Nonlinear Self Dual Solutions for TU-Games*)

by Peter Sudhölter**)

University of Bielefeld, Germany

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Abstract

For cooperative transferable utility games solution concepts are presented which resemble the core-like solution concepts prenucleolus and prekernel. These modified solutions take into account both, the 'power', i.e. the worth, and the 'blocking power' of a coalition, i.e. the amount which the coalition cannot be prevented from by the complement coalition, in a totally symmetric way. As a direct consequence of the corresponding definitions they are self dual, i.e. the solutions of the game and its dual coincide. Sudhölter's recent results on the modified nucleolus are surveyed. Moreover, an axiomatization of the modified kernel is presented.

Key words: TU-game, dual game, nucleolus, kernel.

) Institute of Mathematical Economics, University of Bielefeld, Postfach 100131, D-33501 Bielefeld, Gemany; **E-mail: PSudhoelter@Wiwi.Uni-Bielefeld.De

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0 Introduction

In a series of papers (Sudhölter (1993,1994,1996a,b)) a new solution concept, the modified nucleolus, for cooperative side payment games with a finite set of players is discussed.

The expression 'modified nucleolus' refers to the strong relationship of this solution to the (pre)nucleolus introduced by Schmeidler (1966).

An imputation belongs to the nucleolus of a game, if it successively minimizes the maximal excesses, i.e. the differences of the worths of coalitions and the aggregated weight of these coalitions with respect to (w.r.t.) the imputation, and the number of coalitions attaining them. For the precise definition Section 2 is referred to. By regarding the excesses as a measure of dissatisfaction the nucleolus obtains an intuitive meaning as pointed out by Maschler, Peleg, and Shapley (1979).

The solution discussed in the recent papers constitutes an attempt to treat all coalitions equally as far as this is possible. Therefore it is natural to regard the differences of excesses as a measure of dissatisfaction leading to the following intuitive definition. A preimputation belongs to the modified nucleolus $\Psi(v)$ of a game v, if it successively minimizes the maximal differences of excesses and the number of coalition pairs attaining them. The modified nucleolus takes into account both the 'power', i.e. the worth, and the 'blocking power' of a coalition, i.e. the amount which the coalition cannot be prevented from by the complement coalition. If the power of a coalition is measured by its worth (as usual), then the blocking power of a coalition should be measured by its worth w.r.t. the dual game. Alike the prenucleolus, which only depends on the worths of the coalitions, the modified nucleolus is a singleton.

To give an example look at the glove game with three players, one of them (player **1**) possessing a unique right hand glove whereas the other players (**2** and **3**) possess one single left hand glove each. The worth of a coalition is the number of pairs of gloves of the coalition (i.e. one or zero). If a coalition has positive worth, then **1** is a member of the coalition, i.e. player **1** is a veto player possessing, in some sense, all of the power. Indeed the (pre)nucleolus assigns one to player **1** and zero to the other players. On the other hand both players **2** and **3** together can prevent player **1** from any positive amount by forming a 'syndicate'. Therefore they together have the same blocking power as player **1** has. The modified nucleolus takes care of this fact and assigns 1/2 to the first and 1/4 to each of the other players.

A further motivation to consider the new solution concept is its behaviour on the remarkable class of weighted majority games. For the subclasses of weighted majority constant-sum games on the one hand and for homogeneous games on the other hand the nucleolus (see Peleg (1968)) and the minimimal integer representation (see Ostmann (1987) and Rosenmüller (1987)) respectively can be regarded as canonical representation. Fortunately, the modified nucleolus coincides with the prenucleolus on constant-sum games and, up to normalization, with the weights of the minimal integer representation on homogeneous games. Additionally, it induces a representation for an arbitrary weighted majority game. Therefore the modified nucleolus can be regarded as a canonical representation in the general weighted majority case. For the details Sudhölter (1996b) is referred to.

In general a solution concept which assigns the same preimputations to both, the game and its dual is called **self dual**. Analogously to the prenucleolus the prekernel possesses a self dual modification (see Sudhölter 1993).

This paper is organized as follows:

Section 1 recalls some well-known definitions and necessary notations. In Section 2 the definition and some properties of the modified nucleolus are recalled. The dual game v^* of a game v assigns to each coalition the real number which can be given to it if the worth of the grand coalition is shared and the complement coalition obtains its worth. By looking at complements it turns out that the modified solution concepts of v and v^* coincide (the solutions satisfy **self duality**), this also being a characteristic of the Shapley value. In what follows results of Sudhölter (1996a) are surveyed. The modified nucleolus, e.g., can be viewed as the restriction of the prenucleolus of the dual cover (a certain replication) of the game. The dual cover of a game arises from a game v with player set N by taking the union of two disjoint copies of N to be the new player set and assigning to a coalition S the maximum of the sums of the worths of the intersections of S with the first copy w.r.t. v and the second copy w.r.t. v^* or, conversely, the first copy w.r.t. v^* and the second w.r.t. v. Hence both, the game and its dual, are totally symmetric ingredients of the dual cover. This 'restriction' result enables us to reformulate many properties of the prenucleolus for the modified nucleolus, e.g., the modified nucleolus can be computed by each of the well-known algorithms for the calculation of the prenucleolus (see, e.g., Kopelowitz (1967) or Sankaran (1992)) applied to the dual cover. The coincidence of the pre- and modified nucleolus on constant-sum games is a further interesting property. At the end of this section the behavior of the modified nucleolus on weighted majority games is discussed.

In Sudhölter (1996a) two axiomatizations of the modified nucleolus are presented which are comparable to Sobolev's (1975) characterization of the prenucleolus. In Section 3 one axiomatization of the modified nucleolus is recalled.

In Section 4 two self dual modifications of the prekernel are introduced. The proper modified kernel contains the modified nucleolus and is a subset of the modified kernel. The application to glove games shows that the new solutions concepts do no necessarily coincide.

In the last section an axiomatization of the modified kernel is presented which is similar to Peleg's (1986) axiomatization of the prekernel.

1 Notation and Definitions

A **cooperative game with transferable utility** — a **game** — is a pair $G = (N, v)$, where N is a finite nonvoid set and

$$v : 2^N \to \mathbb{R},\ v(\emptyset) = 0$$

is a mapping. Here $2^N = \{S \subseteq N\}$ is the set of **coalitions** of G.

If $G = (N, v)$ is a game, then N is the **grand** coalition or the set of **players** and v is called **characteristic** (or **coalitional**) **function** of G. Since the nature of G is determined by the characteristic function, v is called game as well.

If $G = (N, v)$ is a game, then the **dual** game (N, v^*) of G is defined by

$$v^*(S) = v(N) - v(N \setminus S)$$

for all coalitions S. The set of **feasible payoff vectors** of G is denoted by

$$X^*(N, v) = X^*(v) = \{x \in \mathbb{R}^N \mid \; x(N) \leq \; v(N)\},$$

whereas

$$X(N, v) = X(v) = \{x \in \mathbb{R}^N \mid \; x(N) = v(N)\}$$

is the set of **preimputations** of G (also called set of **Pareto optimal** feasible payoffs of G). Here

$$x(S) = \Sigma_{i \in S} x_i \;\; (x(\emptyset) = 0)$$

for each $x \in \mathbb{R}^N$ and $S \subseteq N$. Additionally, let x_S denote the restriction of x to S, i.e.

$$x_S = (x_i)_{i \in S} \in \mathbb{R}^S,$$

whereas $A_S = \{x_S \mid x \in A\}$ for $A \subseteq \mathbb{R}^N$. For disjoint coalitions $S, T \subseteq N$ and $x \in \mathbb{R}^N$ let $(x_S, x_T) = x_{S \cup T}$.

