MATHEMATICS OF OPERATIONS RESEARCH
Vol. 32, No. 4, November 2007, pp. 857–872
ISSN 0364-765X | EISSN 1526-5471 | 07 | 3204 | 0857

DOI 10.1287/moor.1070.0275

Bargaining Sets of Majority Voting Games

Ron Holzman
Department of Mathematics, Technion-Israel Institute of Technology, 32000 Haifa, Israel,
holzman@techunix.technion.ac.il

Bezalel Peleg
Institute of Mathematics and Center for the Study of Rationality, The Hebrew University of Jerusalem,
Feldman Building, Givat Ram, 91904 Jerusalem, Israel, pelegba@math.huji.ac.il

Peter Sudhölter
Department of Business and Economics, University of Southern Denmark, Campusvej 55,
5230 Odense M, Denmark, psu@sam.sdu.dk

Let A be a finite set of m alternatives, let N be a finite set of n players, and let R^N be a profile of linear orders on A of the players. Let u^N be a profile of utility functions for R^N. We define the nontransferable utility (NTU) game V_{u^N} that corresponds to simple majority voting, and investigate its Aumann-Davis-Maschler and Mas-Colell bargaining sets. The first bargaining set is nonempty for $m \leq 3$, and it may be empty for $m \geq 4$. However, in a simple probabilistic model, for fixed m, the probability that the Aumann-Davis-Maschler bargaining set is nonempty tends to one if n tends to infinity. The Mas-Colell bargaining set is nonempty for $m \leq 5$, and it may be empty for $m \geq 6$. Furthermore, it may be empty even if we insist that n be odd, provided that m is sufficiently large. Nevertheless, we show that the Mas-Colell bargaining set of any simple majority voting game derived from the k-fold replication of R^N is nonempty, provided that $k \geq n+2$.

Key words: NTU game; voting game; majority rule; bargaining set
MSC2000 subject classification: Primary: 91A12; secondary: 91B12
OR/MS subject classification: Primary: games/group decisions, cooperative; secondary: mathematics, combinatorics
History: Received November 17, 2005; revised August 6, 2006.

1. Introduction. Mas-Colell [15] has introduced a bargaining set that is also defined for finite games. Although the existence problem was settled for certain classes of games, it remained notoriously open for the important class of finite superadditive NTU games. In Peleg and Sudhölter [17] this fundamental question was answered in the negative, by showing that there exists a four-person NTU majority voting game with 10 alternatives, whose Mas-Colell bargaining set is empty. We note that majority voting games are superadditive. That example has led naturally to the following two questions:

(i) What is the minimum number of alternatives for which there exists an NTU majority voting game with an empty Mas-Colell bargaining set?

(ii) Is it possible to find an NTU majority voting game with an odd number of players, whose Mas-Colell bargaining set is empty?

We have investigated the foregoing two problems. In the following paragraphs we present our solutions and comment on their importance.

In Theorem 5.1 we prove that for all NTU majority voting games with at most five alternatives, the Mas-Colell bargaining set is nonempty. Furthermore, Example 5.1 provides a four-person NTU majority voting game with six alternatives and an empty Mas-Colell bargaining set. Thus, we have a complete solution to the first question.

Let A be a set of m alternatives, $m \in \{3, 4, 5\}$, and let N be a set of (an odd number of) voters. To choose one alternative out of A by the majority rule we need to construct a binary voting tree (see Banks [7]). This is an ad hoc procedure. If we focus on a particular binary voting tree, then the final outcome may be sensitive to the chosen agenda. In contrast, the Mas-Colell bargaining set indicates for each profile of linear orders the choice of at least one compromise alternative. Furthermore, it is neutral with respect to the alternatives and works for an even number of voters as well. Because choice problems with at most five alternatives are quite frequent, Theorem 5.1 may be quite useful.

We now turn to the second question, whether or not there is an NTU majority voting game with an odd number of players and an empty Mas-Colell bargaining set. Unfortunately, it is shown in §6 that the answer is positive. However, the very sophisticated proof is innovative and may be helpful in future investigations of NTU games.

A few remarks are in order about the modelling of a social choice problem as an NTU game. Whereas in the social choice setting preferences over the set A of alternatives are ordinal, we represent them by a cardinal utility function in order to pass to utility space, the natural realm of NTU games. This by itself is innocuous. For everything considered in this paper, the numbers that appear in a payoff vector of the NTU game do not matter in themselves; only the ordinal comparisons between them and the utility values of the alternatives matter. In this sense, our analysis is purely ordinal. What may seem more problematic is our assumption of free disposal of utility. That is, a payoff vector is considered feasible in the NTU game that is associated with the social choice problem not only if it corresponds, via the utility representation, to some alternative in A, but also if it is obtained from such a vector by arbitrarily reducing some of its entries. This is necessary in order to satisfy the comprehensiveness property of NTU games, and is in fact common practice in applications of NTU games (see, e.g., Scarf [18], Aumann [4]). Indeed, our positive results depend significantly on this feature: In many instances where we point out a payoff vector in the bargaining set, this vector entails the choice of an alternative in A, accompanied by a reduction of the utilities of some (but not all, as we insist on weak Pareto optimality) of the players.[1,2] To interpret such outcomes in the social choice context, one can think of voters who contribute to the electoral campaign of a candidate in order to secure his election. Our results indicate that such contributions are often essential to the stability (in the sense of the bargaining set) of the outcome. Note that most legal systems allow campaign contributions, but do not allow the transfer of payments to voters. This lends support to both the free disposal of utility and the nontransferability of utility that are inherent in the NTU model.

