Algorithm

A B S T R A C T

Minimal balanced collections are a generalization of partitions of a finite set of n elements and have important applications in cooperative game theory and discrete mathematics. However, their number is not known beyond $n = 4$. In this paper we investigate the problem of generating minimal balanced collections and implement the Peleg algorithm, permitting to generate all minimal balanced collections till $n = 7$. Secondly, we provide practical algorithms to check many properties of coalitions and games, based on minimal balanced collections, in a way which is faster than linear programming-based methods. In particular, we construct an algorithm to check if the core of a cooperative game is a stable set in the sense of von Neumann and Morgenstern. The algorithm implements a theorem according to which the core is a stable set if and only if a certain nested balancedness condition is valid. The second level of this condition requires generalizing the notion of balanced collection to balanced sets.

1. Introduction

The term "balanced collection" has been coined by Shapley in his seminal paper [23], giving necessary and sufficient conditions for the core of a cooperative game to be nonempty. Cooperative game theory aims at defining rational ways (called solutions of a game) in sharing a benefit obtained by cooperation of a set of players. The core of a game (Gillies [15]) is one of the most popular concepts of solution of a game, and appears in other domains like decision theory and combinatorial optimization.

A balanced collection is a collection of subsets of a finite set N, and can be seen as a generalization of partitions of N, in the sense that weights are assigned to each subset in the collection, in order that each element of N receives a total weight equal to 1. In fact, mainly minimal balanced collections are of interest, that is, balanced collections for which no proper subcollection is balanced. Apart from their use in cooperative game theory, balanced collections appear in several domains of discrete mathematics (e.g., hypergraphs), combinatorics (e.g., combinatorial design theory), while their counterpart, namely, unbalanced collections (more precisely, maximal unbalanced collections), appear in quantum physics.

Being a generalization of partitions, (minimal) balanced collections are highly combinatorial objects, and their number is so far not known beyond $|N| = 4$. However, in the domain of cooperative game theory, Peleg [20] has proposed a recursive algorithm to generate them, which, as far as we know, has never been implemented.

☆ The authors are grateful for valuable remarks of a referee and the area editor who helped to improve this article.
* Corresponding author.
E-mail address: dylan.laplace.mermoud@gmail.com (D. Laplace Mermoud).

https://doi.org/10.1016/j.dam.2023.07.025
0166-218X/

The first achievement of this paper is to provide algorithms generating minimal balanced collections. We begin by implementing Peleg's algorithm and generating all minimal balanced collections till $n = 7$. We show also that minimal balanced collections can be generated via a vertex enumeration method, hereby relating the problem of generating minimal balanced collections to one of the fundamental open problems in geometry [7]. Applying Avis and Fukuda's method for vertex enumeration [1], we find the former method by Peleg more efficient.

The second achievement of the paper is to show that minimal balanced collections are a central concept in cooperative game theory that can be applied to check a variety of properties, and most importantly, core stability. The question to check whether the core of a game is a stable set in the sense of von Neumann and Morgenstern has remained an open problem for a long time, and was recently solved by Grabisch and Sudhölter [16]. It turns out that the test of core stability amounts to a complex nested balancedness condition, which needs in particular to identify exact coalitions, strictly vital exact coalitions, extendable coalitions, and feasible collections of coalitions. We show that all these notions can be tested or generated via the use of minimal balanced collections, and that this way is faster than linear programming methods. The main reason is that minimal balanced collections do not depend on the game considered but only on the number of players. As a consequence, minimal balanced collections need to be generated only once, and can be used repeatedly for any game. We have implemented all these algorithms as computer programs, that allow to obtain finally a general algorithm to check core stability.

The third achievement is a direct consequence of the nested balancedness condition of core stability. It happens that this condition requires a more general notion of balanced collection, which we call (minimal) balanced sets. A finite set in the nonnegative orthant is balanced if the characteristic vector of N is a positive linear combination of its elements. We show that this generalized notion causes difficulties and does not seem to be easily generated by methods applied for minimal balanced collections. Again, this problem can be related to a vertex enumeration problem.

The paper is organized as follows. Section 2 introduces cooperative games, stable sets and balanced collections. Section 3 shows where minimal balanced collections appear in different domains and introduces unbalanced collections, which are used in quantum physics. Section 4 focuses on the generation of minimal balanced collections, while Section 5 describes applications in cooperative game theory in detail. Section 6 introduces minimal balanced sets, which is a key notion in the nested balancedness condition for testing core stability, the object of Section 7. Section 8 concludes the paper. Details on the use of maximal unbalanced collections in quantum physics are given in Appendix A, while Appendix B illustrates by examples the Peleg algorithm. Finally, all algorithms presented in the paper are given in pseudo-code, mostly in Appendix C, or in the main text.

2. Background on TU-games and balanced collections

2.1. TU-games

Let N be a finite nonempty set of n players and let 2^N denote its power set, i.e., the set of all subsets of N. In most cases, we tacitly assume that $N = \{1, \dots, n\}$. A nonempty subset of N is called a *coalition*. A *(cooperative TU) game* is a pair (N, v) with $v : 2^N \to \mathbb{R}$ such that $v(\varnothing) = 0$.

Let (N, v) be a game. An *allocation* or *payoff vector* is an n-dimensional vector $x = (x_i)_{i\in N} \in \mathbb{R}^N$, representing the distribution of payoffs among the players. Denote by $x(S) = \sum_{i\in S} x_i$ the payoff received by coalition $S \subseteq N$ with x.

An allocation is said to be *efficient* for the game (N, v) if $x(N) = v(N)$, and an efficient allocation is called a *preimputation*. The set of preimputations is denoted by $X(N, v)$.

An allocation x is *individually rational* for the game (N, v) if $x_i \geq v(\{i\})$ for every player $i \in N$. An individually rational preimputation is called an *imputation*, and the set of imputations is denoted by $I(N, v)$.

An allocation x is *coalitionally rational* for the game (N, v) if $x(S) \geq v(S)$ for every coalition $S \in 2^N$. The *core* [14,15] of the game (N, v) is the set of coalitionally rational preimputations, i.e.,

$$C(N, v) = \{x \in \mathbb{R}^N \mid x(S) \geq v(S), \forall S \subseteq N, x(N) = v(N)\}.$$

2.2. Stable sets

A preimputation $x \in X(N, v)$ *dominates via a coalition* $S \subseteq N$ another preimputation y if $x(S) \leq v(S)$ and $x_i > y_i$ for every $i \in S$, written $x \operatorname{dom}_S y$. If there exists a coalition S such that $x \operatorname{dom}_S y$, then x *dominates* y, which is denoted by $x \operatorname{dom} y$.

Based on this notion, von Neumann and Morgenstern [25] introduced the concept of stable sets for cooperative games. A set $U \subseteq I(N, v)$ is a *stable set* if it satisfies

(i) *Internal stability*: if $y \in U$ is dominated by $x \in I(N, v)$, then $x \notin U$,
(ii) *External stability*: $\forall y \in I(N, v) \setminus U, \exists x \in U$ such that $x \operatorname{dom} y$.

Due to their important stability properties, von Neumann and Morgenstern regard stable sets as the main solution concept for cooperative games and call them, hence, *solutions*. However, although intuitively appealing, considering stable sets is problematic. Indeed, they may be not unique, and there exist games without stable sets (see Lucas [19]). Moreover, they are in general difficult to identify. According to Deng and Papadimitriou [10], the existence of a stable set may be undecidable. These difficulties have led to the development of other solution concepts, among which the core (Gillies [15], see above) is the most popular. The computation of the core is relatively easy but expensive, due to the large number of inequalities defining it, and it is empty for large classes of games.

Both stable sets and the core have their own merits as solution concepts. Indeed, the notions of domination and stability are highly intuitive, "coalitional rationality" is a desirable property, and its easy computability supports the core. By definition, the core is contained in each stable set. Hence, if the core is (externally) stable, it must be the unique stable set. Therefore, it is an interesting and important problem to characterize the set of games for which the above-mentioned solution concepts coincide, i.e., to provide necessary and sufficient conditions for external stability of the core. This is what Grabisch and Sudhölter have achieved [16]. We will come back to this result in Section 7.

2.3. Balanced collections

We use throughout the paper the following notation: For any $T \subseteq N$, its *characteristic vector* $\mathbf{1}^T \in \mathbb{R}^N$ is defined by $\mathbf{1}^T_i = 1$ if $i \in T$, and $\mathbf{1}^T_i = 0$ otherwise.

Definition 2.1. A collection $\mathcal{B}$ of coalitions in 2^N is *balanced* if there exists a system of positive weights $(\lambda_S)_{S\in\mathcal{B}}$, called *balancing weights*, such that $\sum_{S\in\mathcal{B}} \lambda_S \mathbf{1}^S = \mathbf{1}^N$.

Therefore, $\mathcal{B}$ is balanced if and only if the vector $\mathbf{1}^N$ is in the relative interior of the cone generated by the vectors $\mathbf{1}^S, S \in \mathcal{B}$.

Let us give some examples of balanced collections:

1. Every partition of N is a balanced collection, where every set belonging to the partition has balancing weight equal to 1;
2. For $n = 3$: $\{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$ is balanced with weights $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$. More generally, every anti-partition (the set of complements of a partition of $s \geq 2$ blocks) is a balanced collection, where the balancing weight of each element is $\frac{1}{s-1}$;
3. For $n = 4$: $\{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3, 4\}\}$ is balanced with weights $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}\right)$.

A balanced collection is *minimal* if it does not contain a balanced proper subcollection. We denote by $\mathbb{B}(N)$ the set of minimal balanced collections on N. It is well known that a balanced collection is minimal if and only if its system of balancing weights is unique. Note that all the examples above are minimal balanced collections.

As the subsequent sections will show, minimal balanced collections are highly combinatorial objects, much more than partitions, and they appear in many domains. Their number for a given n is not known beyond $n = 4$. Although Peleg [20] proposed an inductive algorithm to generate them, up to our knowledge it has never been implemented, so that there is no available list of minimal balanced collections for $n = 5$ and larger. Thanks to our own implementation of Peleg's algorithm, we are able to give the list of minimal balanced collections till $n = 7$ (see Section 4.1).

3. Balanced collections in other domains

Balanced collections are known in other fields of discrete mathematics under different names, especially in hypergraphs and combinatorial design theory. We start with hypergraphs.

3.1. Hypergraphs

(see Berge [2]) An *(undirected) hypergraph* $\mathcal{H}$ is a pair $\mathcal{H} = (X, E)$ where X is the set of elements called *nodes* or *vertices*, and E is a collection of non-empty subsets of X, called *hyperedges* or *edges*. A hypergraph is *simple* if there is no repetition in the collection of hyperedges. To get rid of repetitions, it is convenient to consider any hypergraph as a simple hypergraph (X, E) to which we assign a weight (or multiplicity) function $w : E \to \mathbb{N}$ on hyperedges, defining the degree of a vertex x by

$$d^w_{\mathcal{H}(x)} = \sum_{e\in\mathcal{H}(x)} w(e).$$

We denote by $\mathcal{H}(x)$ the family of hyperedges $e \in E$ containing the vertex x.

Definition 3.1. A hypergraph (X, E) is *d-regular* if every vertex has *degree d*. A hypergraph is *regular* if it is d-regular for some d.

It is easy to see that regular hypergraphs are in bijection with balanced collections. Indeed, consider a regular hypergraph $\mathcal{H} = (X, E)$ with weight (multiplicity) function w. Denote by $\delta := d^w_{\mathcal{H}(x)}$ the degree of any vertex $x \in X$, and define the weight system $(\lambda_e)_{e\in E}$ by

$$\lambda_e := \frac{w(e)}{\delta}, \quad \forall e \in E.$$

Then, E, viewed as a set of subsets of X, is a balanced collection on X with balancing weight system $(\lambda_e)_{e\in E}$. Conversely, any balanced collection on X can be seen as a regular hypergraph.

In [23], Shapley calls the value δ the *depth* of the associated balanced collection, and $w(S)$ the *multiplicity* of the coalition S in the balanced collection.

There is another concept in hypergraph theory which corresponds to balanced collections.

Definition 3.2. A *fractional matching* in a hypergraph $\mathcal{H} = (X, E)$ is a function $\mu : E \to [0, 1]$ such that for every vertex x in X,

$$\sum_{e\in\mathcal{H}(x)} \mu(e) \leq 1.$$

A fractional matching is called *perfect* if for every vertex x in X,

$$\sum_{e\in\mathcal{H}(x)} \mu(e) = 1.$$

It is easy to see that any simple hypergraph on X admitting a perfect fractional matching μ induces a balanced collection on X, which is $\{e \in E \mid \mu(e) > 0\}$. The converse holds as well.

3.2. Combinatorial design theory

Combinatorial design theory is the part of combinatorial mathematics that deals with the existence, construction, and properties of systems of finite sets whose arrangements satisfy generalized concepts of *balance* and/or *symmetry*. The following definitions come from Colbourn and Dinitz [8].

Definition 3.3. A *t-wise balanced design* of type $t - (n, K, \delta)$ is a pair $(N, \mathcal{B})$ where N is a set of n elements, called *points*, and $\mathcal{B}$ is a collection of subsets of N, called *blocks*, with the property that the size of every block belongs to the set K, and every subset of size t of N is contained in exactly δ blocks.

In the context of combinatorial design, the value δ is called the *replication number*. With a slightly different notation, t-wise balanced designs are also called *t-balanced incidence structures*.

By a mechanism similar to the one used with hypergraphs, it is easy to see that 1-wise balanced designs are balanced collections, and the converse is also true provided repetition is allowed in $\mathcal{B}$.

3.3. Unbalanced collections and hyperplanes arrangements

A collection of subsets of N which is not balanced is said to be *unbalanced*. It is *maximal* if no supercollection of it is unbalanced. Strangely enough, maximal unbalanced collections are also an important topic of discrete mathematics, with applications in physics (see Billera et al. [3]).