A **solution concept** σ **on** a set Γ of games is a mapping that associates with every game $(N, v) \in \Gamma$ a set $\sigma(N, v) = \sigma(v) \subseteq X^*(v)$.

If $\bar{\Gamma}$ is a subset of Γ, then the canonical restriction of a solution concept σ on Γ is a solution concept on $\bar{\Gamma}$. We say that σ is a solution concept on $\bar{\Gamma}$, too. If Γ is not specified, then σ is a solution concept on every set of games.

Some convenient and well-known properties of a solution concept σ on a set Γ of games are as follows.

(1) σ is **anonymous** (satisfies **AN**), if for each $(N, v) \in \Gamma$ and each bijective mapping $\tau : N \to N'$ with $(N', \tau v) \in \Gamma$

$$\sigma(N', \tau v) = \tau(\sigma(N, v))$$

holds (where $(\tau v)(T) = v(\tau^{-1}(T))$, $\tau_j(x) = x_{\tau^{-1} j}$ $(x \in \mathbb{R}^N,\, j \in N',\, T \subseteq N')$).

In this case v and τv are **equivalent** games.

(2) σ satisfies the **equal treatment property (ETP)**, if for every $x \in \sigma(N, v)$ $(v \in \Gamma)$ interchangeable players $i, j \in N$ are treated equally, i.e. $x_i = x_j$. Here i and j are **interchangeable**, if $v(S \cup \{i\}) = v(S \cup \{j\})$ for $S \subseteq N \setminus \{i, j\}$.

(3) σ **respects desirability** if for every $(N, v) \in \Gamma$ every $x \in \sigma(N, v)$ satisfies $x_i \geq x_j$ for a player i who is at least as desirable as player j. Here i is **at least as desirable** as j if $v(S \cup \{i\}) \geq v(S \cup \{j\})$ for $S \subseteq N \setminus \{i, j\}$.

(4) σ satisfies the **null player property (NPP)** if for every $(N,v) \in \Gamma$ every $x \in \sigma(N,v)$ satisfies $x_i = 0$ for every nullplayer $i \in N$. Here i is **nullplayer** if $v(S \cup \{i\}) = v(S)$ for $S \subseteq N$.

(5) σ is **covariant under strategic equivalence** (satisfies **COV**), if for $(N,v),(N,w) \in \Gamma$ with $w = \alpha v + \beta$ for some $\alpha > 0, \beta \in \mathbb{R}^N$

$$\sigma(N,w) = \alpha\sigma(N,v) + \beta$$

holds. The games v and w are called **strategically equivalent**.

(6) σ is **single valued** (satisfies **SIVA**), if $\mid \sigma(v) \mid = 1$ for $v \in \Gamma$.

(7) σ satisfies **nonemptiness (NE)**, if $\sigma(v) \neq \emptyset$ for $v \in \Gamma$.

(8) σ is **Pareto optimal** (satisfies **PO**), if $\sigma(v) \subseteq X(v)$ for $v \in \Gamma$.

(9) σ satisfies **reasonableness (on both sides) (REAS)**, if

(a)
$$x_i \geq \min\{v(S \cup \{i\}) - v(S) \mid S \subseteq N \setminus \{i\}\}$$

and

(b)
$$x_i \leq \max\{v(S \cup \{i\}) - v(S) \mid S \subseteq N \setminus \{i\}\}$$

for $i \in N, (N,v) \in \Gamma$, and $x \in \sigma(N,v)$.

Note that both equivalence and strategical equivalence commute with duality, i.e. $(\tau v)^* = \tau(v^*)$, $(\alpha v + \beta)^* = \alpha v^* + \beta$, where τ, α, β are chosen according to the definitions given above. With the help of assertion (9b) Milnor (1952) defined his notion of reasonableness.

It should be remarked (see Shapley (1953)) that the Shapley value φ (to be more precise the solution concept σ given by $\sigma(v) = \{\varphi(v)\}$) satisfies all above properties.

Some more notation will be needed. Let (N,v) be a game and $x \in \mathbb{R}^N$. The **excess of** a coalition $S \subseteq N$ **at** x is the real number

$$e(S,x,v) = e(S,x) = v(S) - x(S).$$

Let $\mu(x,v) = \mu(x)$ be the **maximal excess at** x, i.e. $\mu(x,v) = \max\{e(S,x) \mid S \subseteq N\}$. For **different** players $i,j \in N$ let

$$s_{ij}(x,v) = s_{ij}(x) = \max\{e(S,x) \mid i \in S \subseteq N \setminus \{j\}\}$$

denote the **maximal surplus of** i **over** j **at** x.

2 A Self Dual Modification of the Nucleolus

This section serves to define a self dual modification of the classical prenucleolus. Some well-known properties of this solution concept are recalled and an example is presented. For detailed proofs of all assertions in this section Sudhölter (1996a,b) is referred to.

The nucleolus of a game was introduced by Schmeidler (1966). Some corresponding definitions and results are recalled: Let $\vartheta : \bigcup_{n\in I\!N} I\!R^n \to \bigcup_{n\in I\!N} I\!R^n$ be defined by

$$\vartheta(x) = y \in I\!R^n \quad (x \in I\!R^n),$$

where y is the vector which arises from x by arranging the components of x in a nonincreasing order. The **nucleolus of** v **w.r.t** X, where $X \subseteq I\!R^N$, is the set

$$\mathcal{N}(X,v) = \{x \in X \mid \vartheta((e(S,x,v))_{S\subseteq N}) \leq_{lex} \vartheta((e(S,y,v))_{S\subseteq N}) \text{ for all } y \in X\}.$$

The **prenucleolus of** (N,v) is defined to be the nucleolus w.r.t. the set of feasible payoff vectors and denoted $\mathcal{PN}(v)$, i.e., $\mathcal{PN}(v) = \mathcal{N}(X^*(v),v)$. The prenucleolus of a game is a singleton and it is clearly Pareto optimal (see again Schmeidler (1966)). The unique element $\nu(v)$ of $\mathcal{PN}(v)$ is again called **prenucleolus** (**point**).

For completeness reasons we recall that the nucleolus of (N,v) is the set $\mathcal{N}(X,v)$, where $X = \{x \in X(v) \mid x_i \geq v(\{i\})\}$ is the set of **imputations** of v. Maschler, Peleg and Shapley (1979) tried to give an intuitive meaning to the definition of the (pre)nucleolus by regarding the excess of a coalition as a measure of dissatisfaction which should be minimized. If the excess of a coalition can be decreased without increasing larger excesses, this process will also increase some kind of 'stability', they argued. Nevertheless, Maschler (1992) asked: "What is more 'stable', a situation in which a few coalitions of highest excess have it as low as possible, or one where such coalitions have a slightly higher excess, but the excesses of many other coalitions is substantially lowered?" Anyone, like the present author, who is not convinced by the first or latter, may try to search for a completely different solution concept. The concept which will be introduced in this paper constitutes an attempt to treat all coalitions equally w.r.t. excesses as far as this is possible. Therefore, instead of minimizing the highest excess, then minimizing the number of coalitions with highest excess, minimizing the second highest excess and so on - the highest difference of excesses is minimized, then the number of pairs of coalitions with highest difference of excesses is minimized... Here is the notation.