We now comment on the relevance of bargaining sets to our model. A feasible payoff vector x is in the core if no coalition has a profitable deviation from (objection against) x. An objection y against x is justified, roughly, if there is no objection to y also taking into consideration payoffs at x. The payoff vector x is in the Mas-Colell [15] bargaining set if there exist no justified objections against it. Thus, the Mas-Colell bargaining set focuses on credibility of objections, and it is a natural extension of the core. It was introduced as a variant of the more classical Aumann-Davis-Maschler bargaining set, which is defined similarly (with different notions of objections and counterobjections[3]). It is natural to study the analogues of the questions above also for the Aumann-Davis-Maschler bargaining set, and indeed we do so in this paper (the answers turn out to be easier). There are several other variants of the definition of the bargaining set that we do not study here. We briefly mention two of them. Asscher [2] presented an ordinal bargaining set for NTU games. However, an element of the Asscher bargaining set allows the existence of justified objections. (Its definition uses the transitive closure of the binary relation “Player i has a justified objection against Player j.”) Finally, the consistent bargaining set of Dutta et al. [9] is a subset of the Mas-Colell bargaining set. Therefore, it cannot yield wider existence results. Furthermore, it may be empty even for transferable utility (TU) games.

The starting point of many studies in social choice theory is the voting paradox, which arises when every alternative is dominated by some other alternative according to a majority of voters. This phenomenon is equivalent, in terms of the NTU majority voting game, to the emptiness of the core. As explained above, the bargaining sets represent a weaker notion of stability than the core, because they take into account only justified objections. Thus, our study of the bargaining sets of majority voting games determines the extent to which focusing on credible objections restores the existence of stable outcomes to the social choice problem, albeit in a weaker sense. Our negative results, such as the construction of an NTU majority voting game with six or more alternatives, which has an empty Mas-Colell bargaining set, may be viewed as showing the robustness of the voting paradox to this weakening of the stability requirements. Note that in this context the assumption of free disposal of utility discussed above only makes our negative results stronger: The robustness of the voting paradox is still exhibited when the outcome space is enlarged by allowing free disposal of utility.

We shall now review our results.

In §2 we derive the exact form of the cooperative NTU games that correspond to simple majority voting.[4] We start with a game in strategic form and use the standard procedure of Aumann and Peleg [6] to derive our NTU game. We also recall the definitions of the Aumann-Davis-Maschler and Mas-Colell bargaining sets of cooperative NTU games.

[1] See a discussion of this in a concrete example in §3.

[2] In extreme cases, the utilities are reduced down to the minimal utility level (over A) of the players. This admittedly weakens the positive value of the existence results. Still, on the positive side, we draw attention to Theorem 7.2, which shows that typically (in a suitable probabilistic sense), when there are many players, a positive fraction of the players may receive their maximal utility level in the Mas-Colell bargaining set.

[3] We refer the reader to Mas-Colell [15] and Vohra [21] for a comparison of the two definitions, and to Holzman [14] for a comparison of the resulting bargaining sets. We note that the result of that paper, giving conditions under which the Mas-Colell bargaining set must contain the Aumann-Davis-Maschler bargaining set, does not apply to our model, because the nonlevelness condition is violated.

[4] Similar derivations may be carried out for other voting rules. Here we concentrate on the most natural voting rule, simple majority. We refer the reader to an earlier version of this manuscript (available as Discussion Paper http://ratio.huji.ac.il/dp/dp376.pdf #376, Center for the Study of Rationality, The Hebrew University of Jerusalem) for a treatment of plurality voting and approval voting.

The voting paradox with three voters and three alternatives is analyzed in §3 with respect to these two bargaining sets. It turns out that both bargaining sets are nonempty.

Section 4 addresses the existence question for the Aumann-Davis-Maschler bargaining set of a simple majority voting game. We show that it is nonempty when there are at most three alternatives, but may be empty when there are four or more alternatives even if there are just three voters.

Section 5 is devoted to the solution of the first question. The boundary between existence and nonexistence turns out to be somewhat higher in the case of the Mas-Colell bargaining set: As indicated above, we prove existence for up to five alternatives, and give examples of emptiness for six or more alternatives.

In these examples, there is an even number of voters. The second question is addressed in §6. By an elaborate construction using huge numbers of voters and alternatives, it is shown that there exists an NTU majority voting game with an odd number of voters and an empty Mas-Colell bargaining set.

We conclude in §7 with existence results for two models in which there are many voters, whose preferences are drawn in a specified way. In one of them, a simple probabilistic model, we show that both bargaining sets are nonempty, with probability tending to one as the number of voters tends to infinity. In the other, a replication model, we prove that the Mas-Colell bargaining set is nonempty for any k-fold replication with k sufficiently large, whereas the Aumann-Davis-Maschler bargaining set may be empty for any k.

2. Preliminaries. Let $N = \{1, \ldots, n\}$, $n \geq 2$, be a set of voters, also called players, and let $A = \{a_1, \ldots, a_m\}$, $m \geq 2$, be a set of m alternatives. For $S \subseteq N$ we denote by $\mathbb{R}^S$ the set of all real functions on S. Therefore, $\mathbb{R}^S$ is the $|S|$-dimensional Euclidean space. (Here and in the sequel, if D is a finite set, then $|D|$ denotes the cardinality of D.) If $x, y \in \mathbb{R}^S$, then we write $x \geq y$ if $x^i \geq y^i$ for all $i \in S$. Moreover, we write $x > y$ if $x \geq y$ and $x \neq y$, and we write $x \gg y$ if $x^i > y^i$ for all $i \in S$. Denote $\mathbb{R}^S_+ = \{x \in \mathbb{R}^S \mid x \geq 0\}$. A set $C \subseteq \mathbb{R}^S$ is comprehensive if $x \in C$, $y \in \mathbb{R}^S$, and $y \leq x$ implies that $y \in C$. An *NTU game* with the player set N is a pair (N, V) where V is a function that associates with every *coalition* S (that is, $S \subseteq N$ and $S \neq \varnothing$) a set $V(S) \subseteq \mathbb{R}^S$, $V(S) \neq \varnothing$, such that

(i) $V(S)$ is closed and comprehensive;

(ii) $V(S) \cap (x + \mathbb{R}^S_+)$ is bounded for every $x \in \mathbb{R}^S$.