As for minimal balanced collections, there is no closed-form formula to compute the number of maximal unbalanced collections. Their number is known till $n = 9$ (see Table 1). In order to characterize unbalanced collections, we recall a result from Derks and Peters [11].

Table 1
Number of maximal unbalanced collections as a function of n.

n	Nb of maximal unbalanced collections
2	2
3	6
4	32
5	370
6	11,292
7	1,066,044
8	347,326,352
9	419,172,756,930

Proposition 3.4. *A collection $\mathcal{S} \subseteq 2^N$ of nonempty sets is balanced if and only if for every vector $y \in \mathbb{R}^N$ such that $\sum_{i\in N} y_i = 0$, either $\sum_{i\in S} y_i = 0$ for every $S \in \mathcal{S}$ or there exist $S, T \in \mathcal{S}$ such that $\sum_{i\in S} y_i > 0$ and $\sum_{i\in T} y_i < 0$.*

Therefore, a collection $\mathcal{S}$ of nonempty sets is unbalanced if and only if there exists $y \in \mathbb{R}^N$ such that $\sum_{i\in N} y_i = 0$ and $\sum_{i\in S} y_i > 0$ for all $S \in \mathcal{S}$. Let us give two examples of maximal unbalanced collections with a possible vector y:

(i) For $n = 3$: $\{\{1, 2\}, \{1, 3\}, \{1\}\}$, $y = (2, -1, -1)$;
(ii) For $n = 4$: $\{\{1\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}\}$, $y = (3, -1, -1, -1)$.

This very simple characterization permits to see that maximal unbalanced collections are the same as Björner's positive set sum systems [5], and that they are related to hyperplanes arrangements. In the hyperplane $H_N = \{x \in \mathbb{R}^N \mid x(N) = 0\}$, consider the hyperplanes $\{x \in H_N \mid x(S) = 0\}$, for all $S \in 2^N \setminus \{\varnothing, N\}$. Observe that since we are in H_N, the hyperplanes induced by S and $N \setminus S$ are the same, but their normal vectors point in opposite directions. It follows that these $2^{n-1} - 1$ distinct hyperplanes (called by Billera et al. *restricted all-subset arrangement*) define full-dimensional elementary regions $R_{\mathcal{S}}$, each one characterized by the collection $\mathcal{S} = \{S \in 2^N \setminus \{\varnothing, N\} \mid x(S) > 0\}$ where x is any element of $R_{\mathcal{S}}$, and with the property $|\mathcal{S}| = 2^{n-1} - 1$. In addition, each elementary region $R_{\mathcal{S}}$ corresponds to a maximal unbalanced collection $\mathcal{S}$, and conversely. This shows that each maximal unbalanced collection has $2^{n-1} - 1$ sets, and their number is the number of elementary regions induced by the hyperplane arrangement (see Fig. 1).

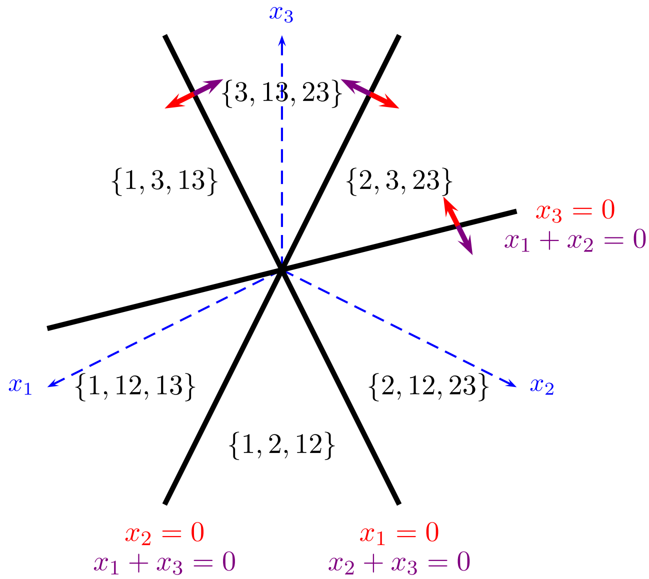

Fig. 1. The restricted all-subset arrangement for $n = 3$ in the plane H_N. Arrows indicate the normal vector to the hyperplane of the same color. The 6 maximal unbalanced collections (subsets are written without comma and braces) correspond to the 6 regions.

As remarked by Billera et al. the above hyperplane arrangement (and consequently, maximal unbalanced collections) appears in the field of thermal quantum physics. We give some details on this application in Appendix A.

4. Generation of minimal balanced collections

So far, there are two known methods for generating minimal balanced collections. The first one, due to Peleg [20], is specifically devoted to the generation of minimal balanced collections and proceeds by induction on the number of players n. The second one uses any vertex enumeration method for convex polyhedra, applied on a specific polytope whose vertices correspond to the minimal balanced collections.

4.1. Peleg's algorithm

Peleg [20] developed an inductive method to construct, from the minimal balanced collections defined on a set N, all those that are defined on the set $N' = N \cup \{p\}$, with p a new player that was not included in N. As far as we know, Peleg's inductive method has never been implemented as an algorithm, perhaps due to the rather abstract way it is described, far from any algorithmic considerations. For this reason, we translate Peleg's method and results from an algorithmic point of view, reproving his results in our new formalism for the sake of clarity and completeness. In the following, the main result is divided into four cases and the fourth one is slightly generalized.

Let $\mathcal{C} = \{S_1, \ldots, S_k\}$ be a balanced collection of k coalitions on N. Denote by $[k]$ the set $\{1, \ldots, k\}$ for any positive integer k. If $\lambda^{\mathcal{C}}$ is a system of balancing weights for $\mathcal{C}$ and $I \subseteq [k]$ is a subset of indices, denote by $\lambda_I^{\mathcal{C}}$ the sum $\sum_{i\in I} \lambda_{S_i}^{\mathcal{C}}$. Also, denote by $A^{\mathcal{C}}$ the $(n \times k)$-matrix formed by the k column vectors $\mathbf{1}^{S_1}, \ldots, \mathbf{1}^{S_k}$. Denote by $\mathrm{rk}(A^{\mathcal{C}})$ the rank of the matrix $A^{\mathcal{C}}$, meaning the dimension of the Euclidean space spanned by its columns viewed as k-dimensional vectors.

First case. Assume that $\mathcal{C}$ is a minimal balanced collection on N. Take $I \subseteq [k]$ such that $\lambda_I^{\mathcal{C}} = 1$. Denote by $\mathcal{C}'$ the new collection in which the coalitions $\{S_i\}_{i\in I}$ contain the new player p as additional member and the other coalitions $\{S_j\}_{j\in[k]\setminus I}$ are kept unchanged.

Lemma 4.1. *$\mathcal{C}'$ is a minimal balanced collection on N'.*

Proof. Because $\mathcal{C}$ is a minimal balanced collection, the equalities $\sum_{S\in\mathcal{C}',S\ni i}\lambda_S^{\mathcal{C}} = 1$ are already satisfied for any player $i \in N$. By definition of I, we also have that $\sum_{S\in\mathcal{C}',S\ni p}\lambda_S^{\mathcal{C}} = 1$. Then $\mathcal{C}'$ is balanced. Because $\mathcal{C}$ is minimal, so is $\mathcal{C}'$. □

Second case. We assume that $\mathcal{C}$ is a minimal balanced collection on N. Take $I \subseteq [k]$ such that $\lambda_I^{\mathcal{C}} < 1$. We denote by $\mathcal{C}'$ the new collection in which the coalitions $\{S_i\}_{i\in I}$ contain the new player p as additional member, the other coalitions $\{S_j\}_{j\in[k]\setminus I}$ are kept unchanged, and in which the coalition $\{p\}$ is added:

$$\mathcal{C}' = \{S_i \cup \{p\} \mid i \in I\} \cup \{S_i \mid i \in [k] \setminus I\} \cup \{\{p\}\}.$$

Lemma 4.2. *$\mathcal{C}'$ is a minimal balanced collection on N'.*

Proof. Because $\mathcal{C}$ is a minimal balanced collection, the equalities $\sum_{S\in\mathcal{C}',S\ni i}\lambda_S^{\mathcal{C}} = 1$ are already satisfied for any player $i \in N$. Define $\lambda^{\mathcal{C}'}$ such that $\lambda_S^{\mathcal{C}'} = \lambda_S^{\mathcal{C}}$ for $S \in \mathcal{C}$ and $\lambda_{\{p\}}^{\mathcal{C}'} = 1 - \lambda_I^{\mathcal{C}}$. Therefore

$$\sum_{\substack{S\in\mathcal{C}'\\ S\ni p}} \lambda_S^{\mathcal{C}'} = \lambda_{\{p\}}^{\mathcal{C}'} + \sum_{i\in I}\lambda_{S_i}^{\mathcal{C}} = 1 - \sum_{i\in I}\lambda_{S_i}^{\mathcal{C}} + \sum_{i\in I}\lambda_{S_i}^{\mathcal{C}} = 1.$$

Then $\mathcal{C}'$ is balanced. We cannot obtain a balanced subcollection of $\mathcal{C}'$ by discarding one of the $\{S_i\}_{i\in[k]}$ because $\mathcal{C}$ is minimal, and we can also not either discard coalition $\{p\}$ because $\lambda_I^{\mathcal{C}} < 1$ and the balancing weights for $\mathcal{C}$ are unique. So $\mathcal{C}'$ is minimal. □

Third case. We assume that $\mathcal{C}$ is a minimal balanced collection on N. Take a subset $I \subseteq [k]$ and an index $\delta \in [k] \setminus I$ such that $1 > \lambda_I^{\mathcal{C}} > 1 - \lambda_{S_\delta}^{\mathcal{C}}$. We denote by $\mathcal{C}'$ the new collection in which the coalitions $\{S_i\}_{i\in I}$ contain the new player p as additional member, the other coalitions $\{S_j\}_{j\in[k]\setminus I}$ are kept unchanged, and in which the coalition $S_\delta \cup \{p\}$ is added:

$$\mathcal{C}' = \{S_i \cup \{p\} \mid i \in I\} \cup \{S_i \mid i \in [k] \setminus I\} \cup \{S_\delta \cup \{p\}\}.$$

Lemma 4.3. *$\mathcal{C}'$ is a minimal balanced collection on N'.*

Proof. Define $\lambda^{\mathcal{C}'}$ by $\lambda_S^{\mathcal{C}'} = \lambda_S^{\mathcal{C}}$ for $S \in \mathcal{C} \setminus \{S_\delta\}$,

$$\lambda_{S_\delta\cup\{p\}}^{\mathcal{C}'} = 1 - \lambda_I^{\mathcal{C}} \text{ and } \lambda_{S_\delta}^{\mathcal{C}'} = \lambda_{S_\delta}^{\mathcal{C}} - \lambda_{S_\delta\cup\{p\}}^{\mathcal{C}'}.$$

Let $i \in N$ be a player. If $i \notin S_\delta$, by balancedness of $\mathcal{C}$, $\sum_{S\in\mathcal{C}',S\ni i}\lambda_S^{\mathcal{C}'} = 1$. If $i \in S_\delta$, then

$$\sum_{\substack{S\in\mathcal{C}'\\ S\ni i}} \lambda_S^{\mathcal{C}'} = \lambda_{S_\delta\cup\{p\}}^{\mathcal{C}'} + \lambda_{S_\delta}^{\mathcal{C}'} + \sum_{\substack{S\in\mathcal{C}\setminus\{S_\delta\}\\ S\ni i}} \lambda_S^{\mathcal{C}'} = \lambda_{S_\delta}^{\mathcal{C}} + \sum_{\substack{S\in\mathcal{C}\setminus\{S_\delta\}\\ S\ni i}} \lambda_S^{\mathcal{C}} = \sum_{\substack{S\in\mathcal{C}\\ S\ni i}} \lambda_S^{\mathcal{C}},$$

that is equal to 1 by balancedness of $\mathcal{C}$. Concerning player p,

$$\sum_{\substack{S\in\mathcal{C}'\\ S\ni p}} \lambda_S^{\mathcal{C}'} = \lambda_{S_\delta\cup\{p\}}^{\mathcal{C}'} + \lambda_I^{\mathcal{C}'} = 1 - \lambda_I^{\mathcal{C}} + \lambda_I^{\mathcal{C}} = 1.$$

Then $\mathcal{C}'$ is balanced. Because none of the coalitions $S \in \mathcal{C}$ or $S_\delta \cup \{p\}$ can be discarded to obtain a balanced subcollection, the proof is finished. □

Last case. In this case, assume that $\mathcal{C}$ is the union of two different minimal balanced collections on N, $\mathcal{C}^1$, and $\mathcal{C}^2$, such that the rank of $A^{\mathcal{C}}$ is $k-1$. Define two systems of balancing weights for $\mathcal{C}$, by

$$\mu_S = \begin{cases} \lambda_S^{\mathcal{C}^1} \text{ if } S \in \mathcal{C}^1, \\ 0 \text{ otherwise.} \end{cases} \qquad \nu_S = \begin{cases} \lambda_S^{\mathcal{C}^2} \text{ if } S \in \mathcal{C}^2, \\ 0 \text{ otherwise.} \end{cases}$$

Take a subset $I \subseteq [k]$ such that $\mu_I \neq \nu_I$ and

$$t^I = \frac{1-\mu_I}{\nu_I - \mu_I} \in]0, 1[.$$

Denote by $\mathcal{C}'$ the new collection in which the coalitions $\{S_i\}_{i\in I}$ contain the new player p as additional member and the other coalitions $\{S_j\}_{j\in[k]\setminus I}$ are kept unchanged.