Definition 2.1 *Let (N,v) be a game. For each $x \in I\!R^N$ define $\tilde\Theta(x,v) = \vartheta((e(S,x,v) - e(T,x,v))_{(S,T)\in 2^N\times\ 2^N}) \in I\!R^{2^{2|N|}}$. The* **modified nucleolus of** *v is the set*

$$\Psi(v) = \{x \in X(v) \mid \tilde\Theta(x,v) \leq_{lex} \tilde\Theta(y,v) \text{ for all } y \in X(v)\}.$$

Remark 2.2 *Let (N,v) be a game.*

(1) If x is any preimputation of the game v, then the following equality holds by definition and Pareto optimality:

$$e(T,x,v^*) = -e((N\setminus T),x,v).$$

With $\bar{\Theta}(y,v) = \vartheta((e(S,y,v)+e(T,y,v^*))_{(S,T)\in 2^N\times 2^N})$ for $y \in \mathbb{R}^N$ this equality directly implies for $x \in X(v)$ that $\bar{\Theta}(x,v) = \tilde{\Theta}(x,v)$ holds true. Note that x has to be Pareto optimal for this equation. Nevertheless the modified nucleolus can be redefined as

$$\Psi(v) = \{x \in X^*(v) \mid \bar{\Theta}(x,v) \leq_{lex} \bar{\Theta}(y,v) \text{ for all } y \in X^*(v)\}, \tag{2.1}$$

because Pareto optimality is, now, automatically satisfied. Indeed, this property can be verified by observing that for every nonvoid coalition both, the excess w.r.t. v and w.r.t. v^*, strictly decrease if all components of a feasible payoff vector can be strictly increased.

(2) *The alternate definition of $\Psi(v)$ in the last assertion (see (2.1)) directly shows that Ψ is* **self dual**, *i.e. $\Psi(v) = \Psi(v^*)$ holds. Note that Ψ shares this property with the Shapley value.*

In what follows two kinds of replicated games are defined. The first one will be used to present a property which allows an axiomatization of the modified nucleolus, which is the restriction of the prenucleolus of the second kind of replication.

Definition 2.3 *Let (N,v) be a game and $\bar{N} = N \times \{0,1\}$. We identify $N \times \{0\}$ with N and $N \times \{1\}$ with N^* in the canonical way, thus $\bar{N} = N \cup N^*$.*

(1) *The game $(N \cup N^*, \bar{v})$, defined by*

$$\bar{v}(S \cup T^*) = v(S) + v^*(T)$$

for all $S, T \subseteq N$ is the **dual replication of** v.

(2) *The game $(N \cup N^*,\ \tilde{v})$, defined by*

$$\tilde{v}(S \cup T^*) = \max\{v(S) + v^*(T), v(T) + v^*(S)\}$$

for all $S, T \subseteq N$ is the **dual cover of** v.

Sudhölter (1996a) proved the following result which shows a strong relation between the modified nucleolus and the prenucleolus of the dual cover of the game.

Theorem 2.4 *The modified nucleolus of a game (N,v) is the restriction of the prenucleolus of $(N \cup N^*, \tilde{v})$ to N; i.e. $\psi(v) = \nu(\tilde{v})_N$. Moreover, $\nu_i(\tilde{v}) = \nu_{i^*}(\tilde{v})$ for $i \in N$.*

In view of Theorem 2.4 the modified nucleolus of a game v is a singleton denoted by $\psi(v)$, i.e. $\{\psi(v)\} = \Psi(v)$. The unique point $\psi(v)$ of $\Psi(v)$ is again called **modified nucleolus** (**point**).

Some properties of the modified nucleolus are presented in the following remark. For the necessary proofs Sudhölter (1996a,b) is referred to.

Remark 2.5 *Let (N, v) be a game.*

(1) If $\nu(v) = \nu(v^*)$, then $\psi(v) = \nu(v)$.

(2) If v is a constant-sum game (i.e. v coincides with v^*), then $\psi(v) = \nu(v)$.

(3) If v is convex (i.e. $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$ for $S, T \subseteq N$), then the modified nucleolus is contained in the core of v.

(4) The modified nucleolus satisfies REAS, COV, AN, NPP, ETP, and it respects desirability.

(5) The modified nucleolus of the dual replication $(N \cup N^*, \bar{v})$ arises from the modified nucleolus of (N, v) by replication, i.e. $\psi_i(\bar{v}) = \psi_i(v) = \psi_{i^*}(\bar{v})$ for $i \in N$ (written $\psi(\bar{v}) = (\psi(v), \psi(v)^*)$).

To illustrate the notion of the modified nucleolus its behavior on weighted majority games is sketched.

Example 2.6 *A game (N, v) is a* **weighted majority game***, if there is a pair $(\lambda; m)$ satisfying*

(1) $\lambda \in I\!R_{>0}$, $m \in I\!R^N_{\geq 0}$, and $m(N) \geq \lambda$,

(2) $v(S) = \begin{cases} 1 & \text{, if } m(S) \geq \lambda \\ 0 & \text{, otherwise} \end{cases}$.

In this case $(\lambda; m)$ is a **representation** *of the game.*

For an arbitrary weighted majority constant-sum game (N, v) Peleg (1968) showed that the nucleolus $\nu = \nu(v)$ **induces** *a representation, i.e. $(1 - \mu(\nu, v); \nu)$ is a representation of (N, v). By Remark 2.5 (2) the same property holds for the modified nucleolus. For general weighted majority games the nucleolus does not necessarily induce a representation (see, e.g., the glove game presented in the introduction which can be represented by $(3; 2, 1, 1)$ and possesses a nucleolus assigning 0 to players* **2** *and* **3***). In Sudhölter (1996b) the following assertion is proved.*

If (N, v) is a weighted majority game and ψ is its modified nucleolus, then $(1 - \mu(\psi, v); \psi)$ is a representation of (N, v).

For completeness reasons we present a proof of this assertion: Let $(\lambda; m)$ be a representation of (N, v) which is **normalized***, i.e. $m(N) = 1$ (i.e. m is a preimputation of the game). Then*

$$0 \leq e(S, m, v) \leq 1 - \lambda \text{ for } S \in 2^N \text{ with } v(S) = 1$$

and

$$-\lambda < e(S, m, v) \leq 0 \text{ for } S \in 2^N \text{ with } v(S) = 0,$$

thus

$$e(S,m,v) - e(T,m,v) < 1 \text{ for } S,T \in 2^N.$$

By Remark 2.5 (4) $\psi = \psi(v) \geq 0$. Let x be any preimputation of v satisfying $x \geq 0$ which does **not** *induce a representation of (N,v). Take $S,T \in 2^N$ with $v(S) = 1,\ v(T) = 0$, and $x(S) \leq x(T)$. Then*

$$e(S,x,v) - e(T,x,v) = 1 - x(S) + x(T) \geq 1 > \max_{S,T \in 2^N} e(S,m,v) - e(T,m,v),$$

thus $x \neq \psi$ by definition. Additionally, this observation shows that the maximal excess at ψ is attained by some winning coalitions only, thus $(1 - \mu(\psi,v);\psi)$ is a representation of v. *q.e.d.*

For a proof (which is more involved) showing that ψ coincides with the normalized vector of weights of the unique minimal integer representation in the homogeneous case Sudhölter (1996b) is referred to.