We shall now assume that each $i \in N$ has a linear order R^i on A. Thus, for every $i \in N$, R^i is a complete, transitive, and antisymmetric binary relation on A. Moreover, let u^i, $i \in N$, be a utility function that represents R^i. We shall always assume that

$$\min_{\alpha \in A} u^i(\alpha) = 0 \quad \text{for all } i \in N. \tag{1}$$

Let $u^N = (u^i)_{i \in N}$ be a utility profile that satisfies (1). We consider the strategic game in which every player votes for some alternative in A. If a strict majority of voters agrees on $\alpha \in A$, then the outcome is α, and every voter i gets utility $u^i(\alpha)$. Otherwise, if no majority forms, a deadlock results and every voter gets utility zero. These rules associate with every strategy profile $(\alpha^i)_{i \in N} \in A^N$ and every player $k \in N$ a utility (or payoff) that we denote by $v^k((\alpha^i)_{i \in N})$. We shall now recall the notion of $\boldsymbol{\alpha}$-effectiveness (see Aumann and Peleg [6] and Aumann [3]). Every coalition S, $\varnothing \neq S \subseteq N$, may adopt any strategy profile $(\alpha^i)_{i \in S} \in A^S$ by a binding agreement and, hence, S may guarantee $x \in \mathbb{R}^S$, that is, S is $\boldsymbol{\alpha}$-effective[5] for x, if there exists a $(\alpha^i)_{i \in S} \in A^S$ such that, for every strategy profile $(\beta^j)_{j \in N \setminus S} \in A^{N \setminus S}$, $x^k \leq v^k((\alpha^i)_{i \in S}, (\beta^j)_{j \in N \setminus S})$ for each $k \in S$. Let $V_{u^N}(S) = \{x \in \mathbb{R}^S \mid S$ is $\boldsymbol{\alpha}$-effective for $x\}$. The NTU game (N, V_{u^N}) associated with choice by simple majority voting is called the *simple majority voting game* (see Aumann [4]). Let S be a coalition and $i \in S$. If $|S| \leq n/2$ and if all members of $N \setminus S$ select i's worst alternative, then S cannot guarantee any positive payoff to player i, because it is not possible to reach a majority for any but i's worst alternative. If $|S| > n/2$ and if each member of S selects the same alternative $\alpha \in A$, then S guarantees $u^i(\alpha)$ to any $i \in S$. Thus,

$$V_{u^N}(S) = \{x \in \mathbb{R}^S \mid x \leq 0\} \quad \text{if } S \subseteq N, \quad 1 \leq |S| \leq \frac{n}{2}; \tag{2}$$

$$V_{u^N}(S) = \{x \in \mathbb{R}^S \mid \exists \alpha \in A \text{ such that } x \leq u^S(\alpha)\} \quad \text{if } S \subseteq N, \quad |S| > \frac{n}{2}, \tag{3}$$

where $u^S(\alpha) = (u^i(\alpha))_{i \in S}$. It should be noted that there also exists the notion of $\boldsymbol{\beta}$-effectiveness: A coalition S is $\boldsymbol{\beta}$-effective for $x \in \mathbb{R}^S$ if, for each strategy profile of $N \setminus S$, there exists a strategy profile of S such that k's payoff is at least x^k for all $k \in S$. A careful inspection of the foregoing reasoning shows that $\boldsymbol{\beta}$-effectiveness again leads to the same NTU game (N, V_{u^N}) in our case.

[5] The symbol $\boldsymbol{\alpha}$ does not refer to an element $\alpha \in A$.

NOTATION 2.1. In the sequel, let $L = L(A)$ denote the set of linear orders on A. For $R \in L$ and for $k \in \{1, \ldots, m\}$, let $t_k(R)$ denote the kth alternative in the order R. If $R^N \in L^N$ and $\alpha, \beta \in A$, $\alpha \neq \beta$, then α *dominates* β (abbreviated $\alpha \succ_{R^N} \beta$) if $|\{i \in N \mid \alpha\ R^i\ \beta\}| > n/2$. We shall say that an alternative $\alpha \in A$ is a *weak Condorcet winner* (with respect to R^N) if $\beta \not\succ_{R^N} \alpha$ for all $\beta \in A$. Also, if $R^N \in L^N$, then denote

$$\mathcal{U}^{R^N} = \{(u^i)_{i \in N} \mid u^i \text{ is a representation of } R^i \text{ satisfying (1) } \forall i \in N\}.$$

Let (N, V) be an NTU game. The pair (N, V) is *zero normalized* if $V(\{i\}) = -\mathbb{R}_+^{\{i\}}$ for all $i \in N$. Also, (N, V) is *superadditive* if for every pair of disjoint coalitions S, T, $V(S) \times V(T) \subseteq V(S \cup T)$. It should be remarked that the NTU games defined by (2) and (3) are zero normalized and superadditive.

Now we shall recall the definitions of two bargaining sets introduced by Davis and Maschler [8] and by Mas-Colell [15], following the general approach delineated by Aumann and Maschler [5]. Let (N, V) be a zero-normalized NTU game and $x \in \mathbb{R}^N$. We say that x is

- *individually rational* if $x \geq 0$;
- *Pareto optimal* (in $V(N)$) if $x \in V(N)$ and if $y \in V(N)$ and $y \geq x$ imply $x = y$;
- *weakly Pareto optimal* (in $V(N)$) if $x \in V(N)$ and if for every $y \in V(N)$ there exists $i \in N$ such that $x^i \geq y^i$;
- a *preimputation* if x is weakly Pareto optimal in $V(N)$;
- an *imputation* if x is an individually rational preimputation.

We also use the natural analogue of the Pareto optimality notion with respect to $V(S)$, where $\emptyset \neq S \subseteq N$.