Lemma 4.4. *$\mathcal{C}'$ is a minimal balanced collection on N'.*

Proof. Define $\lambda = (\lambda_S)_{S\in\mathcal{C}'}$ by $\lambda_S = (1-t^I)\mu_S + t^I\nu_S$. Because λ is a convex combination of two systems of balancing weights of $\mathcal{C}$, $\sum_{S\in\mathcal{C}',S\ni i}\lambda_S = 1$ for all the players $i \in N$. Concerning player p,

$$\sum_{\substack{S\in\mathcal{C}'\\ S\ni p}} \lambda_S = \lambda_I = (1-t^I)\mu_I + t^I\nu_I = \mu_I + t^I(\nu_I - \mu_I) = \mu_I + 1 - \mu_I = 1.$$

We conclude that $\mathcal{C}'$ is a balanced collection. Now, let us prove the minimality of $\mathcal{C}'$ as a balanced collection. Because $\text{rk}\left(A^{\mathcal{C}}\right) = k-1$, the set of systems of balancing weights for $\mathcal{C}$ is the set of convex combinations of μ and ν, and therefore the set of systems of balancing weights for $\mathcal{C}'$ is a subset of this. More precisely, it is the subset $\{\lambda \in \text{conv}(\mu,\nu) \mid \lambda_I = 1\}$, equivalently $\{t \in [0,1] \mid (1-t)\mu_I + t\nu_I = 1\} = T$, and therefore the condition is on the variable t. By assumption, $\mu_I \neq \nu_I$, and then $\mu_I < 1 \leq \nu_I$ without loss of generality. Because the map $f : t \mapsto (1-t)\mu_I + t\nu_I$ is linear and $f(0) < 1$ and $f(1) \geq 1$, there is a unique $t^* \in T$ such that $f(t^*) = 1$, then this unique t^* must be t^I. □

Final algorithm. It is now possible to construct, from the set of minimal balanced collections on a set N, the set of minimal balanced collections on another set $N' = N \cup \{p\}$ (see Algorithm 1).

Algorithm 1 AddNewPlayer

Require: A set of minimal balanced collection $\mathbb{B}(N)$ on a set N
Ensure: A set of minimal balanced collection $\mathbb{B}(N')$ on a set $N' = N \cup \{p\}$

procedure AddNewPlayer($\mathbb{B}(N), p$)
  for $(\mathcal{C}^1, \mathcal{C}^2) \in \mathbb{B}(N) \times \mathbb{B}(N)$ **do**
    $\mathcal{C} \leftarrow \mathcal{C}^1 \cup \mathcal{C}^2$ and $k \leftarrow |\mathcal{C}|$
    if $\text{rk}(A^{\mathcal{C}}) = k-1$ **then**
      for $I \subseteq [k]$ **such that** $t^I \in]0,1[$ **do**
        for $i \in I$ **do** add $S_i \cup \{p\}$ with weights $(1-t^I)\mu_{S_i} + t^I\nu_{S_i}$ to $\mathcal{C}'$
        for $i \notin I$ **do** add S_i with weights $(1-t^I)\mu_{S_i} + t^I\nu_{S_i}$ to $\mathcal{C}'$
        add $\mathcal{C}'$ to $\mathbb{B}(N')$
  for $\mathcal{C} \in \mathbb{B}_N$ **do**
    $k \leftarrow |\mathcal{C}|$
    for $I \subseteq [k]$ **such that** $\lambda_I^{\mathcal{C}} \leq 1$ **do**
      $C' \leftarrow \varnothing$
      for $i \in I$ **do** add $S_i \cup \{p\}$ with weights $\lambda_{S_i}^{\mathcal{C}}$ to $\mathcal{C}'$
      for $i \notin I$ **do** add S_i with weights $\lambda_{S_i}^{\mathcal{C}}$ to $\mathcal{C}'$
      if $\lambda_I^{\mathcal{C}} < 1$ **then** add $\{p\}$ with weight $1 - \lambda_I^{\mathcal{C}}$ to $\mathcal{C}'$
      add $\mathcal{C}'$ to $\mathbb{B}(N')$
      for $\delta \in [k] \setminus I$ **such that** $\lambda_{S_\delta} > 1 - \lambda_I^{\mathcal{C}}$ **do**
        $C' \leftarrow \varnothing$
        for $i \in I \setminus \{\delta\}$ **do** add $S_i \cup \{p\}$ with weights $\lambda_{S_i}^{\mathcal{C}}$ to $\mathcal{C}'$
        for $i \notin I \cup \{\delta\}$ **do** add S_i with weights $\lambda_{S_i}^{\mathcal{C}}$ to $\mathcal{C}'$
        add $S_\delta \cup \{p\}$ with weight $1 - \lambda_I^{\mathcal{C}}$ to $\mathcal{C}'$
        add S_δ with weight $\lambda_{S_\delta}^{\mathcal{C}} + \lambda_I^{\mathcal{C}} - 1$ to $\mathcal{C}'$
        add $\overline{\mathcal{C}}$ to $\mathbb{B}(N')$
  return $\mathbb{B}(N')$

Theorem 4.5. *The algorithm* AddNewPlayer, *which takes as an input the set of all minimal balanced collections on a set N, generates all the minimal balanced collections on the set $N' = N \cup \{p\}$.*

Proof. Thanks to the four previous lemmas, the algorithm generates only minimal balanced collections on N'. It remains to prove that every minimal collection is generated by this algorithm. Let $\mathcal{B}$ be a minimal balanced collection on N'. If the player p is removed from each coalition of $\mathcal{B}$, the collection is still balanced. Denote by $\mathcal{B}_{-p}$ this new collection.

- If $\{p\} \in \mathcal{B}$: as $\mathcal{B}$ is a minimal balanced collection, $\{\mathbf{1}^S \mid S \in \mathcal{B}\}$ is linearly independent (in $\mathbb{R}^{N'}$). Hence, because $\{p\} \in \mathcal{B}$, there does not exist $S \in \mathcal{B}$ such that $p \in S$ and $S \setminus \{p\} \in \mathcal{B}$. Therefore,

$$\left\{\mathbf{1}^{S\setminus\{p\}} \mid S \in \mathcal{B}, S \neq \{p\}\right\} = \left\{\mathbf{1}^T \mid T \in \mathcal{B}_{-p}\right\}$$

  is linearly independent in $\mathbb{R}^N$. We conclude that the balanced collection $\mathcal{B}_{-p}$ must be a minimal balanced collection so that $\mathcal{B}$ is generated by the second case.

- If $\{p\} \notin \mathcal{B}$ and there exists $S \in \mathcal{B}$ such that $p \in S$ and $T := S \setminus \{p\} \in \mathcal{B}$: then $T \in \mathcal{B}_{-p}$ and for such pairs the weights of T in the balanced collection $\mathcal{B}_{-p}$ must be the sum of the balancing weights of S and T in the balanced collection $\mathcal{B}$. Doing so, the minimality of $\mathcal{B}$ implies the minimality of $\mathcal{B}_{-p}$. Then $\mathcal{B}$ is generated by the third case.
- Assume now that there is no singleton $\{p\}$ in $\mathcal{B}$, and that there is no $S \in \mathcal{B}$ that satisfies $p \in S$ and $S \setminus \{p\} \in \mathcal{B}$.
  - ▹ If $\mathcal{B}_{-p}$ is a minimal balanced collection, $\mathcal{B}$ is generated by the first case.
  - ▹ Assume now that $\mathcal{B}_{-p}$ is not a minimal balanced collection. Because $\mathcal{B}$ is a minimal balanced collection of k coalitions, $\mathrm{rk}(A^{\mathcal{B}}) = k$, and therefore $\mathrm{rk}(A^{\mathcal{B}_{-p}}) = k - 1$. Consequently, the set of solutions of the following system of inequalities

$$A^{\mathcal{B}_{-p}}\lambda = \mathbf{1}^N, \quad \lambda_i \geq 0, \forall i \in [k] \tag{1}$$

is one-dimensional and has the form $\lambda = \lambda^0 + t\lambda^1$, where λ^0 is a system of balancing weights for $\mathcal{B}_{-p}$, t is a real number and λ^1 is a nonzero solution of the homogeneous system

$$A^{\mathcal{B}_{-p}}\lambda = 0, \quad \lambda_i \geq 0, \forall i \in [k].$$

The set of solutions of (1) being bounded and one-dimensional, it is a non-degenerate segment $[\alpha, \beta]$ that consists of all the solutions of the above system. Let $U_\alpha = \{i \mid \alpha_i > 0\}$ and $U_\beta = \{i \mid \beta_i > 0\}$. Clearly, U_α and U_β are proper subsets of $\{1, \ldots, k\}$ and $U_\alpha \cup U_\beta = \{1, \ldots, k\}$. Denote $\mathcal{B}^\alpha = \{B_i \in \mathcal{B} \mid i \in U_\alpha\}$ and $\mathcal{B}^\beta = \{B_i \in \mathcal{B} \mid i \in U_\beta\}$. α^*, the restriction of α to U_α, is a system of balancing weights for $\mathcal{B}^\alpha$, and β^*, the restriction of β to U_β, is a system of balancing weights for $\mathcal{B}^\beta$. Since α and β are extremal solutions of the system (1), $\mathcal{B}^\alpha$ and $\mathcal{B}^\beta$ must be minimal balanced collections. Then $\mathcal{B}$ is the union of $\mathcal{B}^\alpha$ and $\mathcal{B}^\beta$, and is generated by the fourth case. □

With the procedure AddNewPlayer used recursively, all the minimal balanced collections on any fixed set N are generated from the ones on $\{1, 2\}$. This is achieved by the procedure Peleg (see Algorithm 2).

Algorithm 2 Minimal balanced collections computation

Require: A number of players $n \geq 3$
Ensure: The set of minimal balanced collections on the set $[n]$

```
procedure PELEG(n)
    B({1, 2}) ← {{{1, 2}}, {{1}, {2}}}
    for i ∈ {3, ..., n} do
        B([i]) ← ADDNEWPLAYER(B([i − 1]), i)
    return B([n])
```

An example of generation of minimal balanced collections, illustrating the different cases, is given in Appendix B.

Remark 4.6. It is possible to adapt Algorithm 1 to compute the minimal balanced collections on every set system $\mathcal{F} \subseteq 2^N$ on which the game is defined. The only difference for the implementation is to check, when a new minimal balanced collection is created, that every coalition is a subset of an element of the set system. If it is not the case, just ignore the newly created collection and continue the computation.

Results and performance. We implemented the above algorithms in Python,[1] and found the following results and performance, given in Table 2. We have checked if this sequence of numbers was already known in the OEIS (On Line Encyclopedia of Integer Sequences [24]). As it was not the case, we added it to the Encyclopedia, and it can be accessed under the number A355042.[2] Moreover, we have stored all minimal balanced collections till $n = 7$.[3]

Table 2
Number of minimal balanced collections as a function of n.

n	Nb of minimal balanced collections	CPU time
1	1	–
2	2	0.00 s
3	6	0.01 s
4	42	0.03 s
5	1,292	1.05 s
6	200,214	4 min 4 s
7	132,422,036	63 h

Compared to the number of maximal unbalanced collections (see Table 1), we see that the latter increase much slower than the number of minimal balanced collections.

1 Computing device: Intel Xeon W-1250, CPU 3.30 GHz, 32 GB RAM
2 See https://oeis.org/A355042.
3 Available on request from the corresponding author.

4.2. Vertex enumeration method

Consider the polytope $W(N)$ defined by

$$W(N) = \left\{ \lambda \in \mathbb{R}^{2^N \setminus \{\emptyset\}} \mid \sum_{S \in 2^N \setminus \{\emptyset\}} \lambda_S \mathbf{1}^S = \mathbf{1}^N, \lambda_S \geq 0, \forall S \in 2^N \setminus \{\emptyset\} \right\} \tag{2}$$

It is easy to check that the vertices of $W(N)$ are in bijection with the minimal balanced collections on N. Indeed, an element of $W(N)$ is a vertex if and only if its support is minimal balanced with the corresponding balancing weights (see, e.g., [22, Corollary 3.1.9]). The reason is essentially the following. Consider λ an element of W_N. By definition, $\mathcal{B} := \{S \in 2^N \setminus \{\emptyset\} \mid \lambda_S > 0\}$ is a balanced collection with balancing weight system λ. If λ is a vertex, it cannot be obtained as a convex combination of other vectors in $W(N)$, hence the balancing weight system is unique and the corresponding balanced collection is minimal.

Consequently, generating all minimal balanced collections of N amounts to finding all vertices of $W(N)$. As described in [7], vertex enumeration of a polytope remains one of the open problem in geometry. We have used the classical Avis–Fukuda (1992) method for enumerating all vertices, available in the pycddlib package in Python. Running the algorithm for $n = 6$ gave the following (the performance of our implementation of Peleg's method is recalled), see Table 3.[4] The comparison clearly shows that Peleg's method outperforms the Avis–Fukuda algorithm.

Table 3
Comparison of the computation times of both methods with $n = 6$.

Algorithm used	Algorithm based on Peleg's method	Avis–Fukuda algorithm
Computation time	244 s	1764 s

5. Applications in game theory

This section is devoted to various applications of minimal balanced collections in cooperative game theory. We will start by recalling the famous problem of nonemptiness of the core and its classical solution brought by Bondareva and Shapley independently, which can be viewed as the very starting point of minimal balanced collections. Then, we will prove some original results, showing that minimal balanced collections can be used to prove nontrivial properties of coalitions.

An important general remark for all the subsequent results is that the set of minimal balanced collections does not depend on the game under consideration, but only on n. Therefore, there is no need to generate them at each application, but just to export them from some storage device. Till $n = 7$, this gives a computational advantage compared to other methods based on linear programming and polyhedra, as it will be shown with the example of the nonemptiness of the core.