3 An Axiomatization of the Modified Nucleolus

In Sudhölter (1996a) two axiomatizations of the modified nucleolus are presented. We will present one of them.

First of all the characterizing axioms for the prenucleolus will be recalled.

Definition 3.1

(1) For a set U let $\Gamma_U = \{(N,v) \mid N \subseteq U\}$ denote the set of games with player set contained in U.

(2) Let (N,v) be a game, $x \in I\!R^N$, and $\bar{S}$ be a nonvoid coalition of N. The game $(\bar{S}, v^{\bar{S},x})$, where

$$v^{\bar{S},x}(S) = \begin{cases} v(N) - x(N \setminus \bar{S}), & \text{if } S = \bar{S} \\ 0, & \text{if } S = \emptyset \\ \max\{v(S \cup Q) - x(Q) \mid Q \subseteq N \setminus \bar{S}\}, & \text{otherwise} \end{cases},$$

is the **reduced game of** *v w.r.t. x and $\bar{S}$.*

(3) A solution concept σ on a set Γ of games satisfies **consistency** *(***CONS***) if $(N,v) \in \Gamma, x \in \sigma(v), \emptyset \subset \bar{S} \subseteq N$ implies $(\bar{S}, v^{\bar{S},x}) \in \Gamma$ and $x_{\bar{S}} \in \sigma(\bar{S}, v^{\bar{S},x})$.*

The notion of a reduced game was introduced by Davis and Maschler (1965). For the axiom CONS - also called **reduced game property** - and for the following axiomatization of the prenucleolus Sobolev (1975) is referred to. Note that the condition $(\bar{S}, v^{\bar{S},x}) \in \Gamma$ in the definition of the reduced game can be dropped in Sobolev's result, because the considered set of games (Γ_U) is rich enough, i.e. each reduced game w.r.t. each feasible payoff vector automatically is an element of this set.

Theorem 3.2 *(Sobolev) If U is an infinite set, then the prenucleolus is the unique solution concept on Γ_U satisfying SIVA, AN, COV, and CONS.*

For the definition of SIVA, AN, COV Section 1 is referred to. Moreover, Ψ does not satisfy CONS on Γ_U, because it does not coincide with ν. In what follows it turns out that the modified nucleolus can be characterized by replacing the reduced game property and the anonymity by three additional axioms. Some notation is needed.

Definition 3.3 *Let (N, v) be a game.*

(1) For $x \in I\!R^N$ let $\Lambda(x, v)$ be defined by

$$\Lambda(x, v) = \min\{v(T) - v^*(T) \mid \emptyset \subset T \subset N\} - \mu_0(x, v),$$

where $\mu_0(x, v) = max\{e(S, x, v) \mid \emptyset \subset S \subset N\}$ denotes the maximal nontrivial excess at x. Here $\min \emptyset = \infty$ and $\max \emptyset = -\infty$ as usual and, in addition, $\Lambda(x, v) = 0$ for a 1-person game.

(2) The game v has the **large excess difference property** *(satisfies* **LED***) w.r.t. $x \in I\!R^N$, if $\Lambda(x, v) \geq 0$.*

(3) A solution concept σ on a set Γ of games satisfies **large excess difference consistency** *(***LEDCONS***), if $(S, v^{S,x}) \in \Gamma$ and $x_S \in \sigma(v^{S,x})$, whenever $(N, v) \in \Gamma, x \in \sigma(v)$, and v satisfies LED w.r.t. x.*

In case a game (N, v) satisfies LED w.r.t. a vector x the excess of a nontrivial coalition S (i.e. $\emptyset \subset S \subset N$) w.r.t. v weakly dominates the excess of S w.r.t. the dual game v^*, even if this number is enlarged by the maximal excess of nontrivial coalitions w.r.t. v. Intuitively, the modified nucleolus is 'stable' against objections of coalitions S argueing that the own excess should be diminished if compared to the smaller excesses of further coalitions T. In case of LED 'stability' of x is checked as soon as 'stability' of excess differences of pairs (S, T) with $T = \emptyset$ or $T = N$ can be verified. To be more precise, the modified nucleolus and the prenucleolus coincide, whenever the game satisfies LED w.r.t. the latter (see Remark 3.5 (1)).

An interpretation of LEDCONS will be given together with a verbal description of a further 'derived' game defined as follows with the help of the initial game, its dual, and a given payoff vector.

Definition 3.4 *Let σ be a solution concept on a set Γ of games, let (N, v) be a game and $x \in I\!R^N$.*

(1) Define a game (N, v^x) by

$$v^x(S) = \begin{cases} v(S) & \text{, if } S \in \{\emptyset, N\} \\ \max\{v(S) + \mu + 2\mu^*, v^*(S) + \mu^* + 2\mu\} & \text{, otherwise} \end{cases}$$

for $S \subseteq N$, where $\mu = \mu(x, v)$ and $\mu^* = \mu(x, v^*)$.

(2) σ satisfies **excess comparability** *(***EC***), if $v \in \Gamma$, $x \in \sigma(v)$, and $v^x \in \Gamma$ imply $x \in \sigma(v^x)$.*

The idea of the game v^x is as follows. Assume that x is Pareto optimal, i.e. x constitutes a rule how to share $v(N)$. Moreover, assume that the players agree that this rule should take into account the worth $v(S)$ of each coalition S and the amount which S can be given, if the complement coalition $N \setminus S$ obtains its own worth $v(N \setminus S)$. Now the problem to compare these numbers $v(S)$ and $v^*(S)$ is solved here by adding constants to both, $v(S)$ and $v^*(S)$, such that the arising modified maximal excesses w.r.t. v and v^* coincide (as long as both initial maximal excesses are attained by nontrivial coalitions). Excess comparability now means that the solution x has not to be changed if the game v is replaced by v^x, i.e. by a game which contains v and its dual as totally symmetric ingredients in its definition such that the coalitions with maximal initial excesses possess coinciding new excesses (except if one maximal excess is attained by the empty and grand coalition only).

If $x = \nu(\tilde{v})_N$ is the restriction of the prenucleolus of the dual cover of the game, then v^x coincides - up to adding a constant to the worth of every nontrivial coalition - with the reduced game of the dual cover w.r.t. the initial player set and the prenucleolus, hence $x = \nu(v^x)$ in this case. Moreover, v^x satisfies LED w.r.t. x, hence x coincides with the modified nucleolus of v^x. Therefore $\psi(v^{\psi(v)}) = \psi(v)$ holds true. For these properties Remark 3.5 is referred to.

The large excess difference property can be interpreted with the help of v^x as follows. If v satisfies LED w.r.t. the Pareto optimal vector x, then $\mu(x, v^*) = 0$. Due to the definition of LED we obtain $v(S) - v^*(S) - \mu(x, v) \geq 0$, thus

$$v(S) + 2\mu(x, v^*) + \mu(x, v) \geq v^*(S) + 2\mu(x, v) + \mu(x, v^*)$$

for $\emptyset \subset S \subset N$. This motivates the notion of a shift game.