A pair (P, y) is an *objection* at x if $\emptyset \neq P \subseteq N$, y is Pareto optimal in $V(P)$, and $y > x^P$. An objection (P, y) is *strong* if $y \gg x^P$. The pair (Q, z) is a *weak counterobjection* to the objection (P, y) if $Q \subseteq N$, $Q \neq \emptyset, P$, if $z \in V(Q)$, and if $z \geq (y^{P \cap Q}, x^{Q \setminus P})$. A weak counterobjection (Q, z) is a *counterobjection* to the objection (P, y) if $z > (y^{P \cap Q}, x^{Q \setminus P})$. A strong objection (P, y) is *justified in the sense of the bargaining set* if there exist players $k \in P$ and $l \in N \setminus P$ such that there does not exist any weak counterobjection (Q, z) to (P, y) satisfying $l \in Q$ and $k \notin Q$. The *bargaining set* of (N, V), $\mathcal{M}(N, V)$, is the set of all imputations x that do not have strong justified objections at x in the sense of the bargaining set (see Davis and Maschler [8]). An objection (P, y) is *justified in the sense of the Mas-Colell bargaining set* if there does not exist any counterobjection to (P, y). The *Mas-Colell bargaining set* of (N, V), $\mathcal{MB}(N, V)$, is the set of all imputations x that do not have a justified objection at x in the sense of the Mas-Colell bargaining set (see Mas-Colell [15]).

REMARK 2.1. We recall that $x \in \mathbb{R}^N$ is in the *core* of the NTU game (N, V), $\mathcal{C}(N, V)$, if $x \in V(N)$ and for any coalition S and any $y \in V(S)$ there exists $i \in S$ such that $x^i \geq y^i$.

(i) Note that $\mathcal{C}(N, V) \subseteq \mathcal{M}(N, V)$. Indeed, $x \in \mathcal{C}(N, V)$ is weakly Pareto optimal, individually rational, and it has no strong objection.

(ii) The Mas-Colell bargaining set need not contain the core (see the following example). Still, the Mas-Colell bargaining set of a simple majority voting game must contain a core element if the core is nonempty. To see this, let $R^N \in L(A)^N$, $u^N \in \mathcal{U}^{R^N}$, and $V = V_{u^N}$. Note that $\mathcal{C}(N, V) \neq \emptyset$ if and only if A has a weak Condorcet winner with respect to R^N, and in this case also $\mathcal{C}(N, V) \cap \mathcal{MB}(N, V) \neq \emptyset$. Indeed, if α is a weak Condorcet winner, then $u^N(\alpha) \in \mathcal{C}(N, V) \cap \mathcal{MB}(N, V)$, because $u^N(\alpha)$ has no objection in this case. Also, if A has no weak Condorcet winner and $x \in \mathbb{R}_+^N$ satisfies $x \leq u^N(\beta)$ for some $\beta \in A$, then there exists $\gamma \in A$, $\gamma \succ_{R^N} \beta$. Let $S = \{i \in N \mid \gamma R^i \beta\}$ and observe that $|S| > n/2$ implies that $y = u^S(\gamma) \in V(S)$. Because $y^i > x^i$ for all $i \in S$, it follows that $x \notin \mathcal{C}(N, V)$.

(iii) We conclude that if a simple majority voting game has a nonempty core, then its Aumann-Davis-Maschler bargaining set and its Mas-Colell bargaining set are nonempty as well.

EXAMPLE 2.1. Let $n = 4$ and let R^N be given by Table 1.

Then $x = (\min\{u^i(b), u^i(a)\})_{i \in N} \in \mathcal{C}(N, V)$ because there is no strong objection at x. However, $x \notin \mathcal{MB}(N, V)$ because $(N, u^N(a))$ is a justified objection in the sense of the Mas-Colell bargaining set at x.

REMARK 2.2. The original definition of Mas-Colell considered preimputations, not just imputations. In restricting our attention to imputations, we follow Vohra [21]. In any case, all our results about existence and nonexistence are valid for both variants of the definition.

TABLE 1. Preference profile of a four-person voting problem.

R^1	R^2	R^3	R^4
a	a	c	c
b	b	b	b
c	c	a	a

TABLE 2. Preference profile of the 3×3 voting paradox.

R^1	R^2	R^3
a	c	b
b	a	c
c	b	a

REMARK 2.3. For a given $R^N \in L^N$, the particular choice of a representation $u^N \in \mathcal{U}^{R^N}$ is essentially immaterial: Different representations lead to NTU games that are derived from each other by ordinal transformations, and so are their bargaining sets.

3. The 3 × 3 voting paradox. In this section we shall compute the bargaining sets of the voting paradox of three voters and three alternatives and interpret the results.

Let $A = \{a, b, c\}$, let $n = 3$, and let $R^N \in L^N$ be given by Table 2.

For $i \in N$, let u^i be a utility representation of R^i satisfying (1) and let $V = V_{u^N}$ (see (2) and (3)).

We claim that $\mathcal{M}(N, V) = \{0\}$. Indeed, it is straightforward to verify that $0 \in \mathcal{M}(N, V)$. To show the opposite inclusion, let $x \in \mathcal{M}(N, V)$. Then there exists $\alpha \in A$ such that $x \le u^N(\alpha)$. Without loss of generality, we may assume that $\alpha = a$. Assume, on the contrary, that $x > 0$. If $x^1 > 0$, then $(\{2, 3\}, u^{\{2,3\}}(c))$ is a justified objection of 3 against 1 at x in the sense of the bargaining set. If $x^1 = 0$ and, hence, $x^2 > 0$, then $(\{1, 3\}, u^{\{1,3\}}(b))$ is a justified objection of 1 against 2.

To compute the Mas-Colell bargaining set, we define $x = (u^1(b), u^2(a), 0)$ and claim that $x \in \mathcal{MB}(N, V)$. Indeed, let (P, y) be an objection at x. Then $|P| \ge 2$. Because y is Pareto optimal in $V(P)$, $y \in \{u^P(\alpha) \mid \alpha \in A\}$. If $y = u^P(a)$, then (P, y) is countered by $(\{2, 3\}, u^{\{2,3\}}(c))$. If $y = u^P(b)$, then $y > x^P$ implies that $P = \{1, 3\}$. In this case (P, y) is countered by $(\{1, 2\}, u^{\{1,2\}}(a))$. Finally, if $y = u^P(c)$, then $y > x^P$ implies that $P = \{2, 3\}$ and that (P, y) is countered by $(\{1, 3\}, u^{\{1,3\}}(b))$.