5.1. Nonemptiness of the core

Let (N, v) be a game. The question is whether the core $C(N, v)$ of this game is nonempty. Consider the following linear program:

$$\begin{aligned} &\min x(N) \\ &\text{s.t. } x(S) \geq v(S), \forall S \in 2^N \setminus \{\emptyset\}. \end{aligned}$$

Clearly, $C(N, v) \neq \emptyset$ if and only if the optimal value of this LP is $x(N) = v(N)$. Therefore, one simple way to check nonemptiness of the core is to solve this LP and compute its optimal value.

Another way is to take the dual program of this LP. This was done by Bondareva [6] and Shapley [23], and directly leads to minimal balanced collections and the following classical result.

Theorem 5.1. *A game (N, v) has a nonempty core if and only if for any minimal balanced collection $\mathcal{B}$ with balancing vector $(\lambda_S^{\mathcal{B}})_{S \in \mathcal{B}}$, we have*

$$\sum_{S \in \mathcal{B}} \lambda_S^{\mathcal{B}} v(S) \leq v(N). \tag{3}$$

Moreover, none of the inequalities is redundant, except the one for $\mathcal{B} = \{N\}$.

4 Computing device: Apple M1 chip, CPU 3.2 GHz, 16 GB RAM.

Table 4
Comparison of the computation time for checking 5000 games with $n = 6$, for both methods.

Algorithm used	Algorithm based on BS theorem	Revised simplex method
Cumulated computation time	0.96 s	24.85 s

This result shows that nonemptiness of the core can be checked by a simple algorithm inspecting inequality (3) for each minimal balanced collection. The test can be stopped once we find a minimal balanced collection for which the inequality is violated.

In order to compare both approaches, we fixed $n = 6$ and generated 5000 different games in the following way: the values $v(S)$ for all coalitions but the grand coalition N are drawn at random in the interval $[0, 5]$, while $v(N)$ is fixed to 50. Doing so, each generated game has a nonempty core, as $(50/6, \dots, 50/6)$ is a core element for any generated game. Therefore, in the algorithm based on the Bondareva–Shapley theorem, all inequalities have to be checked in order to conclude for nonemptiness of the core (most defavorable case). For solving the LP, we have used the revised simplex method, already implemented natively in Python. Both algorithms being implemented in the same language, the comparison is fair. The results are given in Table 4.[5]

We conclude that the algorithm based on minimal balanced collections (provided they are generated off line) is much faster than a direct approach based on linear programming.

5.2. Properties of coalitions and collections

Throughout the section let (N, v) be a *balanced* game, i.e., a game with a nonempty core, S be a coalition, and $S^c := N\setminus S$. Denote by H_S the hyperplane of the set of preimputations defined by

$$H_S = \{x \in X(N, v) \mid x(S) = v(S)\}.$$

Denote by (S, v) the subgame on S, in which only the subcoalitions of S are considered, and by (N, v^S) the game that may differ from (N, v) only inasmuch as $v^S(S^c) = v(N) - v(S)$. This definition can be extended to a collection of coalitions $\mathcal{S}$, with $v^{\mathcal{S}}(S^c) = v(N) - v(S)$ for all $S \in \mathcal{S}$ and $v^{\mathcal{S}}(T) = v(T)$ otherwise.

All algorithms pertaining to this section are relegated in Appendix C.

Exactness. A coalition S is *exact* (for (N, v)) if there exists a core element $x \in C(N, v)$ such that $x(S) = v(S)$.

Hence, a coalition S is exact if and only if the hyperplane H_S intersects the core. The following result permits us to build an algorithm that checks exactness.

Proposition 5.2. *Let (N, v) be a balanced game. A coalition S is exact if and only if (N, v^S) is balanced.*

Proof. Assume that (N, v^S) is balanced. Then, for all $x \in C(N, v^S)$, $x(N) = v(N)$ and $x(S^c) \geq v(N) - v(S)$. It implies that $x(S) = x(N) - x(S^c) \leq v(S)$. But, because x belongs to the core of (N, v^S), it follows that $x(S) \geq v(S)$, and therefore $x(S) = v(S)$.

Assume now that S is exact. Therefore, there exists $x \in C(N, v)$ such that $x(S) = v(S)$. Because $x(N) = v(N)$, then $x(S^c) = v(N) - v(S)$, and $x \in C(N, v^S)$. $\square$

Effectiveness. A coalition S is *effective* (for (N, v)) if $x(S) = v(S)$ for all core elements x. We denote by $\mathcal{E}(N, v)$ the set of coalitions that are effective for (N, v).

Equivalently, $S \in \mathcal{E}(N, v)$ if and only if $C(N, v) \subseteq H_S$. The following result allows to obtain all effective coalitions of a game.

Lemma 5.3. *$\mathcal{E}(N, v)$ is the union of all minimal balanced collections $\mathcal{B}$ such that*

$$\sum_{S\in\mathcal{B}} \lambda_S^{\mathcal{B}} v(S) = v(N).$$

Proof. Let $\mathcal{B}$ be a minimal balanced collection such that $\sum_{S\in\mathcal{B}} \lambda_S^{\mathcal{B}} v(S) = v(N)$ and x be a core element. Then

$$v(N) = x(N) = \sum_{S\in\mathcal{B}} \lambda_S^{\mathcal{B}} x(S) \geq \sum_{S\in\mathcal{B}} \lambda_S^{\mathcal{B}} v(S) = v(N).$$

As $\lambda_S^{\mathcal{B}} > 0$, $x(S) = v(S)$ for all $S \in \mathcal{B}$, i.e., $\mathcal{B} \subseteq \mathcal{E}(N, v)$.

For the other inclusion, let $S \in \mathcal{E}(N, v)$. As $\{N\}$ is a minimal balanced collection, it may be assumed that $S \neq N$. It remains to show that S is contained in some minimal balanced collection $\mathcal{B}$ that satisfies $\sum_{S\in\mathcal{B}} \lambda_S^{\mathcal{B}} v(S) = v(N)$. Assume

[5] Computing device: Apple M1 chip, CPU 3.2 GHz, 16 GB RAM.

the contrary. Then, by Theorem 5.1, there exists $\varepsilon > 0$ such that the game (N, v^{ε}) that differs from (N, v) only inasmuch as $v^{\varepsilon}(S) = v(S) + \varepsilon$ is still balanced. Hence, for $x \in C(N, v^{\varepsilon})$, it follows $x(S) > v(S)$ and $x \in C(N, v)$, then the desired contradiction has been obtained. □

Strict vital-exactness. A coalition S is *strictly vital-exact* (for (N, v)) if there exists a core element $x \in C(N, v)$ such that $x(S) = v(S)$ and $x(T) > v(T)$ for all $T \in 2^S \setminus \{\emptyset, S\}$ (Grabisch and Sudhölter [16]). Denote by $\mathcal{VE}(N, v)$ the set of strictly vital-exact coalitions.

In particular, an exact singleton is strictly vital-exact.

Remark 5.4. Let (N, v) be a balanced game. Because the core is convex, for any collection $\mathcal{S}$ of coalitions such that $\mathcal{S} \cap \mathcal{E}(N, v)$ is empty, there exists a core element $x^{\mathcal{S}}$ such that $x^{\mathcal{S}}(S) > v(S)$, for all $S \in \mathcal{S}$. Indeed, for every coalition $S \in \mathcal{S}$, there exists a core element x^S such that $x^S(S) > v(S)$ because $S \notin \mathcal{E}(N, v)$. Then, by taking the convex midpoint $\frac{1}{|\mathcal{S}|}\sum_{S\in\mathcal{S}} x^S$, the desired $x^{\mathcal{S}}$ is defined, and it belongs to the core by convexity.

By Remark 5.4, we deduce that the minimal (w.r.t. inclusion) coalitions of $\mathcal{E}(N, v)$ are strictly vital-exact. In view of Lemma 5.3, the following proposition shows that it is possible to compute the set of strictly vital-exact coalitions.

Proposition 5.5. *A coalition S is strictly vital-exact if and only if it is exact and*

$$\mathcal{E}(N, v^S) \setminus \{S\} \subseteq \{R \in 2^N \mid R \cap S^c \neq \emptyset\}.$$

Proof. Assume that S is strictly vital-exact. Then there exists $x \in C(N, v)$ such that $x(S) = v(S)$ and $x(T) > v(T)$ for all $T \in 2^S \setminus \{\emptyset, S\}$. Therefore, no coalition $T \in 2^S \setminus \{\emptyset, S\}$ is included in $\mathcal{E}(N, v^S)$.

Conversely, assume that S is exact and $\mathcal{E}(N, v^S) \setminus \{S\} \subseteq \{R \in 2^N \mid R \cap S^c \neq \emptyset\}$. Thanks to Proposition 5.2, $C(N, v^S)$ is nonempty. The collection $2^S \setminus \{S\}$ does not intersect $\mathcal{E}(N, v^S)$ by hypothesis. Hence, by Remark 5.4 there exists an element $x \in C(N, v^S)$ such that $x(T) > v(T)$ for all $T \in 2^S \setminus \{\emptyset, S\}$. □

Extendability. (Kikuta and Shapley [17]) A coalition S is called *extendable* (w.r.t. (N, v)) if, for any $x \in C(S, v)$, there exists $y \in C(N, v)$ such that $x = y_S$, where y_S is the restriction of y to coordinates in S. A game (N, v) is *extendable* if all coalitions are extendable.

To check whether a coalition is extendable, by convexity of the core, it is sufficient to check if each vertex of $C(S, v)$ can be extended to a core element. To this end, the reduced game property of the core is used. Let S be a coalition, and $z \in \mathbb{R}^{S^c}$. Recall that the traditional *reduced game* (Davis and Maschler [9]) of (N, v) w.r.t. S and z, $(S, v_{S,z})$, is the game defined by

$$v_{S,z}(T) = \begin{cases} v(N) - z(S^c), & \text{if } T = S,\\ \max_{Q\subseteq S^c} v(T \cup Q) - z(Q), & \text{if } \emptyset \neq T \subsetneq S. \end{cases}$$

According to Peleg [21] the core satisfies the *reduced game property*, i.e., if $x \in C(N, v)$, then $x_S \in C(S, v_{S,x_{S^c}})$.

Lemma 5.6. *Let (N, v) be a balanced game, S be a coalition and $y \in C(S, v)$. Then there exists $x \in C(N, v)$ such that $x_S = y$ if and only if $(S^c, v_{S^c,y})$ is balanced.*

Proof. The only if part is due to the reduced game property. For the if part choose an arbitrary $z \in C(S^c, v_{S^c,y})$. It suffices to show that the only allocation $x \in \mathbb{R}^N$ such that $x_S = y$ and $x_{S^c} = z$ belong to the core. Assume, on the contrary, that $x \notin C(N, v)$. As $x(S^c) = v_{S^c,y}(S^c) = v(N) - x(S)$ by definition, $x(N) = v(N)$. Therefore, there exists $T \subsetneq N$ such that $x(T) < v(T)$. As (N, v) is balanced, $v(S^c) \leq v(N) - v(S) = v(N) - x(S)$, so that $T \neq S^c$. Moreover, as $y \in C(S, v)$, $T \cap S^c \neq \emptyset$. Therefore, $v_{S^c,y}(T \cap S^c) = \max_{Q\subseteq S} v((T \cap S^c) \cap Q) - x(Q) \geq v((T \cap S^c) \cup (T \cap S)) - x(T \cap S)$. Hence, $x(T \cap S^c) < v(T) - x(T \cap S) \leq v_{S^c,y}(T \cap S^c)$, which contradicts $x_{S^c} = z \in C(S^c, v_{S^c,y})$. □

Lemma 5.6 gives us a necessary and sufficient condition for the existence of an extension of an element of $C(S, v)$ to an element of $C(N, v)$, based upon a balancedness check. If there exists an extension for each extreme point of $C(S, v)$, by convexity of the core, any element of $C(S, v)$ can be extended. Algorithm 6 in Appendix C checks for each extreme point whether the reduced game of (N, v) w.r.t. the complement of S and the currently considered extreme point is balanced.

Feasible collections. Let (N, v) be a balanced game and $\mathcal{F} \subseteq 2^N$ be a core-describing collection of coalitions, i.e.,

$$C(N, v) = \{x \in X(N, v) \mid x(S) \geq v(S), \forall S \in \mathcal{F}\}.$$

Let us consider a subcollection $\mathcal{S} \subseteq \mathcal{F}$, and consider the following subset of $X(N, v)$:

$$X_{\mathcal{S}} = X^{\mathcal{F}}_{\mathcal{S}} = \left\{ x \in X(N, v) \,\middle|\, \begin{array}{l} x(S) < v(S) \text{ for all } S \in \mathcal{S},\\ x(T) \geq v(T) \text{ for all } T \in \mathcal{F} \setminus \mathcal{S} \end{array} \right\}.$$

We may call $X_{\mathcal{S}}$ a *region* of $X(N, v)$, remarking that in the hyperplane $X(N, v)$, the hyperplanes H_S, $S \in \mathcal{F}$, form a hyperplane arrangement, inducing (elementary) regions (see Section 3). The collection $\mathcal{S}$ is $\mathcal{F}$-*feasible* if the corresponding region $X^{\mathcal{F}}_{\mathcal{S}}$ is nonempty. The regions form a partition of $X(N, v)$, with $C(N, v) = X_{\{\emptyset\}}$. If no ambiguity occurs, the collection is simply said to be feasible, and the region is simply denoted by $X_{\mathcal{S}}$. Here are some properties about the feasible collections.

Lemma 5.7 (*Grabisch and Sudhölter [16]*). *Let (N, v) be a balanced game and let $\mathcal{S} \subseteq \mathcal{F}$. The following holds.*

(i) If $\mathcal{S}$ is feasible, then it does not contain a balanced collection.
(ii) For $S, S' \in \mathcal{S}$ such that $S \cup S' = N$, no $x \in X_{\mathcal{S}}$ is dominated via S or S'.

Interestingly, (i) shows that a feasible $\mathcal{S}$ must be unbalanced (see Section 3 for a similar result).