The game (N, w) is a **shift** game of the game (N, v) if there is a real number $\alpha \in I\!R$ such that

$$w(S) = \begin{cases} v(S) + \alpha & \text{, if } \emptyset \subset S \subset N \\ v(S) & \text{, otherwise} \end{cases} .$$

In this case w is the α**-shift** game of v, denoted ${}^\alpha v$.

In this sense v^x coincides with a shift game of v (provided v satisfies LED w.r.t. x) and, hence, v can be seen as the only significant ingredient of v^x in this case. If the coalitions agree to the 'comparability principle' (i.e. to the replacement of v by v^x), then each coalition should argue with its excess w.r.t. the original game instead of switching to the dual game v^*.

Note that every reduced game w.r.t. x of a game (N, v) which satisfies LED w.r.t. the feasible payoff vector x inherits this property, i.e. $(S, v^{S,x})$ satisfies LED w.r.t. x_S. (see Remark 3.5 (2)).

For the following remark Sudhölter (1996a), Lemmata 4.5 and 4.8, is referred to.

Remark 3.5 *Let (N, v) be a game and $x \in X^*(v)$ be a feasible payoff vector.*

(1) If v satisfies LED w.r.t. the prenucleolus $\nu(v)$, then the prenucleolus coincides with the modified nucleolus ($\nu(v) = \psi(v)$).

(2) If v satisfies LED w.r.t. x, then every reduced game $(S, v^{S,x})$ satisfies LED w.r.t. the restricted vector x_S.

(3) If x is Pareto optimal, then v^x satisfies LED w.r.t. x.

(4) If $\nu = \nu(\tilde{v})$ is the prenucleolus of the dual cover $(N \cup N^*, \tilde{v})$, then the reduced game $(N, \tilde{v}^{N,\nu})$ is a shift game of v^{ν_N}.

(5) The prenucleolus of every shift game of v coincides with the prenucleolus of v.

A further axiom which requires, roughly speaking, that the solution concept of the dual replication arises from the solution concept of the initial game by replication (see Remark 2.5 (5)), implies self duality and will be used in the axiomatization.

Definition 3.6 *A solution concept σ on a set Γ of games satisfies the* **dual replication property (DRP)***, if the following is true: If $v \in \Gamma, \tau : N \cup N^* \to \tilde{N}$ is a bijection such that $(\tilde{N}, w) \in \Gamma$, where $w = \tau\bar{v}, x \in \sigma(v)$, then $\tau(x, x^*) \in \sigma(w)$.*

This definition means that the replication of an element of the solution has to be a member of the solution of the dual replication of the game in case both, the game and its dual replication belong to the considered set of games. In order to get a strong instrument which can also be applied if dual replications of games do not belong to Γ we also demand the property just described in case there is a game which is only equivalent to the dual replication. It is straightforward (see Sudhölter (1996a)) to verify that both, the Shapley value and the modified nucleolus satisfy DRP.

Theorem 3.7 *Let U be an infinite set. Then the modified nucleolus is the unique solution concept on Γ_U satisfying SIVA, COV, LEDCONS, EC, and DRP.*

A proof of this theorem contained in Sudhölter (1996a). Nevertheless an outline of the proof is presented for completeness reasons:

The modified nucleolus satisfies the desired properties by Theorem 2.4, Remark 2.5, and Remark 3.5. To show uniqueness let σ be a solution concept which satisfies the desired properties. Lemmata 4.7 and 4.9 of Sudhölter (1996a) show that σ satisfies AN and PO. We proceed similarly to Sobolev's proof of Theorem 3.2. Let $(N, v) \in \Gamma_U$ be a game, $\{x\} = \sigma(v)$, and $y = \psi(v)$. As in the classical context we can assume $y = 0$ by COV. By the infinity assumption of the cardinality of U and AN we assume that the dual replication $(N \cup N^*, \bar{v})$ is a member of Γ_U. With $w = \bar{v}^{(x,x^*)}$ it can be shown that $w = {}^{\alpha}(\bar{v}^{(y,y^*)})$ for some nonnegative α (recall that the modified nucleolus minimizes sums of excesses w.r.t. v and v^*). The game w satisfies LED w.r.t. $(y, y^*) = \nu(w)$ by Remark 3.5. By DRP, EC, and SIVA it suffices to show that $(y, y^*) \in \sigma(w)$ holds true. This can be done by applying Sobolev's approach to w. He showed the existence of a game $(\tilde{N}, u) \in \Gamma_U$ with $N \cup N^* \subseteq \tilde{N}$ satisfying

(1) $u^{N \cup N^*, z} = w$ (where $z = 0 \in I\!R^{\tilde{N}}$),

(2) $u(S) \geq \min_{\emptyset \subset T \subset N} w(T)$ for $\emptyset \subset S \subset \tilde{N}$ and $u(\tilde{N}) = 0$, and

(3) u is transitive (i.e. u's symmetry group is transitive).

By AN and PO $z \in \sigma(u)$ can be concluded. The proof is finished by the observation that u satisfies LED w.r.t z. q.e.d.

It should be remarked that Sudhölter (1996a) contains examples which show that all properties in Theorem 3.7 (including the infinity assumption on the cardinality of the univers U of players) are logically independent.

4 Self Dual Modifications of the Prekernel

The (pre)kernel was introduced in Davis and Maschler (1965) and Maschler, Peleg, and Shapley (1979) respectively. According to the strong relationship between the prekernel, nucleolus, and least core the second paper is referred to and, in the homogeneous case, Peleg and Rosenmüller (1992). For the prenucleolus the corresponding modified solution concept is already defined, whereas the definition of the modified least core is straightforward (see Sudhölter (1996b)). The notion of the modified kernel is given as follows. Analogously to the prekernel the modified kernel will not only be used as an auxiliary solution concept but will be given an intuitive meaning with the help of an axiomatization. At first the definition of the prekernel is recalled. Let (N, v) be a game and $x \in I\!R^N$. The **prekernel** of v is the set of **balanced** preimputations

$$\mathcal{PK}(v) = \{x \in X(v) \mid s_{ij}(x, v) = s_{ji}(x, v) \text{ for } i, j \in N, i \neq j\}.$$

Definition 4.1 *Let (N, v) be a game, $x \in I\!R^N$, and $i, j \in N$ be different players of v.*

(1) Define two numbers

$$\tilde{s}_{ij}(x, v) = \max_{j \notin S \ni i} (e(S, x, v) + \mu(x, v^*), e(S, x, v^*) + \mu(x, v))$$

and

$$\bar{s}_{ij}(x, v) = \max_{i \in S, j \notin T} (e(S, x, v) + e(T, x, v^*), e(S, x, v^*) + e(T, x, v)).$$

Then $\tilde{s}_{ij}$ is the **maximal modified surplus of** i **over** j **at** x.

(2) The **modified kernel of** v *is the set*

$$\mathcal{MK}(v) = \{x \in X(v) \mid \tilde{s}_{ij}(x, v) = \tilde{s}_{ji}(x, v) \textit{ for } i, j \in N, i \neq j\}$$

and the **proper modified kernel of** v *is the set*

$$\mathcal{MK}_0(v) = \{x \in \mathcal{MK}(v) \mid \bar{s}_{ij}(x,v) = \bar{s}_{ji}(x,v) \text{ for } i,j \in N, i \neq j\}.$$

The proper modified kernel is a subset of the modified kernel of the game and Example 4.4 shows that these concepts do not necessarily coincide.