In order to show that every $\hat{x} \in \mathbb{R}^N$ satisfying $0 \le \hat{x} \le x$ is an element of $\mathcal{MB}(N, V)$, it should be noted that each objection at $\hat{x}$ is also an objection at x if $\hat{x}^1 > 0$ and $\hat{x}^2 > 0$. If $\hat{x}^1 = 0$ and $\hat{x}^2 > 0$, then the additional objections are of the form $(P, u^P(c))$ for some $P \subseteq N$ and these objections can be countered by $(\{1, 3\}, u^{\{1,3\}}(b))$. Similarly, if $\hat{x}^1 > 0$ and $\hat{x}^2 = 0$, then the additional objections can be countered by $(\{1, 2\}, u^{\{1,2\}}(a))$. Finally, if $\hat{x} = 0$, then each additional objection can be countered by one of the foregoing pairs $(\{1, 3\}, u^{\{1,3\}}(b))$ or $(\{1, 2\}, u^{\{1,2\}}(a))$.

Similarly, for $y = (u^1(b), 0, u^3(c))$ and $z = (0, u^2(a), u^3(c))$ we have that every $\hat{y} \in \mathbb{R}^N$ satisfying $0 \le \hat{y} \le y$ and every $\hat{z} \in \mathbb{R}^N$ satisfying $0 \le \hat{z} \le z$ is in $\mathcal{MB}(N, V)$.

We shall now show that there are no other elements in $\mathcal{MB}(N, V)$. Indeed, any remaining individually rational $\tilde{x} \in V(N)$ must have a coordinate that is higher than the utility of that voter's second-best alternative. Say, without loss of generality, that $\tilde{x}^1 > u^1(b)$. Then $(\{2, 3\}, u^{\{2,3\}}(c))$ is a justified objection in the sense of the Mas-Colell bargaining set at $\tilde{x}$. We conclude that $\mathcal{MB}(N, V)$ is the intersection of $\mathbb{R}^N_+$ and the comprehensive hull of $\{x, y, z\}$.

Discussion: The singleton $\mathcal{M}(N, V)$ tells us that in order to achieve (coalitional) stability the players have to give up any profit above their individually protected levels of utility. There is no hint how an alternative of A will be chosen. The message of $\mathcal{MB}(N, V)$ is much more detailed. For example, the element $x = (u^1(b), u^2(a), 0)$ tells us that the alternative a may be chosen provided player 1 disposes of $u^1(a) - u^1(b)$ utiles. Thus, we also see here that lower utility levels guarantee stability. Actually, x implies that there is an agreement between players 1 and 2, the alternative a is chosen as a result of the agreement, and the utility of 1 is reduced (because of the agreement) from $u^1(a)$ to $u^1(b)$. Note that cooperative game theory does not specify the details of agreements that support stable payoff vectors.

In this example (and indeed in many other examples), the Mas-Colell bargaining set is much larger than the Aumann-Davis-Maschler one. However, it is interesting to note that $\mathcal{MB}(N, V)$ need not contain $\mathcal{M}(N, V)$ in general, as can be concluded from Remark 2.1 and Example 2.1. Nevertheless, it can be shown that when the number of alternatives is three and there is no weak Condorcet winner, then in the associated NTU game (N, V) we have $\mathcal{M}(N, V) \subseteq \mathcal{MB}(N, V)$.[6]

[6] We refer the reader to the discussion paper mentioned in Footnote 4, where this fact is derived from a detailed (though incomplete) description of the bargaining sets of simple majority voting games in the case of three alternatives.

TABLE 3. Preference profile of a 4-alternative voting problem.

R^1	R^2	R^3
a	c	b
b	a	c
d	d	d
c	b	a

4. The bargaining set. Throughout this section, let $R^N \in L(A)^N$, $\succ\, = \,\succ_{R^N}$, $u^N \in \mathscr{U}^{R^N}$ (see Notation 2.1), $V = V_{u^N}$ (see (2) and (3)).

THEOREM 4.1. *If* $|A| \leq 3$, *then* $\mathscr{M}(N, V) \neq \varnothing$.

PROOF. If there exists a weak Condorcet winner $\alpha \in A$, then $u^N(\alpha) \in \mathscr{M}(N, V)$. Therefore, we may assume that $|A| = 3$ and for every $\alpha \in A$ there exists $\beta \in A$ such that $\beta \succ \alpha$. We claim that for any $\alpha \in A$ there exists $i \in N$ such that $t_3(R^i) = \alpha$. Indeed, if $\alpha \in \{t_1(R^i), t_2(R^i)\}$ for all $i \in N$ and if $\beta \succ \alpha$, then $|\{i \in N \mid \beta = t_1(R^i)\}| > n/2$ and β is a Condorcet winner that was excluded. We conclude that $0 \in \mathbb{R}^N$ is weakly Pareto optimal. As any objection (P, y) at 0 has the weak counterobjection $(\{l\}, 0)$ for any $l \in N \setminus P$, we conclude that $0 \in \mathscr{M}(N, V)$. □

EXAMPLE 4.1. Let $A = \{a, b, c, d\}$, let $n = 3$, and let R^N be given by Table 3.

We claim that $\mathscr{M}(N, V) = \varnothing$. Let x be an imputation of (N, V). In order to show that $x \notin \mathscr{M}(N, V)$, we may assume without loss of generality that $x^1 \geq u^1(d)$. We distinguish the following possibilities:

(i) $x \leq u^N(a)$ or $x \leq u^N(d)$. Then $(\{2, 3\}, u^{\{2,3\}}(c))$ is a justified objection (in the sense of the bargaining set) of 3 against 1.

(ii) $x \leq u^N(b)$. If $x^3 < u^3(c)$, then we may use the same justified strong objection as in the first possibility. If $x^3 \geq u^3(c)$, then $(\{1, 2\}, u^{\{1,2\}}(a))$ is a justified objection of 2 against 3.

Example 4.1 shows the tension between (weak) Pareto optimality and stability may result in an empty bargaining set.