A collection $\mathcal{S}$ that contains only two coalitions satisfying condition (*ii*) above is called a *blocking feasible collection*. A characterization that can be translated into an algorithm is needed to compute the set of feasible collections. In the sequel, denote $\mathcal{S}^c = \{S^c \mid S \in \mathcal{S}\}$.

Lemma 5.8. *A collection $\mathcal{S} \subseteq \mathcal{F}$ is feasible (w.r.t. $\mathcal{F}$) if and only if for every minimal balanced collection $\mathcal{B} \subseteq \mathcal{F}' = (\mathcal{F} \setminus \mathcal{S}) \cup \mathcal{S}^c$,*

$$\sum_{T \in \mathcal{B}} \lambda_T^{\mathcal{B}} v^{\mathcal{S}}(T) \begin{cases} \leq v(N), & \\ < v(N), & \text{if } \mathcal{B} \cap \mathcal{S}^c \neq \varnothing. \end{cases} \tag{4}$$

Proof. For $\varepsilon, \alpha \in \mathbb{R}$ define $(N, v^{\mathcal{S}}_{\varepsilon,\alpha})$ by, for all coalitions T,

$$v^{\mathcal{S}}_{\varepsilon,\alpha}(T) = \begin{cases} v^{\mathcal{S}}(T) + \varepsilon & \text{if } T \in \mathcal{S}^c, \\ v(T) & \text{if } T \in \mathcal{F} \setminus (\mathcal{S} \cup \mathcal{S}^c) \text{ or } T = N, \\ \alpha & \text{otherwise.} \end{cases}$$

A collection $\mathcal{S}$ is feasible if and only if there exist $x \in \mathbb{R}^N$ and $\varepsilon > 0$ such that $x(S) \geq v(S)$ for all $S \in \mathcal{F} \setminus \mathcal{S}$, $x(N) = v(N)$, and $x(P) \leq v(P) - \varepsilon$, i.e., $x(N \setminus P) = x(N) - x(P) = v(N) - x(P) \geq v(N) - v(P) + \varepsilon$ for all $P \in \mathcal{S}$. Therefore, for $\alpha \leq \min_{R \in 2^N} x(R)$, $x \in C(N, v^{\mathcal{S}}_{\varepsilon,\alpha})$ so that if part of the proof is finished by Theorem 5.1.

For the only if part we again employ Theorem 5.1. Indeed, by (4), there exist $\varepsilon > 0$ and $\alpha \in \mathbb{R}$ small enough such that $(N, v^{\mathcal{S}}_{\varepsilon,\alpha})$ is balanced. The existence of a core element of $(N, v^{\mathcal{S}}_{\varepsilon,\alpha})$ guarantees that $\mathcal{S}$ is feasible. □

6. Generalization: Minimal balanced sets

There is a straightforward generalization of balanced collections on a finite set N of n elements, obtained by viewing subsets of N through their characteristic vectors in $\{0, 1\}^N$. Indeed, it suffices to replace vectors in $\{0, 1\}^N$ by vectors in $\mathbb{R}^N_+$. This leads to the following definition.

Definition 6.1. Let $Z \subseteq \mathbb{R}^N_+ \setminus \{0\}$ be a finite set. Z is a *balanced set* if there exists a system $(\delta_z)_{z \in Z}$ of positive weights (called *balancing weights*) such that

$$\sum_{z \in Z} \delta_z z = \mathbf{1}^N.$$

This notion has been introduced by Grabisch and Sudhölter in [16], as it appears to play a central role in the study of the stability of the core (see Section 7).

The classical notions and elementary results for balanced collections straightforwardly extend to balanced sets. A balanced set is *minimal* if it does not contain a proper subset that is balanced. Note that a balanced set is minimal if and only if it has a unique system of balancing weights. Also, a minimal balanced set must be linearly independent. Hence, it contains at most n elements.

The above observations lead to a direct method for checking if a finite set $Z = \{z^1, \ldots, z^q\} \subseteq \mathbb{R}^N_+$ of at most n elements is a minimal balanced set. Consider the matrix A^Z made by the column vectors $z^1, \ldots, z^q$. Then Z is a minimal balanced set if the following linear system

$$A^Z \delta = \mathbf{1}^N \tag{5}$$

has a unique solution $\delta \in \mathbb{R}^q$ which is positive. By standard results in linear algebra, the existence of a unique solution amounts to check that $\mathrm{rk}(A^Z) = q = \mathrm{rk}[A^Z \; \mathbf{1}^N]$.

An interesting question is how minimal balanced sets are related to minimal balanced collections, or how to make one from the other, by either replacing in a vector $z \in Z$ all nonzero coordinates by 1, or conversely by replacing in a vector $\mathbf{1}^S$, S belonging to a minimal balanced collection, coordinates equal to 1 by a some nonzero numbers. We will show that none of these transformations is always successful.

One direction is trivial: consider $n = 3$ and $Z = \{(1, \; 0.4, \; 0), (0, \; 0.6, \; 1)\}$. Then Z is a minimal balanced set, but the corresponding collection is $\{\{1, 2\}, \{2, 3\}\}$, which is not balanced. The other direction is more interesting: Let $\mathcal{B}$ be a minimal balanced collection and consider $S' \in \mathcal{B}$. Build $Z = \{\mathbf{1}^S, S \in \mathcal{B} \setminus \{S'\}\} \cup \{z^{S'}\}$, with $z^{S'} \in \mathbb{R}^N_+$ such that $z^{S'} \neq \mathbf{1}^{S'}$ and $z^{S'}_i > 0$ iff $i \in S'$. Under which conditions on $z^{S'}$ is Z a minimal balanced set?

First of all, Z must be a linearly independent set. Let us write the decomposition in direct sum, using previous notation:

$$\mathbb{R}^N = \mathrm{Im}(A^{\mathcal{B} \setminus \{S'\}}) \oplus \mathrm{Ker}(A^{\mathcal{B} \setminus \{S'\}})$$

where Im and Ker denote the image and kernel of the considered linear mappings. Let $\{y^1, \ldots, y^{n-q+1}\}$ be a basis of $\mathrm{Ker}(A^{\mathcal{B}\setminus\{S'\}})$, where $q = |\mathcal{B}|$. Then $z^{S'}$ is linearly independent of $\{\mathbf{1}^S, S \in \mathcal{B} \setminus \{S'\}\}$ iff

$$z^{S'} \cdot y^i \neq 0, \quad \text{for some } i \in \{1, \ldots, n-q+1\}, \tag{6}$$

where $\cdot$ indicates the scalar product. Now, we have the following result.

Proposition 6.2. *Let $\mathcal{B}$ be a minimal balanced collection on N. Consider a set $Z = \{\mathbf{1}^S, S \in \mathcal{B}\setminus\{S'\}\}\cup\{z^{S'}\}$, with $\mathbf{1}^{S'} \neq z^{S'} \in \mathbb{R}^N_+$ and $z^{S'}_i > 0$ iff $i \in S'$. A necessary condition for Z to be a minimal balanced set is that $z^{S'}$ satisfies (6) and $z^{S'} \in \mathrm{Im}(A^{\mathcal{B}})$.*

Proof. Consider Z as above, and the matrices $A^{\mathcal{B}}$ and A^Z. Z is a minimal balanced set iff the system

$$A^Z\delta = \mathbf{1}^N$$

has a unique solution which is positive for each coordinate. The system has a unique solution iff $\mathrm{rk}[A^Z\ \mathbf{1}^N] = \mathrm{rk}(A^Z)$, i.e., $\mathbf{1}^N$ is in the span of Z and A^Z has full rank $q := |Z| = |\mathcal{B}|$. As the condition (6) is satisfied, Z is an independent set, hence A^Z has full rank. Moreover, $\mathcal{B}$ is minimal, hence $A^{\mathcal{B}}$ has full rank q. Now, as $\mathcal{B}$ is balanced, $\mathbf{1}^N$ is in the span of $\{1^S, S \in \mathcal{B}\}$, and by linear independence of Z and the assumption that $z^{S'}$ is in the span of $\{1^S, S \in \mathcal{B}\}$, the vectors in Z and $\{1^S, S \in \mathcal{B}\}$ span the same space. □

Unfortunately, the condition fails to be sufficient as nothing ensures the positivity of the solution. This is shown by the following example:

Example 6.3. Take $n = 4$, and consider $\mathcal{B} = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$ and

$$Z = \{(1,\ 1,\ 1,\ 0), (1,\ 1,\ 0,\ 1), (1,\ 0,\ 0.1,\ 1), (0,\ 0.2,\ 0.1,\ 0.5)\}.$$

Then the unique solution δ of (5) is

$$\delta = (25/31,\ -4/31,\ 10/31,\ 50/31).$$

Nevertheless, the necessary condition of Proposition 6.2 can be useful to discard potential candidates for being a minimal balanced set, as shown in the next example.

Example 6.4. Take $n = 5$, the minimal balanced collection

$$\mathcal{B} = \{\{3, 4, 5\}, \{1, 2, 4, 5\}, \{2, 3\}, \{1, 3\}\}$$

and choose $S' = \{1, 2, 4, 5\}$, letting $z^{S'} = (\alpha,\ \beta,\ 0,\ \gamma,\ \delta)$, with $\alpha, \beta, \gamma, \delta > 0$. We obtain that any vector $y \in \mathrm{Ker}(A^{\mathcal{B}\setminus\{\{1,2,4,5\}\}})$ has the form

$$y = (-\theta,\ -\theta,\ \theta,\ \theta - \rho,\ \rho), \quad \theta, \rho \in \mathbb{R},$$

hence $(-1,\ -1,\ 1,\ 1,\ 0)$ and $(0,\ 0,\ 0,\ -1,\ 1)$ form a basis of $\mathrm{Ker}(A^{\mathcal{B}\setminus\{\{1,2,4,5\}\}})$. Then, condition (6) reads:

$$\gamma \neq \alpha + \beta \text{ or } \gamma \neq \delta.$$

Let us examine now the second condition. A vector $y \in \mathrm{Im}(A^{\mathcal{B}\setminus\{\{1,2,4,5\}\}})$ has the form

$$y = (\lambda_2 + \lambda_4,\ \lambda_2 + \lambda_3,\ \lambda_1 + \lambda_3 + \lambda_4,\ \lambda_1 + \lambda_2,\ \lambda_1 + \lambda_2),$$

with $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}$, showing that the two last coordinates must be equal. Combining both conditions, we find that the necessary condition of Proposition 6.2 becomes

$$\gamma \neq \alpha + \beta, \quad \gamma = \delta.$$

As for minimal balanced collections, an enumeration problem arises if one reduces the choice of a possible Z as a subset of some finite set $\Omega \subseteq \mathbb{R}^N_+$ (for minimal balanced collections, $\Omega = \{1^S, S \in 2^N \setminus \{\varnothing\}\}$). This will be indeed the case when studying the stability of the core (see Section 7).

Unfortunately, the basic inductive mechanism of Peleg's algorithm does not seem to be adaptable to this general framework. Then we are left with two methods: either check by the direct method (see Eq. (5)) if every subset of Ω of at most n elements is a minimal balanced set, possibly taking advantage of Proposition 6.2 to eliminate candidates, or use the polyhedral approach. Indeed, it is easy to see that minimal balanced sets of Ω are in one-to-one correspondence with the vertices of the following polytope:

$$W(\Omega) = \left\{\delta \in \mathbb{R}^{\Omega} \mid \sum_{z\in\Omega} \delta_z z = \mathbf{1}^N, \delta_z \geq 0, \forall z \in \Omega\right\}$$

which is defined very similarly to (2). Therefore, finding the minimal balanced sets is equivalent to a vertex enumeration problem, which is a fundamental open problem in geometry. For more details about vertex enumeration problems of polytopes and polyhedra defined by a set of inequalities and related problems, we refer the reader to [7].

7. Core stability by nested balancedness

Checking the stability of the core of a game constitutes a nice and complex application of both minimal balanced collections and minimal balanced sets. The method is based on a result by Grabisch and Sudhölter [16], giving a sufficient and necessary condition for core stability. The aim of this section is to focus on the algorithmic aspects of the result, and the actual implementation as a computer program. The result of [16] provides a theoretical characterization of core stability, but does not provide a way to compute all the intermediary objects involved in it, such as minimal balanced collections, minimal balanced sets, nor a way to find exact coalitions, strictly vital exact coalitions, feasible collections of coalitions, etc. The results we have established above permit to find them solely by means of the minimal balanced collections, thus providing faster algorithms than usual linear programming methods, so that finally we come up with a complete algorithm able to check core stability. Moreover, we complete the algorithmic characterization of core stability by adding some checking procedures of necessary or sufficient conditions for core stability at an early stage of the algorithm.

Essentially, the method of Grabisch and Sudhölter uses a double balancedness condition, called "nested balancedness". We briefly explain why such a condition arises.

Recall that testing nonemptiness of the core is a problem involving one quantifier on a variable in an uncountable set, and linear inequalities:

$$\exists x \in X(N, v), x(S) \geq v(S), \forall S.$$

The Bondareva–Shapley theorem permits to reduce this problem to a finite number of tests, by using minimal balanced collections. On the other hand, checking core stability involves two quantifiers on two variables in uncountable sets, and linear inequalities:

$$\forall y \in X(N, v) \setminus C(N, v), \exists x \in C(N, v), \exists S \in 2^N, x_i > y_i, \forall i \in S, x(S) = v(S).$$

Intuitively, one can get rid of each quantifier by a balancedness condition, and therefore to have *in fine* a finite number of tests. As it will be explained, the second balancedness condition will involve minimal balanced sets.

Throughout this section, (N, v) is a balanced game. All definitions and results in Sections 7.2 to 7.4 are due to Grabisch and Sudhölter [16].