There is a strong relationship between the prekernel of the dual cover of a game and the proper modified kernel of the game, implying nonemptiness.

Lemma 4.2 *Let* (N,v) *be a game. Then*

$$\mathcal{MK}_0(v) = \{x \in I\!R^N \mid (x,x^*) \in \mathcal{PK}(\tilde{v})\}.$$

Proof: Let $x \in I\!R^N$ and $i,j \in N$ with $i \neq j$. By definition $s_{ij}((x,x^*),\tilde{v}) = \tilde{s}_{ij}(x,v)$ and $s_{ij^*}((x,x^*),\tilde{v}) = \bar{s}_{ij}(x,v)$ hold true. Hence $x \in \mathcal{MK}_0(v)$, iff $(x,x^*) \in \mathcal{PK}(\tilde{v})$. q.e.d.

As a consequence of this lemma we obtain $\psi(v) \in \mathcal{MK}_0(v) \subseteq \mathcal{MK}(v)$ for each game v. The set $\{(x,x^*) \in \mathcal{PK}(\tilde{v})\} = \mathcal{SPK}(\tilde{v})$ could be called **symmetric** prekernel of the dual cover $\tilde{v}$ of v.

Remark 4.3

(1) $\mathcal{MK}(v) \supseteq \mathcal{MK}_0(v) \supseteq \mathcal{PK}(v) \cap \mathcal{PK}(v^*)$ *holds true by definition. Moreover, both versions of the modified kernel coincide with the prekernel on constant-sum games.*

(2) The (proper) modified kernel satisfies reasonableness on both sides and respects desirability. A proof of these assertion is straightforward (see Sudhölter (1993)), because both, the game and its dual possess the same 'desirability structure' and the same maximal and minimal marginal contributions.

(3) Both modified kernels satisfy covariance, anonymity, the equal treatment property, and the nullplayer property.

Example 4.4 *For glove games the proper modified kernel is a proper subset of the modified kernel. A game* (N,v) *is a* **glove game**, *if the player set can be partitioned into the sets* R *of 'right hand glove owners' and* L *of 'left hand glove owners' (i.e.* $R \cup L = N$, $R \cap L = \emptyset$, $R \neq \emptyset \neq L$*), whereas the coalitional function* v *counts the number of pairs of gloves owned by the coaltions (i.e.* $v(S) = \min\{\mid R \cap S \mid, \mid L \cap S \mid\}$*). Without loss of generality we may assume* $r = \mid R \mid \leq \mid L \mid = l$*. Moreover, we restrict our attention to the case* $l \geq 2$*, because for two-person games both modified kernels coincide with the Shapley value (*$\mathcal{MK}$*,* $\mathcal{MK}_0$ *and* Ψ *are* **Standard solutions***) by NE, PO, COV, and ETP. We are going to show the following claims:*

(1) The proper modified kernel of v *is the singleton which treats the groups of left hand glove owners and right hand glove owners equally, i.e.*

$$\mathcal{MK}_0(v) = \{z^E\} \text{ where } z_i^E = \begin{cases} 1/2 & , \text{if } i \in R \\ r/2l & , \text{if } i \in L \end{cases}.$$

(2) *If $r < l$, then the modified kernel is the convex hull of the* **equal treatment vector** *z^E and the nucleolus z^R, defined by*

$$z_i^R = \begin{cases} 1 & , \text{ if } i \in R \\ 0 & , \text{ if } i \in L \end{cases} .$$

(3) *If $r = l$, then the modified kernel coincides with the core, i.e. with the convex hull of z^R and z^L. (Here z^L is defined analogously to z^R by $z_i^L = \begin{cases} 1 & , \text{ if } i \in L \\ 0 & , \text{ if } i \in R \end{cases}$.)*

Proof: *If $z \in \mathcal{MK}(v)$ and $i, j \in R$ or $i, j \in L$, then $z_i = z_j$ by ETP (see Remark 4.3 (3)). Moreover, $0 \leq z_i \leq 1$ for $i \in N$ by Remark 4.3 (2). With $z^\alpha \in I\!R^N$ defined by*

$$z^\alpha = \begin{cases} \alpha & , \text{ if } i \in R \\ r(1-\alpha)/l & , \text{ if } i \in L \end{cases}$$

Pareto optimality implies that

$$\mathcal{MK}(v) \subseteq \{z^\alpha \mid 0 \leq \alpha \leq 1\} = Z$$

holds true. For every $z^\alpha \in Z$ and $i, j \in R$ or $i, j \in L$ it is straightforward to verify

$$\begin{array}{l} \tilde{s}_{ij}(x^\alpha, v) = \tilde{s}_{ji}(x^\alpha, v) \\ \bar{s}_{ij}(x^\alpha, v) = \bar{s}_{ji}(x^\alpha, v) \end{array} \quad \text{for } i, j \in R \text{ or } i, j \in L. \tag{4.1}$$

(1) *If $\alpha < 1/2$, then R is the unique coalition attaining $\mu(z^\alpha, v^*)$. In view of (2) we can assume that $r = l$ holds true. Then we have $\mu(z^\alpha, v) = e(S, z^\alpha, v)$ for every coalition S satisfying $\mid S \cap R \mid = \mid S \cap L \mid = 1$. This observation implies*

$$\bar{s}_{ij}(z^\alpha, v) = \mu(z^\alpha, v) + \mu(z^\alpha, v^*) > \bar{s}_{ij}(z^\alpha, v) \text{ for } i \in R,\ j \in L,$$

thus $z^\alpha \notin \mathcal{MK}_0(v)$.

If $\alpha > 1/2$ and $r = l$, the proof can be finished analogously by interchanging the rôles of R and L. If $\alpha > 1/2$ and $r < l$, then L is the unique coalition attaining $\mu(z^\alpha, v^*)$, whereas $\mu(z^\alpha, v)$ is attained by coalitions S satisfying $R \subseteq S$ and $\mid L \cap S \mid = r$. The observation

$$\bar{s}_{ij}(z^\alpha, v) = \mu(z^\alpha, v) + \mu(z^\alpha, v^*) > \bar{s}_{ji}(z^\alpha, v) \text{ for } i \in L,\ j \in R$$

finishes the proof of (1).

(2) If $\alpha \geq 1/2$, then L attains $\mu(z^\alpha, v^*)$ and $R \cup T$ with $T \subset L$ such that $| T |= r$ attains $\mu(z^\alpha, v)$, thus

$$\tilde{s}_{ij}(z^\alpha, v) = \tilde{s}_{ji}(z^\alpha, v) = \mu(z^\alpha, v) + \mu(z^\alpha, v^*) \textit{ for } i \in R,\ j \in L.$$

This equality together with (4.1) implies that $z^\alpha \in \mathcal{MK}(v)$ holds true.

If $\alpha < 1/2$, then R is the unique coalition attaining $\mu(z^\alpha, v^*)$ and every coalition S attaining $\mu(z^\alpha, v)$ contains R, thus $\tilde{s}_{ij}(z^\alpha, v) = \mu(z^\alpha, v) + \mu(z^\alpha, v^*) > \tilde{s}_{ji}(z^\alpha, v)$ for $i \in R,\ j \in L$. This observation shows that z^α cannot be a member of the modified kernel in this case.