Example 4.1 may be generalized to any number $m \geq 4$ of alternatives. Indeed, let $A = \{a, b, c, d_1, \ldots, d_k\}$, where $k = m - 3$, and define R^N by

$$R^1 = (a, b, d_1, \ldots, d_k, c),$$
$$R^2 = (c, a, d_1, \ldots, d_k, b),$$
$$R^3 = (b, c, d_1, \ldots, d_k, a),$$

and note that $\mathscr{M}(N, V) = \varnothing$. More interestingly, Example 4.1 can be generalized to yield an empty bargaining set for simple majority voting games on four alternatives with infinitely many numbers of voters.

EXAMPLE 4.2 (EXAMPLE 4.1 GENERALIZED). Let

$$R_1 = (a, b, d, c), \qquad R_2 = (a, c, d, b), \qquad R_3 = (b, a, d, c),$$
$$R_4 = (b, c, d, a), \qquad R_5 = (c, a, d, b), \qquad R_6 = (c, b, d, a),$$

and let $k \in \mathbb{N}$. Let $N = \{1, \ldots, 6k - 3\}$ and let $R^N \in L^N$ satisfy

$$|\{j \in N \mid R^j = R_i\}| = \begin{cases} k, & \text{if } i = 1, 4, 5, \\ k - 1, & \text{if } i = 2, 3, 6. \end{cases}$$

Then $\mathscr{M}(N, V) = \varnothing$. Indeed, $k = 1$ coincides with Example 4.1. The reader may check, e.g., the case $k = 2$ (see Table 4) by repeating the arguments of Example 4.1.

TABLE 4. Preference profile for $k = 2$.

R^1	R^2	R^3	R^4	R^5	R^6	R^7	R^8	R^9
a	a	b	b	c	c	a	c	b
b	c	a	c	a	b	b	a	c
d	d	d	d	d	d	d	d	d
c	b	c	a	b	a	c	b	a

5. The Mas-Colell bargaining set. We shall show that $\mathcal{MB}$ is nonempty for any simple majority voting game on less than six alternatives. Also, we shall show that there is a simple majority voting game on six alternatives whose Mas-Colell bargaining set is empty. We shall always assume that $R^N \in L(A)^N$, $\succ = \succ_{R^N}$, $u^N \in \mathcal{U}^{R^N}$, and $V = V_{u^N}$. We start with the following simple lemma.

Lemma 5.1. *Assume that there is no weak Condorcet winner. If $x \in \mathbb{R}^N_+$ satisfies $x^i \le u^i(t_{m-1}(R^i))$ for all $i \in N$ and if x is weakly Pareto optimal in $V(N)$, then $x \in \mathcal{MB}(N, V)$.*

Proof. If (S, y) is an objection at x, then $|S| > n/2$ and there exists $\alpha \in A$ such that $u^S(\alpha) = y$. Choose $\beta \in A$ such that $\beta \succ \alpha$. Then there exists $T \subseteq N$, $|T| > n/2$ such that $u^T(\beta) \gg u^T(\alpha)$. Thus, $(T, u^T(\beta))$ is a counterobjection. □

Theorem 5.1. *If $|A| \le 5$, then $\mathcal{MB}(N, V) \ne \varnothing$.*

Proof. If $|A| \le 3$, the proof that we gave for $\mathcal{M}$ (Theorem 4.1) also works for $\mathcal{MB}$. In order to prove the theorem for $m = 4$, we may assume that there is no weak Condorcet winner. Then, for each $\alpha \in A$,

$$\text{there exists } i \in N \text{ such that } \alpha \in \{t_3(R^i), t_4(R^i)\}. \tag{4}$$

Indeed, if for some $\alpha \in A$, $\alpha \in \{t_1(R^i), t_2(R^i)\}$ for all $i \in N$, then $\beta \succ \alpha$ implies that β is a Condorcet winner that was excluded. For $\alpha \in A$, define $x_\alpha = (\min\{u^i(\alpha), u^i(t_3(R^i))\})_{i \in N}$. By Lemma 5.1, $x_\alpha \in \mathcal{MB}(N, V)$, if x_α is weakly Pareto optimal. Hence, in order to complete the proof for $m = 4$, it suffices to show that there exists $\alpha \in A$ such that x_α is weakly Pareto optimal. Two possibilities may occur: If there exists $\alpha \in A$ such that $\alpha \ne t_4(R^i)$ for all $i \in N$, then by (4), x_α is weakly Pareto optimal. Otherwise, any x_α is weakly Pareto optimal.

Now, let $m = 5$, let $A = \{a_1, \ldots, a_5\}$, and assume that $\mathcal{MB}(N, V) = \varnothing$. Then, for each $\alpha \in A$

(i) there exists $\beta \in A$ such that $\beta \succ \alpha$;

(ii) $u^N(\alpha)$ is Pareto optimal (because $\mathcal{MB}$ is nonempty when we restrict our attention to the game corresponding to the restriction of u^N to $A \backslash \{\alpha\}$).

For $\alpha \in A$, denote $l(\alpha) = \max\{k \in \{1, \ldots, 5\} \mid \exists i \in N\colon t_k(R^i) = \alpha\}$. Let $l_{\min} = \min_{\alpha \in A} l(\alpha)$. We distinguish cases:

(1) $l_{\min} \ge 4$: Then there exists a weakly Pareto optimal $x \in V(N)$ such that $x^i \le u^i(t_4(R^i))$ for all $i \in N$, which is impossible by Lemma 5.1.

(2) $l_{\min} \le 2$: Let $\alpha, \beta \in A$ such that $l(\alpha) = l_{\min}$ and $\beta \succ \alpha$. Then β is a Condorcet winner, which is impossible by (i).