7.1. Simple conditions for stability

We start with giving some necessary or sufficient conditions for core stability, yielding to simple tests (all involving previously introduced notions and checkable by some balancedness conditions) permitting to give a quick answer without invoking the full method.

Proposition 7.1 (*Gillies [15]*)**.** *A balanced game has a stable core only if each singleton is exact.*

As the set of exact coalitions can be computed (see Section 5.2), the necessary condition of Gillies can be easily checked. Another interesting consequence of this result is the expansion of the space in which the core is externally stable if it is a stable set. Indeed, if a balanced game satisfies this necessary condition, for all player $i \in N$, the core element $x \in C(N, v)$ such that $x_i = v(\{i\})$ dominates every element y of $X(N, v)$ such that $y_i < v(\{i\})$, via $\{i\}$. Therefore, in the definition of stability (see Section 2.2) for the core, the set $I(N, v)$ may be replaced by $X(N, v)$.

Recall that $\mathcal{VE}(N, v)$ denotes the set of strictly vital-exact coalitions (see Section 5.2).

Proposition 7.2. *Let (N, v) be a balanced game. The core is stable only if $\mathcal{VE}(N, v)$ is core-describing, i.e.,*

$$C(N, v) = \{x \in X(N, v) \mid x(S) \geq v(S), \forall S \in \mathcal{VE}(N, v)\}.$$

Proof. Assume that the core is stable, and suppose by contradiction that there exists $y \in X(N, v) \setminus C(N, v)$ such that $y(S) \geq v(S)$ for all $S \in \mathcal{VE}(N, v)$. Because the core is stable, there exists $x \in C(N, v)$ such that x dom y. Choose a minimal (w.r.t. inclusion) coalition S such that $x \text{ dom}_S\, y$. Then, $v(S) = x(S) > y(S)$, and $x(T) > v(T)$ for all $T \in 2^S \setminus \{\emptyset, S\}$. Therefore, S is strictly vital-exact, a contradiction. □

This important result shows that checking core stability should begin by finding all strictly vital-exact coalitions (using Algorithm 5), and to check if these coalitions determine the core. If yes, one should work on $\mathcal{F} = \mathcal{VE}(N, v)$ instead of 2^N, as this considerably reduces the combinatorial aspect of the method. Hence, for the rest of Section 7, we put $\mathcal{F} = \mathcal{VE}(N, v)$.

Kikuta and Shapley [17] have provided a sufficient condition for a game to have a stable core via extendability.

Theorem 7.3 (*Kikuta and Shapley [17]*)**.** *An extendable game has a nonempty and stable core.*

Extendability of a game can be checked by using Algorithm 6, but is time-consuming. However, this property can be considerably weakened as follows. Say that a game (N, v) is $\mathcal{F}$-*weakly extendable* if each $\mathcal{F}$-feasible collection of coalitions (see Section 5.2) of $\mathcal{F}$ contains a minimal (w.r.t. inclusion) coalition that is extendable.

Proposition 7.4. *A $\mathcal{F}$-weakly extendable game has a nonempty and stable core.*

Proof. Let $\mathcal{S}$ be a $\mathcal{F}$-feasible collection and S be extendable and minimal w.r.t. inclusion in $\mathcal{S}$. Take $y \in X_{\mathcal{S}}$. Then $y(S) < v(S)$ and $y(T) \geq v(T)$ for all $T \subset S$, $T \in \mathcal{F}$. Define $z_S \in \mathbb{R}^S$ by $z_S = y_S + \frac{1}{|S|}(v(S) - y(S))\mathbf{1}^S$. Clearly, $z_S \in C(S, v)$ and $(z_S)_i > y_i$ for all $i \in S$. As S is extendable, there exists $x \in C(N, v)$ such that $x_S = z_S$. Then x dom$_S$ y. □

7.2. Association, admissibility, and outvoting

We first recall the definition of *outvoting*, a transitive subrelation of domination, that was inspired by a definition given by Kulakovskaja [18]. In view of Proposition 7.2, throughout we assume that (N, v) is a balanced game for which the collection of strictly vital-exact coalitions is core-describing.

Definition 7.5. A preimputation y *outvotes* another preimputation x via $P \in \mathcal{F}$, written $y \succ_P x$, if y dom$_P$ x and $y(S) \geq v(S)$ for all $S \notin 2^P$. Also, y outvotes x, $(y \succ x)$ if there exists a coalition $P \in \mathcal{F}$ such that $y \succ_P x$.

Denote by $M(v) = \{x \in X(N, v) \mid y \not\succ x, \forall y \in X(N, v)\}$ the set of preimputations that are maximal w.r.t. outvoting.

Proposition 7.6. *Let (N, v) be a balanced game. Then $C(N, v) = M(v)$ if and only if $C(N, v)$ is stable.*

All results are based on this new characterization. To present the main result, some definitions are needed.

Definition 7.7. Let S be a strictly vital-exact coalition and $\mathcal{B}$ be a minimal balanced collection in $\mathcal{F}$. $\mathcal{B}$ is *associated with S* if there exists $i \in S$ such that $\{i\} \in \mathcal{B}$ and

$$\mathcal{B} \subseteq \{\{j\} \mid j \in S\} \cup \{S^c\} \cup (\mathcal{F} \setminus 2^S).$$

Denote by $\mathbb{B}^S(N)$ the set of minimal balanced collections on N associated with S.

Example 7.8. Let $N = \{1, 2, 3, 4\}$, $\mathcal{F} = \mathcal{VE}(N, v) = 2^N \setminus \{\emptyset\}$ and $S = \{1, 2\}$. Therefore, the minimal balanced collection $\mathcal{B} = \{\{1\}, \{2\}, \{3, 4\}\}$ is included in $\mathbb{B}^S(N)$. Indeed, the coalitions $\{1\}$ and $\{2\}$ are singletons of S, and $\{3, 4\}$ is the complement of S. Moreover, $\mathcal{B}$ is also associated with $\{1, 2, 3\}$ for example.

Let S be a strictly vital-exact coalition and $\mathcal{B}$ be a minimal balanced collection associated with S. Denote by $\mathcal{B}_S^*$ the collection $\mathcal{B}_S^* = \mathcal{B} \setminus \{\{i\} \mid i \in S\}$. Thanks to the notions of association and outvoting, the following result holds.

Theorem 7.9. *Let x be a preimputation, and S a strictly vital-exact coalition. Then x is outvoted by some preimputation via S if and only if*

$$\forall \mathcal{B} \in \mathbb{B}^S(N), \qquad \sum_{\substack{i \in S \\ \{i\} \in \mathcal{B}}} \lambda_{\{i\}}^{\mathcal{B}} x_i + \sum_{T \in \mathcal{B}_S^*} \lambda_T^{\mathcal{B}} v^S(T) < v(N). \tag{7}$$

This result can be sharpened, with the use of the following notion.

Definition 7.10. Let $\mathcal{S}$ be a nonempty collection of strictly vital-exact coalitions, $S \in \mathcal{S}$, and $\mathcal{B}$ be a minimal balanced collection associated with S. $\mathcal{B}$ is *admissible* for $\mathcal{S}$ if $\mathcal{B}_S^* \cap \mathcal{S} \neq \emptyset$ or $\mathcal{B}_S^* \cap \mathcal{S}^c = \emptyset$. Denote by $\mathbb{B}_{\mathcal{S}}^S(N)$ the set of minimal balanced collections associated with S and admissible for $\mathcal{S}$.

Example 7.11. Let $N = \{1, 2, 3, 4\}$, $\mathcal{F} = \mathcal{VE}(N, v) = 2^N \setminus \{\emptyset\}$ and $S = \{1, 4\}$. Therefore, the collection $\mathcal{B} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{4\}\}$ is associated with S, and $\mathcal{B}_S^* = \mathcal{B} \setminus \{\{4\}\}$. Let $\mathcal{S} = \{\{2, 3\}, \{1, 4\}\}$. The first condition of the definition is satisfied: $\mathcal{B}_S^* \cap \mathcal{S} = \{\{2, 3\}\} \neq \emptyset$, therefore $\mathcal{B}$ is admissible for $\mathcal{S}$.

For each nonempty collection $\mathcal{S}$ of strictly vital-exact coalitions, denote

$$\mathbb{C}(\mathcal{S}) = \bigtimes_{S \in \mathcal{S}} \mathbb{B}_{\mathcal{S}}^S(N).$$

The concept of admissibility allows to sharpen the previous result, and then to reduce the algorithmic complexity of the core stability checking, thanks to the following result.

Corollary 7.12. *Let $\mathcal{S}$ be a feasible collection. $M(v) \cap X_{\mathcal{S}} \neq \emptyset$ if and only if there exists a system of balanced collections $(\mathcal{B}_S)_{S \in \mathcal{S}} \in \mathbb{C}(\mathcal{S})$ and $x \in X_{\mathcal{S}}$ such that*

$$\forall S \in \mathcal{S}, \qquad \sum_{\substack{i \in S \\ \{i\} \in \mathcal{B}}} \lambda_{\{i\}}^{\mathcal{B}} x_i + \sum_{T \in \mathcal{B}_S^*} \lambda_T^{\mathcal{B}} v^S(T) \geq v(N).$$

7.3. Minimal balanced subsets

For the study of core stability, minimal balanced subsets of a specific set must be computed. Let $\mathcal{S}$ be a feasible collection and $(\mathcal{B}_S)_{S\in\mathcal{S}} \in \mathbb{C}(\mathcal{S})$. For $S \in \mathcal{S}$, let $z^S \in \mathbb{R}^N$ be given by

$$z_j^S = \begin{cases} \lambda^{\mathcal{B}}_{\{i\}}, & \text{if } j = i \text{ for some } i \in S \text{ such that } \{i\} \in \mathcal{B}_S, \\ 0, & \text{for all other } j \in N. \end{cases}$$

Define the sets

$$\Omega_A(\mathcal{S}) = \left\{\mathbf{1}^{S^c} \mid S \in \mathcal{S}\right\}, \quad \Omega_B(\mathcal{S}) = \left\{\mathbf{1}^T \mid T \in \mathcal{F}\setminus\mathcal{S}\right\},$$
$$\Omega_C(\mathcal{S}, (\mathcal{B}_S)_{S\in\mathcal{S}}) = \left\{z^S \mid S \in \mathcal{S}\right\},$$
$$\text{and} \quad \Omega := \Omega(\mathcal{S}, (\mathcal{B}_S)_{S\in\mathcal{S}}) = \Omega_A(\mathcal{S}) \cup \Omega_B(\mathcal{S}) \cup \Omega_C(\mathcal{S}, (\mathcal{B}_S)_{S\in\mathcal{S}}).$$

Let LinAlgSolve be a procedure that takes as an input a $(n \times k)$-matrix A and returns a k-dimensional vector λ such that $A\lambda = \mathbf{1}^N$. Denote by $\mathbb{B}(\Omega)$ the set of minimal balanced subsets of Ω. Algorithm 10 in Appendix D generates $\mathbb{B}(\Omega)$.

7.4. Nested balancedness

Finally, for each $z \in \Omega$, we define $a_z = a_z(\mathcal{S}, (\mathcal{B}_S)_{S\in\mathcal{S}}) = \max(A \cup B \cup C)$, where

$$A = \{v(N) - v(S) \mid S \in \mathcal{S}, \mathbf{1}^{S^c} = z\},$$
$$B = \{v(T) \mid T \in \mathcal{F}\setminus\mathcal{S}, \mathbf{1}^T = z\},$$
$$C = \{v(N) - \sum_{T\in\mathcal{B}_S^*} \lambda_T^{\mathcal{B}_S} v^S(T) \mid S \in \mathcal{S}, z = z^S\}.$$

Note that A and B are empty or singletons, but C can be multi-valued because distinct coalitions can generate the same z. Let $N = \{1, 2, 3\}$, $S = \{1, 2\}$, $T = \{1, 3\}$ and $\mathcal{B}_S = \mathcal{B}_T = \{\{1\}, \{2, 3\}\}$. Then, $z^S = (1, 0, 0) = z^T$. To summarize,

$$a_z = \begin{cases} \max C & \text{if } C \neq \emptyset = A, \\ \max\{A, C\} & \text{if } C \neq \emptyset \neq A, \\ v(N) - v(S) & \text{if } z = \mathbf{1}^{S^c} \text{ for some } S \in \mathcal{S}, C = \emptyset, \\ v(T) & \text{if } z = \mathbf{1}^T \text{ for some } T \in \mathcal{F}\setminus\mathcal{S}, A = \emptyset = C. \end{cases}$$

Recall that $\mathbb{B}(\Omega)$ is the set of all minimal balanced sets $Z \subseteq \Omega$ and denote by $\mathbb{B}_0(\Omega)$ the subset of $\mathbb{B}(\Omega)$ such that, for all $Z \in \mathbb{B}_0(\Omega)$, there exists $S \in \mathcal{S}$ such that $z = \mathbf{1}^{S^c} \in Z$ and $a_z = v(N) - v(S)$.

Theorem 7.13 (*Grabisch and Sudhölter, 2021*)**.** *A balanced game* (N, v) *has a stable core if and only if, for every feasible collection* $\mathcal{S}$ *and for every* $(\mathcal{B}_S)_{S\in\mathcal{S}} \in \mathbb{C}(\mathcal{S})$,

$$\exists Z \in \mathbb{B}(\Omega)\setminus\mathbb{B}_0(\Omega) \text{ such that } \sum_{z\in Z} \lambda_z^Z a_z > v(N), \text{ or}$$
$$\exists Z \in \mathbb{B}_0(\Omega) \text{ such that } \sum_{z\in Z} \lambda_z^Z a_z \geq v(N).$$

For each $Z \in \mathbb{B}(\Omega)$, let $\psi(Z) = \sum_{z\in Z} \lambda_z^Z a_z$. Algorithm 11 in Appendix D checks whether a game has a stable core.