(3) For $0 \leq \alpha \leq 1$ the maximal excess $\mu(z^\alpha, v)$ is attained by all coalitions S satisfying $| S \cap R |= 1 =| S \cap L |$, because z^α is a member of the core (recall that $r = l$ is assumed). Using the assumption $l > 1$, i.e. $r > 1$ is automatically satisfied by $r = l$, we come up with

$$\tilde{s}_{ij}(z^\alpha, v) = \mu(z^\alpha, v) + \mu(z^\alpha, v^*) \textit{ for } i, j \in N,$$

thus the proof is finished.

q.e.d.

Applied to the modified nucleolus this example shows that $\psi(v)$ assigns the same amount to both groups R and L. Glove games can be seen as **two-sided assignment games** *as discussed, e.g., in Shapley and Shubik (1972). It can be shown (see Sudhölter (1994)) that both sides of an assignment game are treated equally by the modified nucleolus in general.*

The following lemma is used to show that the (proper) modified kernel satisfies excess comparability as well as LEDCONS.

Lemma 4.5 *Let (N, v) be a game and $x \in X^*(v)$ be a feasible payoff vector. Assume v satisfies LED w.r.t. x. Then the following properties are valid.*

(1) If $i, j \in N$ with $i \neq j$, then $s_{ij}(x, v) = s_{ji}(x, v)$ iff $\tilde{s}_{ij}(x, v) = \tilde{s}_{ji}(x, v)$.

(2) If $s_{ij}(x, v) = s_{ji}(x, v)$ for all $i, j \in N$ with $i \neq j$, then $\bar{s}_{ij}(x, v) = \bar{s}_{ji}(x, v)$ for all $i, j \in N$ with $i \neq j$.

Proof: Assume w.l.o.g. $| N | \geq 2$ (otherwise both assertions are trivially satisfied). Analogously to Remark 2.2 (1) it is obvious that

$$e(S, x, v) = -e(N \setminus S, x, v^*) + v(N) - x(N) \tag{4.2}$$

for all $S \subseteq N$ holds true. Using (4.2), LED and

$$\Lambda(x, v) = \min\{\min\{e(S, x, v), e(T, x, v)\} - e(S, x, v) - e(T, x, v^*) \mid \emptyset \neq S, T \neq N\} \tag{4.3}$$

(for a proof of equation (4.3) see Sudhölter (1996a)) it can easily be seen that

$$e(S,x,v^*) \leq 0 \text{ for all } S \neq N, \tag{4.4}$$

thus

$$e(S,x,v) \geq v(N) - x(N) \geq 0 \text{ for all } S \neq \emptyset. \tag{4.5}$$

Therefore we come up with

$$\mu(x,v) = \mu_0(x,v), \mu(x,v^*) = v(N) - x(N). \tag{4.6}$$

Let $i,j \in N, i \neq j$ and $j \notin T \ni i$ for some $T \subseteq N$. Then

$$\begin{aligned} e(T,x,v) + \mu(x,v^*) &= e(T,x,v) + v(N) - x(N) \text{ (by (4.6))} \\ &\geq e(T,x,v) \text{ (by the feasibility of } x) \\ &\geq e(T,x,v^*) + \mu_0(x,v) \text{ (by (4.3))} \\ &= e(T,x,v^*) + \mu(x,v) \text{ (by 4.6)),} \end{aligned}$$

thus $\tilde{s}_{ij}(x,v) = s_{ij}(x,v) + v(N) - x(N)$; hence the first assertion is established.

In order to show the second one, observe that

$$\begin{aligned} &\bar{s}_{ij}(x,v) \\ &= \max_{i\in S, j\notin T}(e(S,x,v) + e(T,x,v^*), e(S,x,v^*) + e(T,x,v)) \\ &= \max\{e(S,x,v) \mid i \in S\} \cup \{e(T,x,v) + v(N) - x(N) \mid j \notin T\} \text{ (by (4.4))} \\ &\leq \mu(x,v) + v(N) - x(N) \text{ (by definition).} \end{aligned} \tag{4.7}$$

Take any coalition $S \subseteq N$ with $e(S,x,v) = \mu(x,v)$ and $\emptyset \neq S \neq N$ - note that the existence of S is guaranteed by (4.6). If $j \notin S$, then $\bar{s}_{ij}(x,v) = \mu(x,v) + v(N) - x(N)$ (see (4.7)). If $j \in S$, then choose any $k \in N \setminus S$. Now, by assumption, $s_{jk}(x,v) = \mu(x,v) = s_{kj}(x,v)$, thus there is a coalition $T \subseteq N$ with $j \notin T \ni k$ and $e(T,x,v) = \mu(x,v)$. Again $\bar{s}_{ij}(x,v) = \mu(x,v) + v(N) - x(N)$ is concluded in view of (4.7). q.e.d.

Note that Lemma 4.5 yields a relationship between the prekernel, the modified, and the proper modified kernel in case LED is satisfied. Indeed, under the assumptions of this lemma, the vector x is a member of the prekernel of v, iff this is true for the modified kernel. Moreover, modified can be replaced by proper modified. These considerations together with consistency of the prekernel lead to

Corollary 4.6 *The modified and proper modified kernel satisfy LEDCONS and EC on Γ_U for each set U.*

Proof: For both modified solution concepts LEDCONS is directly implied by Lemma 4.5, Remark 3.5 (2), and consistency of the prekernel. By Lemma 4.5 and Remark 3.5 (3) it remains to show that the modified kernel satisfies EC. Let $(N,v) \in \Gamma_U,\ x \in \mathcal{MK}(v)$ and $i,j \in N$. The straightforward observations $\mu(x,v^x) = 2 \cdot (\mu(x,v) + \mu(x,v^*))$ and $\mu(x,(v^x)^*) = 0$ imply

$$\tilde{s}_{ij}(x,v^x) = \tilde{s}_{ij}(x,v) + \mu(x,v) + \mu(x,v^*), \tag{4.8}$$

thus the proof is finished. q.e.d.

5 An Axiomatization of the Modified Kernel

First of all Peleg's (1986) axiomatization of the prekernel is recalled. For a finite set N let $\Pi(N) = \{\{i,j\} \mid i,j \in N, i \neq j\}$ denote the set of player pairs. A solution concept σ on a set Γ of games satisfies **converse consistency** (**COCONS**), if the following condition is satisfied:

If $(N,v) \in \Gamma,\ x \in X(v),\ (S, v^{S,x}) \in \Gamma$, and $x_S \in \sigma(S, v^{S,x})$ for every $S \in \Pi(N)$, then $x \in \sigma(N,v)$.

Theorem 5.1 *(Peleg) If U is a set, then the prekernel is the unique solution concept on Γ_U satisfying NE, PO, ETP, COV, CONS, and COCONS.*

In order to axiomatize $\mathcal{MK}$ one further axiom is needed, which resembles COCONS and which finally leads to an analogon of Peleg's result.

Definition 5.2 *A solution concept σ on a set Γ of games satisfies* **large excess difference converse consistency (LEDCOCONS)***, if the following condition is satisfied:*

If $(N,v) \in \Gamma,\ x \in X(v),\ (S, u^{S,x}) \in \Gamma$, where $u = v^x$, and $x_S \in \sigma(S, u^{S,x})$ for every $S \in \Pi(N)$, then $x \in \sigma(v)$.