(3) $l_{\min} = 3$: Let $B = \{\beta \in A \mid l(\beta) = 3\}$. If $|B| = 3$, then any $\alpha \in A \backslash B$ violates (ii). If $|B| = 2$, let us say $B = \{\alpha, \beta\}$, then we may assume without loss of generality that $\alpha \not\succ \beta$. Let $\gamma \in A$ such that $\gamma \succ \beta$. Then, none of the remaining $\delta \in A \backslash (\{\gamma\} \cup B)$ dominates any of the elements α, β, γ. By (i) we conclude that $\gamma \succ \beta \succ \alpha \succ \gamma$. Then $(\min\{u^i(\alpha), u^i(\beta)\})_{i \in N} \in \mathcal{MB}(N, V)$.

Now we turn to the case $|B| = 1$; let us say $B = \{a_3\}$. Let $\hat{S} = \{i \in N \mid t_3(R^i) = a_3\}$. For any $k \in \hat{S}$ there exists $x_k \in \mathbb{R}^N$ such that x_k is weakly Pareto optimal, $x_k^k = u^k(a_3)$, and $x_k^i \le u^i(t_4(R^i))$ for all $i \in N \backslash \{k\}$. As $x_k \notin \mathcal{MB}(N, V)$, there exists a justified objection $(S, u^S(\alpha))$ for some $S \subseteq N$, $|S| > n/2$, and some $\alpha \in A$. Let $\beta \in A$ such that $\beta \succ \alpha$. Then there exists $T \subseteq N$, $|T| > n/2$, such that $u^{S \cap T}(\beta) \gg u^{S \cap T}(\alpha)$ and $u^{T \backslash S}(\beta) \ge (u^i(t_4(R^i)))_{i \in T \backslash S}$. Because $(T, u^T(\beta))$ is not a counterobjection, we conclude that $k \in T$, $t_4(R^k) = \beta$, and $t_5(R^k) = \alpha$. We conclude that for any $k \in \hat{S}$ the alternative $t_5(R^k)$ is only dominated by $t_4(R^k)$. If n is odd, we may now easily finish the proof by the observation that α dominates all other alternatives except β, and therefore $(\min\{u^i(\alpha), u^i(\beta)\})_{i \in N} \in \mathcal{MB}(N, V)$. Hence, we may assume from now on that n is an even number. As $a_3 \not\succ \alpha$, $\{i \in N \mid u^i(\alpha) > u^i(a_3)\} \cap \{i \in N \mid u^i(\beta) > u^i(\alpha)\} \ne \varnothing$. Thus, there exists $j \in \hat{S}$ such that $t_1(R^j) = \beta$ and $t_2(R^j) = \alpha$. So far, we have for any $k \in \hat{S}$, where $\alpha = t_5(R^k)$, $\beta = t_4(R^k)$:

$$\alpha \text{ is only dominated by } \beta; \tag{5}$$

$$\text{there exists } j \in \hat{S} \text{ such that } t_1(R^j) = \beta, \quad t_2(R^j) = \alpha; \tag{6}$$

$$|\{i \in N \mid u^i(\alpha) > u^i(a_3)\}| \ge \frac{n}{2}. \tag{7}$$

Now, let $k, j \in \hat{S}$ have the foregoing properties—let us say $k = 1$ and $j = 2$. We also may assume that $t_4(R^1) = a_4$, $t_5(R^1) = a_5$, $t_4(R^2) = a_1$, $t_5(R^2) = a_2$ (hence, $R^2 = (a_4, a_5, a_3, a_1, a_2)$). Therefore, for any $k \in \hat{S}$, we have

$$\{t_4(R^k), t_5(R^k)\} = \{a_4, a_5\} \;\Rightarrow\; t_4(R^k) = a_4 \tag{8}$$

$$t_5(R^k) = a_5 \;\Rightarrow\; t_4(R^k) = a_4 \tag{9}$$

$$\{t_4(R^k), t_5(R^k)\} = \{a_1, a_2\} \;\Rightarrow\; t_4(R^k) = a_1 \tag{10}$$

$$t_5(R^k) = a_2 \;\Rightarrow\; t_4(R^k) = a_1. \tag{11}$$

We are now going to show that there exists $k \in \hat{S}$ such that $t_5(R^k) \notin \{a_5, a_2\}$. Assume the contrary. Then $\{i \in N \mid u^i(a_5) > u^i(a_3)\} \cap \{i \in N \mid u^i(a_2) > u^i(a_3)\} = \varnothing$ and, by (7), $a_5 \not\succ a_3$ and $a_2 \not\succ a_3$. Hence, by (i), $a_1 \succ a_3$ or $a_4 \succ a_3$. However, note that by our assumption $u^i(a_1) > u^i(a_3)$ implies $u^i(a_1) > u^i(a_5)$ for all $i \in N$. Thus, if $a_1 \succ a_3$, then $a_1 \succ a_5$, which contradicts (5). Similarly, $a_4 \succ a_3$ can be excluded.

Hence, we may assume without loss of generality that there exists $k \in \hat{S}$ such that $t_5(R^k) = a_1$. We now claim that there exists $j \in \hat{S}$ such that $t_5(R^j) = a_4$. By (5) and the fact that $a_1 \succ a_2$, $t_4(R^k) \in \{a_4, a_5\}$. If $t_4(R^k) = a_4$, then by (6) there exists $j \in \hat{S}$ such that $\{t_4(R^j), t_5(R^j)\} = \{a_2, a_5\}$. By (9), $a_5 \neq t_5(R^j)$, and by (11), $a_2 \neq t_5(R^j)$. Hence, this possibility is ruled out. We conclude that $t_4(R^k) = a_5$. By (6) there exists $j \in \hat{S}$ such that $\{t_4(R^j), t_5(R^j)\} = \{a_2, a_4\}$. By (11), $t_5(R^j) = a_4$. Therefore, our claim has been shown.

So far we have deduced that there exist $k_j \in \hat{S}$, $j = 1, 2, 4, 5$, such that $t_5(R^{k_j}) = a_j$. By (7), $|\{i \in N \mid u^i(a_j) > u^i(a_3)\}| \geq n/2$ for all $j = 1, 2, 4, 5$. We conclude that $a_3 = t_3(R^i)$ for all $i \in N$ and $|\{i \in N \mid u^i(a_j) > u^i(a_3)\}| = n/2$ for all $j = 1, 2, 4, 5$. Therefore, a_3 is not dominated by any alternative, which contradicts (i). $\square$

We shall now present an example of a simple majority voting game on six alternatives, whose Mas-Colell bargaining set is empty.