7.5. Complete algorithm

In this paper, we presented all the subroutines used for the construction of the core stability checking algorithm. Many of them have interest of their own, but altogether they allow us to check the core stability of a given game. The first necessary subroutine is the computation of the feasible collections, that discretizes the set of preimputations. The next subroutine computes the set $\mathbb{C}(\mathcal{S})$, for each feasible collection $\mathcal{S}$. Then, the third one computes the set Ω, for each collection $(\mathcal{B}_S)_{S\in\mathcal{S}} \in \mathbb{C}(\mathcal{S})$. Once the sets Ω are computed, we compute the set of its minimal balanced subsets. The last step is to compute the weighted sums described in Theorem 7.13.

In addition to the necessary subroutines, we described subroutines to avoid useless computation. First, we check the non-emptiness of the core, thanks to the minimal balanced collections. Next, we compute the set $\mathcal{VE}$ of strictly vital-exact coalitions. Indeed, a necessary condition for core stability is that $\mathcal{VE}$ is core determining. Moreover, Theorem 7.13 works for any core-describing collection, and because $\mathcal{VE}$ is the smallest one, we avoid the generation of useless feasible collections. With the subroutine checking the exactness of coalitions, we check the exactness of the singletons (see Proposition 7.1). Then, we check if there is no blocking feasible collection, i.e., a feasible collection $\mathcal{S} = \{S_1, S_2\}$ such that $S_1 \cup S_2 = N$ (see Lemma 5.7). Finally, we compute the set of extendable coalitions and discard the feasible collections that contain a minimal (w.r.t. inclusion) coalition that is extendable. If we discard them all, the core is stable (see Proposition 7.4). The complete algorithm is presented below in pseudocode (for the subroutines, see the appendices).

Algorithm 3 Core stability checking algorithm

Require: A game (N, v)
Ensure: The Boolean value: '(N, v) has a stable core'

```
procedure IsCoreStable((N, v))
    𝔹(N) ← Peleg(|N|)
    for (ℬ, λ^ℬ) ∈ 𝔹(N) do                                   ▷ Checking balancedness
        if Σ_{S∈ℬ} λ_S v(S) > v(N) then
            return False
        for i ∈ N do                                         ▷ Checking exactness of the singletons
            if Σ_{S∈ℬ} λ_S v^{i} > v(N) then
                return False
    𝒱ℰ ← ∅, ℰxt ← ∅
    for S ∈ 2^N \ {∅} do
        if IsStrictlyVitalExact(S, (N, v)) then
            Add S to 𝒱ℰ
        if IsExtendable(S, (N, v)) then
            Add S to ℰxt
    if not IsCoreDescribing(𝒱ℰ) then                         ▷ see Proposition 7.2
        return False
    for 𝒮 ⊆ 𝒱ℰ do
        if IsFeasible(𝒮, 𝒱ℰ, (N, v)) then
            if 𝒮 = {S_1, S_2} such that S_1 ∪ S_2 = N then        ▷ see Lemma 5.7
                return False
            else
                for S ∈ ℰxt do                               ▷ see Proposition 7.4
                    if S minimal (w.r.t. inclusion) in 𝒮 then
                        Go to the next set of coalitions 𝒮
                ℂ(𝒮) ← Admissibles(𝒮, ℱ, 𝔹(N))
                if not IsGSConditionSatisfied(𝒮, ℱ, 𝔹(N), (N, v)) then
                    return False
    return True
```

7.6. Examples

Computing device: Apple M1 chip, CPU 3.2 GHz, 16 GB RAM.

4-Player game. Let (N, v) be the game defined by $N = \{1, 2, 3, 4\}$ and $v(S) = 0.6$ if $|S| = 3$, $v(N) = 1$ and $v(T) = 0$ otherwise. The algorithm returns that the set $\mathcal{E}(N, v)$ only contains N. The set of strictly vital-exact coalitions is $\mathcal{VE}(N, v) = \{\{i\} \mid i \in N\} \cup \{N \setminus \{i\} \mid i \in N\}$. The collection $\{\{1, 3, 4\}, \{1, 2, 3\}\}$ is a blocking feasible collection, so by Lemma 5.7, the core is not stable. The CPU time for this example is 0.1 s.

5-Player game. Let (N, v) be the game defined by Biswas et al. [4], defined on $N = \{1, 2, 3, 4, 5\}$ by $v(S) = \max\{x(S), y(S)\}$ with $x = (2, 1, 0, 0, 0)$ and $y = (0, 0, 1, 1, 1)$. For this game, the set of effective proper coalitions is

$$\mathcal{E}(N, v) \setminus \{N\} = \{\{2, 3\}, \{2, 4\}, \{2, 5\}, \{1, 3, 4\}, \{1, 3, 5\}, \{1, 4, 5\}\}.$$

The set of strictly vital-exact coalitions is $\mathcal{VE}(N, v) = \mathcal{E}(N, v) \cup \{\{i\} \mid i \in N\}$. The feasible collections that do not contain a minimal extendable coalition are the nonempty subsets of $\{\{1, 3, 4\}, \{1, 3, 5\}, \{1, 4, 5\}\}$, so there are 7 feasible collections. The collection $\{\{1, 3, 5\}, \{1, 4, 5\}\}$ does not satisfy the condition of Theorem 7.13, therefore the core of the game is not stable. The CPU time for this example is 1.5 s.

Let (N, v) be the same game, but with $v(N) = 3.1$. The set $\mathcal{E}(N, v)$ becomes $\{N\}$. The set of strictly vital-exact coalitions now contains 14 coalitions, while the previous game had 11 strictly vital-exact coalitions. The additional ones are $\{1, 3\}, \{1, 4\}, \{1, 5\}$. The set of feasible collections that do not contain a minimal extendable coalition considerably increases, with 51 feasible collections, but no blocking feasible collection. The largest feasible collection contains 6 strictly vital-exact coalitions. The estimated time for the algorithm to check if this specific collection satisfies the condition of Theorem 7.13 is greater than 200 h, due to the cardinality of the set $\mathbb{C}(\mathcal{S})$ with $\mathcal{S}$ denoting the specific collection.

6-Player game. Let (N, v) be the game defined by Studený and Kratochvíl (2021), defined on $N = \{1, 2, 3, 4, 5, 6\}$ by

$$\begin{aligned}
v(S) &= 2 \text{ for } S = \begin{cases} \{2,5\}, \{3,5\}, \{1,2,5\}, \{2,3,5\}, \{2,4,5\}, \{2,5,6\}, \{1,2,4,5\} \\ \{1,2,4,6\}, \{1,2,5,6\}, \{2,4,5,6\} \text{ and } \{1,2,4,5,6\}, \end{cases} \\
v(S) &= 3 \text{ for } S = \{3,4,5\}, \\
v(S) &= 4 \text{ for } S = \begin{cases} \{3,6\}, \{1,3,5\}, \{1,3,6\}, \{3,4,6\}, \{3,5,6\}, \{1,2,3,5\}, \\ \{1,3,4,5\}, \{1,3,4,6\}, \{1,3,5,6\}, \{2,3,4,5\} \text{ and } \{1,2,3,4,5\}, \end{cases} \\
v(S) &= 6 \text{ for } S = \begin{cases} \{2,3,6\}, \{1,2,3,6\}, \{2,3,4,6\}, \{2,3,5,6\}, \\ \{1,2,3,4,6\} \text{ and } \{1,2,3,5,6\}, \end{cases} \\
v(S) &= 8 \text{ for } S = \{3,4,5,6\}, \{1,3,4,5,6\}, \{2,3,4,5,6\}, \\
v(N) &= 10 \text{ and } v(T) = 0 \text{ otherwise.}
\end{aligned}$$

The set $\mathcal{E}(N, v)$ is only $\{N\}$. The set of strictly vital-exact coalitions is

$$\{\{i\} \mid i \in N\} \cup \{\{2,5\}, \{3,6\}, \{1,3,5\}, \{2,3,6\}, \{1,2,4,6\}, \{2,3,4,5\}, \{3,4,5,6\}\}$$

and the feasible collections that do not contain a minimal extendable coalition are the nonempty subsets of $\{\{1,3,5\}, \{3,4,5,6\}, \{2,3,4,5\}\}$. The feasible collection $\{\{1,3,5\}, \{3,4,5,6\}\}$ does not satisfy the condition of Theorem 7.13, therefore the core of the game is not stable. The CPU time for this example is 18 min and 12 s, among which 43 s for computing the set of minimal balanced collections on a set of 6 players.

8. Concluding remarks

We have shown in this paper that minimal balanced collections are a central notion in cooperative game theory, as well as in other areas of discrete mathematics, and even in physics. As a balanced collection is merely the expression of a sharing of one unit of resource among subsets, we believe that many more applications should be possible.

Just focusing on the domain of cooperative games, the consequences of our results appear to be of primary importance for the computability of many notions like exactness, extendability, etc. Indeed, a blind application of the definition of these notions leads to difficult problems related to polyhedra, limiting their practical applicability. Thanks to our results, provided minimal balanced collections are generated beforehand (which is possible since they do *not* depend on the considered game), these notions can be checked very easily and quickly, as most of the tests to be done reduce to checking simple linear inequalities.

Generating minimal balanced collections has also permitted implementing an algorithm testing core stability. The examples in Section 7.6 have shown that, even if for many cases, the answer can be obtained quickly, there are instances where the computation time goes beyond tractability, due mostly to the use of minimal balanced sets. Still, further research is needed to investigate more on minimal balanced sets in order to overcome this limitation.

Funding

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

Appendix A. Maximal unbalanced collections in thermal quantum physics

In theoretical physics, *thermal quantum field theory* is a set of methods to calculate expectation values of physical observables of a *quantum field theory* at finite temperature. Quantum field theory is a theoretical framework that combines classical field theory (for example, Newtonian gravitation or Maxwell's equations of electromagnetic fields), special relativity, and quantum mechanics. Quantum field theory treats particles as excited states of their underlying quantum fields, which are more fundamental than the particles.

Key objects of quantum field theory are the *correlators*, also called *Green functions*, that are used to calculate various *observables*, i.e., self-adjoint operators on the Hilbert *space of states* $\mathbb{H}$ that extract some physical properties from a particular state of the studied system. These correlators are all encoded in a generating functional, called the *partition function*, analog to the way a sequence of coefficients in combinatorics is encoded in a generating function.

With the *imaginary time formalism*, the difference between the partition function in thermal quantum field theory and in zero-temperature quantum field theory is a thermal weight $e^{-\beta H}$, which is actually the action of a *time-evolution* operator e^{-iHT}, that operates a shift in time of $-i\beta$. The physicists aim to extract the corresponding correlators from this new partition function. In the computation, a function Φ appears, that takes as an input a set of complex energies $\{z_i\}_{i\in I}$ satisfying $\sum_{i\in I} z_i = 0$, called the *imaginary Matsubara energies*. Physicists are interested in the analytic continuations of Φ, which exist only where

$$\sum_{i\in J} z_i \notin \mathfrak{Re}, \qquad \forall J \subseteq I.$$

We remark that for a given subset of indices $J \subseteq I$, the set $H_J := \{z \in \mathbb{C}^{|I|} \mid \sum_{i\in J} z_i \in \mathfrak{Re}\}$ is a hyperplane of the energy space. It has been proven (Evans, [12]) that the analytic continuations of Φ, which produce solutions called *(thermal) generalized retarded functions* (Evans, [13]), and the regions of the hyperplane arrangement $\{H_J\}_{J\subseteq I}$, called the *restricted all-subset arrangement* (Billera et al. [3]), are in bijection.

Thanks to the discussion in Section 3, we see that the generalized retarded functions, produced with the imaginary time formalism studying n-body problems, are then in bijection with the maximal unbalanced collections on N.

Appendix B. Example of generation of minimal balanced collections

Let $N = \{a, b, c, d\}$ and $N' = N \cup \{e\}$. Let $S_1 = \{a, b\}$, $S_2 = \{a, c\}$, $S_3 = \{a, d\}$ and $S_4 = \{b, c, d\}$. Then $\mathcal{C} = \{S_1, S_2, S_3, S_4\}$ is a minimal balanced collection with the following system of balancing weights $\lambda = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}\right)$.

First case. Remark that the set $I = \{1, 4\}$ satisfies the equation $\lambda_I = 1$. Therefore, a minimal balanced collection can be constructed as follows:

$$\mathcal{C}' = \{\{a, b, e\}, \{a, c\}, \{a, d\}, \{b, c, d, e\}\}, \text{ with } \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}\right).$$

Second case. Let $I = \{4\}$. Then $\lambda_I = \frac{2}{3} < 1$. Therefore,

$$\mathcal{C}' = \{\{a, b\}, \{a, c\}, \{a, d\}, \{b, c, d, e\}, \{e\}\}, \text{ with } \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{1}{3}\right).$$

is a minimal balanced collection on N'.