LEDCOCONS is a modified converse consistency (COCONS) property in the sense of Peleg (1986). Indeed, if (N,v) satisfies LED w.r.t. x, then $v^x = u$ coincides with v up to a nonnegative shift. Moreover, the reduced games $u^{S,x}$ coincide with $v^{S,x}$ up to a shift. For the general case COCONS is hardly comparable with the modified property. Nevertheless, at least together with EC both converse consistency properties are similar.

Theorem 5.3 *Let U be a set. The modified kernel is the unique solution concept on Γ_U satisfying NE, COV, PO, ETP, LEDCONS, LEDCOCONS, and EC.*

Proof: Clearly, $\mathcal{MK}$ satisfies NE, COV, PO, ETP, LEDCONS, and EC by Lemma 4.2, Remark 4.3 (3), definition, and Corollary 4.6. To verify LEDCOCONS, let $(N,v) \in \Gamma_U$ and $x \in X(v)$ such that $x_S \in \mathcal{MK}(S, u^{S,x})$, where $u = x^x$, for every $S \in \Pi(N)$. By Remark 3.5, (2) and (3), and Lemma 4.5 we conclude that $x_S \in \mathcal{PK}(S, u^{S,x})$ holds true for every $S \in \Pi(N)$. By COCONS of the prekernel $x \in \mathcal{PK}(N,u)$. Remark 3.5 (4) and Lemma 4.5 imply $x \in \mathcal{MK}(N,u)$, thus equation 4.8 (which is valid for every Pareto optimal x) shows that $x \in \mathcal{MK}(N,v)$.

In order to show the uniqueness part let σ be a solution concept on Γ_U which satisfies NE, COV, PO, ETP, LEDCONS, LEDCOCONS, and EC. Due to NE, COV, PO, and ETP, we have $\sigma(N,v) = \mathcal{PK}(N,v) = \mathcal{MK}(N,v)$ for all games (N,v) with $N \subseteq U$ and $\mid N \mid = 2$ as in the classical context (see Peleg (1986), Remark 4.4). From now on only games $(N,v) \in \Gamma_U$ satisfying $\mid N \mid \geq 3$ are considered.

First we prove the inclusion $\mathcal{MK}(N,v) \subseteq \sigma(N,v)$. Let $x \in \mathcal{MK}(N,v)$. Then $x \in \mathcal{MK}(N,v^x)$, because $\mathcal{MK}$ satisfies EC. Write $u = v^x$. In view of Corollary 4.6 (The modified kernel satisfies EC.) and Remark 3.5 (3) (The derived game (N,u) satisfies LED w.r.t. x.) we obtain $x_S \in \mathcal{MK}(S,u^{S,x})$ for every $\emptyset \neq S \subseteq N$, in particular for every coalition S with $\mid S \mid= 2$. For two-person games we already know that the solution concept σ coincides with the modified kernel, i.e. $x_S \in \sigma(S,u^{S,x})$ for $S \subset N$ with $\mid S \mid= 2$. We conclude $x \in \sigma(N,v)$, because σ satisfies LEDCOCONS. These considerations complete the proof of the inclusion $\mathcal{MK}(N,v) \subseteq \sigma(N,v)$.

Secondly we prove the inverse inclusion $\sigma(N,v) \subseteq \mathcal{MK}(N,v)$. Let $x \in \sigma(N,v)$. Then $x \in \sigma(N,v^x)$, because σ satisfies EC. Write $u = v^x$. In view of the assumption that σ satisfies EC and of Remark 3.5 (3) (The derived game (N,u) satisfies LED w.r.t. x.) we obtain $x_S \in \sigma(S,u^{S,x})$ for every $\emptyset \neq S \subseteq N$, in particular for every coalition S with $\mid S \mid= 2$. For two-person games we already know that the solution concept σ coincides with the modified kernel, i.e. $x_S \in \mathcal{MK}(S,u^{S,x})$ for $S \subset N$ with $\mid S \mid= 2$. We conclude $x \in \mathcal{MK}(N,v)$, because $\mathcal{MK}$ satisfies LEDCOCONS. These considerations complete the proof of the inclusion $\sigma(N,v) \subseteq \mathcal{MK}(N,v)$. q.e.d.

Note that the universe U of players in Theorem 5.3 may be finite or infinite as in the classical context (Theorem 5.1). For an axiomatization of the proper modified kernel Sudhölter (1993) is referred to. Peleg showed the logical independence of NE, COV, PO, ETP, CONS, and COCONS by defining six solution concepts which do not coincide with the prekernel satisfying all differing five of the preceding properties. Slightly modified, these examples also show the independence of the axioms of Theorem 5.3. Indeed, define $\sigma^i \ \ (i \in \{1,\ldots,7\})$ on Γ_U for each $(N,v) \in \Gamma_U$ by

$\sigma^1(v) = \emptyset,$

$\sigma^2(v) = \{x \in I\!R^N \mid x_i = v(N)/\mid N \mid \text{ for } i \in N\},$

$\sigma^3(v) = \{x \in X^*(v) \mid \tilde{s}_{ij}(x) = \tilde{s}_{ji}(x,v) \text{ for } i,j \in N, i \neq j\},$

$\sigma^4(v) = X(v),$

$\sigma^5(v) = \{x \in X(v) \mid v(\{i\}) - x_i = v(\{j\}) - x_j$ for $i,j \in N$ with $i \simeq_v j\}$, where $\simeq_v$ is the equivalence relation defined by $i \simeq_v j$, if $\max\{v(S \cup \{i\}), v^*(S \cup \{i\})\} - \max\{v(\{i\}), v^*(\{i\}\} = \max\{v(S \cup \{j\}), v^*(S \cup \{j\})\} - \max\{v(\{j\}), v^*(\{j\}\}$ holds true for $S \subseteq N \setminus \{i,j\}$,

$\sigma^6(v) = \Psi(v),$

$$\sigma^7(v) = \begin{cases} \mathcal{MK}(v) & \text{, if } v \text{ satisfies LED w.r.t. } \varphi(v) \\ \mathcal{MK}(v) \cup \{\varphi(v)\} & \text{, otherwise.} \end{cases}$$

Following Peleg's approach (1986,1988/89) it is straightforward to verify that each σ^i satisfies all axioms of Theorem 5.3 up to the i-th one. Clearly, σ^1 violates NE, if $\mid U \mid\geq 1$. The solution concepts σ^2, σ^3, and σ^4 violate COV, PO, and ETP, respectively, if $\mid U \mid\geq 2$.

The modified kernel is contained in $\sigma^5(v)$. Both concepts only coincide in case $\mid U \mid\leq 2$, in which they cannot be distinguished from σ^6 and σ^7. The last two examples show the logical independence of LEDCOCONS and EC, respectively.

In view of the just mentioned solution concepts Theorem 5.3 can be regarded as an axiomatization of the modified kernel.

In Sudhölter (1993) one crucial difference between the modified kernel and the proper modified kernel is observed. Indeed, in contrast to the modified kernel the proper modified kernel satisfies DRP. From the axiomatic viewpoint 'properness' seems to be less intuitive. The application to many examples (see, e.g., Example 4.4) indicates the converse statement.

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