Third case. Let $I = \{1, 2\}$ and $\delta = 4$. Then $\lambda_I = \frac{2}{3}$ and $1 - \lambda_{S_\delta} = \frac{1}{3}$. Therefore, $1 > \lambda_I > 1 - \lambda_{S_\delta}$ and the following minimal balanced collection can be constructed:

$$\mathcal{C}' = \{\{a, b, e\}, \{a, c, e\}, \{a, d\}, \{b, c, d\}, \{b, c, d, e\}\}, \text{ with } \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right).$$

Last case. For the last case, consider another framework. Let $N = \{a, b\}$, and $\mathcal{C}^1 = \{\{a\}, \{b\}\}$, $\mathcal{C}^2 = \{\{a, b\}\}$ be the only two minimal balanced collections on N. Let $\mathcal{C}$ be the union $\mathcal{C} = \{\{a\}, \{b\}, \{a, b\}\}$.

$$\mu = (1, 1, 0) \text{ and } \nu = (0, 0, 1).$$

Observe that

$$\mathrm{rk}(A^{\mathcal{C}}) = \mathrm{rk}\left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\right) = 2 = k - 1.$$

Finally, let $I = \{1, 2\}$. Then $\mu_I = 2$, $\nu_I = 0$, and

$$t^I = \frac{1 - \mu_I}{\nu_I - \mu_I} = \frac{1}{2} \in \,]0, 1[.$$

The following collection may therefore be constructed:

$$\mathcal{C}' = \{\{a, c\}, \{b, c\}, \{a, b\}\}, \text{ with}$$

$$\begin{aligned}
\lambda^{\mathcal{C}'}_{\{a,c\}} &= (1 - t^I)\mu_{\{a,c\}} + t^I \nu_{\{a,c\}} = \tfrac{1}{2}\mu_{\{a,c\}} &&= \tfrac{1}{2},\\
\lambda^{\mathcal{C}'}_{\{b,c\}} &= (1 - t^I)\mu_{\{b,c\}} + t^I \nu_{\{b,c\}} = \tfrac{1}{2}\mu_{\{b,c\}} &&= \tfrac{1}{2},\\
\lambda^{\mathcal{C}'}_{\{a,b\}} &= (1 - t^I)\mu_{\{a,b\}} + t^I \nu_{\{a,b\}} = \tfrac{1}{2}\nu_{\{a,b\}} &&= \tfrac{1}{2}.
\end{aligned}$$

Appendix C. Algorithms for checking various properties of coalitions and collections

These algorithms refer to results given in Section 5.2.

Algorithm 4 Exactness checking subroutine
Require: A coalition S, a balanced game (N, v), the set $\mathbb{B}(N)$
Ensure: The Boolean value: 'S is exact'

procedure IsExact(S, $\mathbb{B}(N)$, (N, v))
    Define v^S such that $v^S(T) = v(T)$ for all $T \in 2^N \setminus \{S^c\}$, and $v^S(S^c) = v(N) - v(S)$
    for $(\mathcal{B}, \lambda) \in \mathbb{B}(N)$ **do**
        if $\sum_{T\in\mathcal{B}} \lambda_T v^S(T) > v(N)$ **then**
            return False
    return True

Algorithm 5 Strict vital-exactness checking algorithm
Require: A coalition S, a balanced game (N, v), the set $\mathbb{B}(N)$
Ensure: The Boolean value: 'S is strictly vital-exact'

procedure IsStrictlyVitalExact(S, $\mathbb{B}(N)$), (N, v)
    for $\mathcal{B} \in \mathbb{B}(N)$ **do**
        if $\sum_{T\in\mathcal{B}} \lambda_T^{\mathcal{B}} v^S(T) > v(N)$ **then**
            return False
        else if $\sum_{T\in\mathcal{B}} \lambda_T^{\mathcal{B}} v^S(T) = v(N)$ **then**
            for $T \in \mathcal{B}$ **do**
                if $T \cap S^c = \emptyset$ **then**
                    return False
    return True

Algorithm 6 Extendability checking algorithm
Require: A coalition S, a balanced game (N, v)
Ensure: The Boolean value: 'S is extendable'

procedure IsExtendable(S, (N, v))
    $\mathbb{B}(S^c) \leftarrow$ Peleg($|S^c|$)
    for $\xi \in \text{ext}(C(S, v))$ **do**
        define the reduced game $v_{S^c,\xi}$
        for $\mathcal{B} \in \mathbb{B}(S^c)$ **do**
            if $\sum_{T\in\mathcal{B}} \lambda_T^{\mathcal{B}} v_{S^c,\xi}(T) > v(N) - v(S)$ **then**
                return False
    return True

Algorithm 7 Feasibility checking algorithm
Require: A balanced game (N, v), its support $\mathcal{F}$, a set $\mathcal{S} \subseteq \mathcal{F}$, the set $\mathbb{B}(N)$
Ensure: The Boolean value: '$\mathcal{S}$ is feasible'

procedure IsFeasible($\mathcal{S}$, $\mathcal{F}$, $\mathbb{B}(N)$, (N, v))
    for $\mathcal{B} \in \mathbb{B}(N)$ **such that** $\mathcal{B} \subseteq (\mathcal{F} \setminus \mathcal{S}) \cup \mathcal{S}^c$ **do do**
        if $\mathcal{B} \cap \mathcal{S}^c \neq \emptyset$ **and** $\sum_{S\in\mathcal{B}} \lambda_S^{\mathcal{B}} v^{\mathcal{S}}(S) \geq v(N)$ **then**
            return False
        else if $\mathcal{B} \cap \mathcal{S}^c = \emptyset$ **and** $\sum_{S\in\mathcal{B}} \lambda_S^{\mathcal{B}} v^{\mathcal{S}}(S) > v(N)$ **then**
            return False
    return True

Appendix D. Algorithms for checking core stability

See Algorithms 8–11.

Algorithm 8 Association/admissibility subroutine
Require: A set of coalition $\mathcal{S}$, the set of minimal balanced collections $\mathbb{B}(N)$, the set $\mathcal{F}$
Ensure: The set $\mathbb{C}(\mathcal{S})$

procedure Admissibles($\mathcal{S}$, $\mathcal{F}$, $\mathbb{B}(N)$)
    for $S \in \mathcal{S}$ **do**
        $\mathcal{A}(S) \leftarrow \emptyset$

for $(\mathcal{B}, \lambda) \in \mathbb{B}(N)$ **do**
  if $\mathcal{B} \subseteq \{\{j\} \mid i \in S\} \cup \{S^c\} \cup (\mathcal{F} \setminus 2^S)$ **and** $\mathcal{B} \cap \{\{j\} \mid j \in S\} \neq \emptyset$ **then**
    $\mathcal{B}^* \leftarrow \mathcal{B} \setminus \{\{j\} \mid j \in S\}$
    if $\mathcal{B}^* \cap \mathcal{S} \neq \emptyset$ **or** $\mathcal{B}^* \cap \mathcal{S}^c = \emptyset$ **then**
      Add $\mathcal{B}$ to $\mathcal{A}(S)$
return $\bigtimes_{S \in \mathcal{S}} \mathcal{A}(S)$

Algorithm 9 Omega computation subroutine

Require: A set of coalition $\mathcal{S}$, a collection $(\mathcal{B}_S)_{S\in\mathcal{S}} \in \mathbb{C}(\mathcal{S})$, the set $\mathcal{F}$
Ensure: The set $\Omega = \Omega_A \cup \Omega_B \cup \Omega_C$

procedure OMEGA($\mathcal{S}$, $(\mathcal{B}_S)_{S\in\mathcal{S}}$)
  $\Omega_A \leftarrow \emptyset$, $\Omega_B \leftarrow \emptyset$, $\Omega_C \leftarrow \emptyset$
  for $S \in \mathcal{F}$ **do**
    if not $S \in \mathcal{S}$ **then**
      Add $\mathbf{1}^S$ to Ω_B
    else
      Add $\mathbf{1}^{S^c}$ to Ω_A and $z \leftarrow \vec{o}_{\mathbb{R}^N}$
      for $i \in S$ **do**
        if $\{i\} \in \mathcal{B}_S$ **then**
          $z_j \leftarrow \lambda^{\mathcal{B}}_{\{i\}}$
      Add z to Ω_C
  return $(\Omega_A, \Omega_B, \Omega_C)$

Algorithm 10 Minimal balanced sets computation algorithm

Require: A set $\Omega = \Omega_A \cup \Omega_B \cup \Omega_C$
Ensure: The sets $\mathbb{B}(\Omega)$ and $\mathbb{B}_0(\Omega)$

procedure ISMINIMALBALANCED(Z)
  if $\mathrm{rk}(A^Z) = \mathrm{rk}(A^Z_{\mathbf{1}}) = |Z|$ **then**
    $\lambda \leftarrow$ LINALGSOLVE(A^Z)
    if $\lambda > 0$ **then**
      return True
  return False
procedure BALANCEDSETS(Ω)
  for $Z \subseteq \Omega$ **such that** $|Z| \leq n$ **do**
    if ISMINIMALBALANCED(Z) **then**
      add Z to $\mathbb{B}(\Omega)$
      for $z \in Z$ **do**
        if $z \in \Omega_A \setminus \Omega_C$ **then**
          Add Z to $\mathbb{B}_0(\Omega)$
  return $(\mathbb{B}(\Omega), \mathbb{B}_0(\Omega))$

Algorithm 11 Nested balancedness checking subroutine

Require: A feasible collection $\mathcal{S}$, the game (N, v), its support $\mathcal{F}$, and the set $\mathbb{B}(N)$
Ensure: The Boolean value: '$\mathcal{S}$ satisfies the conditions of Theorem 7.13'

procedure ISGSCONDITIONSATISFIED($\mathcal{S}$, $\mathcal{F}$, $\mathbb{B}(N)$, (N, v))
  $\mathbb{C}(\mathcal{S}) \leftarrow$ ADMISSIBLES($\mathcal{S}$, $\mathcal{F}$, $\mathbb{B}(N)$)
  for $(\mathcal{B}_S)_{S\in\mathcal{S}} \in \mathbb{C}(\mathcal{S})$ **do**
    $(\Omega_A, \Omega_B, \Omega_C) \leftarrow$ OMEGA($\mathcal{S}$, $(\mathcal{B}_S)_{S\in\mathcal{S}}$)
    $(\mathbb{B}(\Omega), \mathbb{B}_0(\Omega)) \leftarrow$ BALANCEDSETS($\Omega_A \cup \Omega_B \cup \Omega_C$)
    if $\max_{Z\in\mathbb{B}(\Omega)} \psi(Z) \leq v(N)$ **and** $\arg\max_{Z\in\mathbb{B}(\Omega)} \psi(Z) \notin \mathbb{B}_0(\Omega)$ **then**
      return False
    if $\max_{Z\in\mathbb{B}(\Omega)} \psi(Z) < v(N)$ **and** $\arg\max_{Z\in\mathbb{B}(\Omega)} \psi(Z) \in \mathbb{B}_0(\Omega)$ **then**
      return False
  return True

References

[1] D. Avis, K. Fukuda, A pivoting algorithm for convex hulls and vertex enumeration of arrangements and polyhedra, Discrete Comput. Geom. 8 (1992) 295–313.
[2] C. Berge, Hypergraphs: Combinatorics of Finite Sets, North-Holland Mathematical Library, 1989.
[3] L.J. Billera, J.T. Moore, C.D. Moraites, Y. Wang, K. Williams, Maximal unbalanced families, 2012, arXiv preprint arXiv:1209.2309.
[4] A.K. Biswas, T. Parthasarathy, J.A.M. Potters, M. Voorneveld, Large cores and exactness, Games and Economic Behavior 28 (1) (1999) 1–12.
[5] A. Björner, Positive sum systems, in: Combinatorial Methods in Topology and Algebra, Springer, 2015, pp. 157–171.
[6] O. Bondareva, Some applications of linear programming to the theory of cooperative games, Probl. Kibern. 10 (1963) 119–139 (in Russian).
[7] E. Boros, K. Elbassioni, V. Gurvich, K. Makino, Generating vertices of polyhedra and related problems of monotone generation, in: Proceedings of the Centre de Recherches Mathématiques at the Université de Montréal, Special issue on Polyhedral Computation (CRM Proceedings and Lecture Notes), vol. 49, 2009, pp. 15–43.
[8] C.J. Colbourn, J.H. Dinitz (Eds.), HandBook of Combinatorial Designs, second ed., Chapman and Hall/CRC, 2006.
[9] M. Davis, M. Maschler, The kernel of a cooperative game, Nav. Res. Logist. Q. 12 (1965) 223–259.
[10] X. Deng, C.H. Papadimitriou, On the complexity of cooperative solution concepts, Math. Oper. Res. 19 (1994) 257–266.
[11] J. Derks, H. Peters, Orderings, excess functions, and the nucleolus, Math. Social Sci. 36 (1998) 175–182.
[12] T.S. Evans, N-point finite temperature expectation values at real times, Nuclear Phys. B 374 (2) (1992) 340–370.
[13] T.S. Evans, What, Is Being Calculated with Thermal Field Theory? 1994, http://dx.doi.org/10.48550/arXiv.hep-ph/9404262.
[14] D.B. Gillies, Some Theorems on n-Person Games (Ph.D. thesis), Princeton, New Jersey, 1953.
[15] D.B. Gillies, Solutions to general zero-sum games, in: A. Tuckey, R. Luce (Eds.), Contributions to the Theory of Games IV, Princeton University Press, 1959, pp. 47–85.
[16] M. Grabisch, P. Sudhölter, Characterization of TU games with stable core by nested balancedness, Math. Program. (2023) 1–26.
[17] K. Kikuta, L. Shapley, Core stability in n-person games, 1986, Unpublished paper.
[18] T.E. Kulakovskaya, Necessary and sufficient conditions for the coincidence of core and solution in a classical cooperative game, in: Doklady Akademii Nauk, vol. 199, (5) Russian Academy of Sciences, 1971, pp. 1015–1017.
[19] W.F. Lucas, The proof that a game may not have a solution, Trans. Amer. Math. Soc. 137 (1969) 219–229.
[20] B. Peleg, An inductive method for constructing minimal balanced collections of finite sets, Nav. Res. Logist. Q. 12 (1965).
[21] B. Peleg, On the reduced game property and its converse, Int. J. Game Theory 15 (1986) 187–200.
[22] B. Peleg, P. Sudhölter, Introduction to the Theory of Cooperative Games, Kluwer Academic Publisher, 2003.
[23] L.S. Shapley. On balanced sets, cores, Naval research logistics quarterly, Nav. Res. Logist. Q. 14 (1967) 453–460.
[24] N.J.A. Sloane, The OEIS Foundation Inc., The on-line encyclopedia of integer sequences, 2023, https://oeis.org.
[25] J. von Neumann, O. Morgenstern, Theory of Games and Economic Behavior, second ed., Princeton University Press, 1947.