Example 5.1. Let $n = 4$, $A = \{a_1, \ldots, a_4, b, c\}$, and let $R^N \in L^N$ be given by Table 5.

We claim that $\mathcal{MB}(N, V) = \varnothing$. Note that the proof below is similar to the proof of the emptiness of an extension of the Mas-Colell bargaining set of a game derived from a four-person voting problem on 10 alternatives (see Peleg and Sudhölter [17, §3]).

Proof of the Claim. Assume that there exists $x \in \mathcal{MB}(N, V)$. Let $\alpha \in A$ such that $x \leq u^N(\alpha)$. Let

$$S_1 = \{1, 2, 3\}, \qquad S_2 = \{1, 2, 4\}, \qquad S_3 = \{1, 3, 4\}, \qquad S_4 = \{2, 3, 4\}.$$

We distinguish the following possibilities:

(i) $x \leq u^N(a_1)$. In this case, $(S_4, u^{S_4}(a_4))$ is an objection at x. Because there must be a counterobjection to this, we conclude that $(S_3, u^{S_3}(a_3))$ is a counterobjection, and therefore also an objection at x. Hence, $x^1 \leq u^1(a_3)$. To this objection, too, there must be a counterobjection. We conclude that $(S_2, u^{S_2}(a_2))$ is a counterobjection. Hence, $x^2 \leq u^2(a_2)$, and therefore $x \ll u^N(b)$ and the desired contradiction has been obtained in this case.

(ii) The possibilities $x \leq u^N(\alpha)$ for $\alpha \in \{a_2, a_3, a_4\}$ may be treated similarly.

(iii) $x \leq u^N(b)$. Then, $(S_1, u^{S_1}(c))$ is an objection at x. There are several possibilities for a counterobjection to this. Each of them involves player 4 and one of the alternatives a_1, a_4, or c. We conclude that, in any case, $x^4 \leq u^4(a_4)$. Hence, $(S_4, u^{S_4}(a_4))$ is an objection at x. Now we conclude that $(S_3, u^{S_3}(a_3))$ must be a counterobjection and, hence, an objection at x. We continue by concluding that $(S_2, u^{S_2}(a_2))$ must be an objection and that, hence, $(S_1, u^{S_1}(a_1))$ is a counterobjection. Therefore, $x \ll u^N(b)$ and the desired contradiction has been obtained.

(iv) $x \leq u^N(c)$. We consecutively deduce that $(S_4, u^{S_4}(a_4)), \ldots, (S_1, u^{S_1}(a_1))$ are objections. The desired contradiction is again obtained by the observation that $x \ll u^N(b)$. $\square$

Example 5.1 may be generalized to any number $m \geq 6$ of alternatives. Also, it may be generalized to any even number $n \geq 4$ of voters: If $R_i = R^i$ for $i = 1, \ldots, 4$, if

$$R_5 = (a_2, a_1, c, b, a_3, a_4), \qquad R_6 = (a_4, a_3, c, b, a_1, a_2),$$

Table 5. Preference profile leading to an empty $\mathcal{MB}$.

R^1	R^2	R^3	R^4
a_1	a_4	a_3	a_2
a_2	a_1	a_4	a_3
c	c	c	b
b	b	b	a_4
a_3	a_2	a_1	c
a_4	a_3	a_2	a_1

if $n = 4 + 2k$ for some $k \in \mathbb{N}$, if $\tilde{R}^N \in L^N$ such that

$$|\{j \in N \mid \tilde{R}^j = R_i\}| = \begin{cases} k, & \text{if } i = 5, 6, \\ 1, & \text{if } i = 1, 2, 3, 4, \end{cases}$$

and if $\tilde{V} = V_{u^N}$ for some $u^N \in \mathcal{U}^{\tilde{R}^N}$, then $\mathcal{MB}(N, \tilde{V}) = \varnothing$.

6. The Mas-Colell bargaining set for an odd number of voters. The examples that we just gave for emptiness of the Mas-Colell bargaining set have an even number of voters. The most natural setting for simple majority rule is when the number of voters is odd. It is therefore desirable to study the existence question for $\mathcal{MB}$ in the class of simple majority voting games with an odd number of voters. Attempts to construct small explicit counterexamples, similar to those above, seem to fail. We take a different approach that leads to the construction of a profile of preferences with an odd number of voters, whose associated simple majority voting game has an empty Mas-Colell bargaining set.

Throughout this section we shall always assume that A is a finite set of $m \geq 2$ alternatives and that $N = \{1, \ldots, n\}$ for some odd $n \in \mathbb{N}$. Recall that $T = (A, \succ)$ is a *tournament* on A if $\succ$ is an irreflexive, asymmetric, and complete relation on A (that is, $\alpha, \beta \in A$, $\alpha \neq \beta$ implies that exactly one of $\alpha \succ \beta$, $\beta \succ \alpha$ holds).

As our construction is complex and is done in several steps, we first give an outline. One can associate with every profile of linear orders R^N a tournament $T = (A, \succ)$, where $\succ\; = \;\succ_{R^N}$ is the domination relation. For a profile R^N, we can look at any given choice $\vec{\alpha} = (\alpha_i)_{i \in N}$ of an alternative for each voter (we call this a position) and ask if the corresponding vector of utility levels has no justified objection in the sense of the Mas-Colell bargaining set. The first idea of the construction is to work with profiles R^N that are tight, in the sense that the contest between any two alternatives is decided by a one-vote difference. Under this assumption, the above condition on a position $\vec{\alpha}$ (there being no justified objection) has a relatively simple expression in terms of the profile R^N and the tournament T; we call such $\vec{\alpha}$ nonenhancing.

As a first step we show that tightness does not restrict in any way the tournaments that can be realized. Recall that McGarvey [16] proved that every tournament may be obtained as the domination relation $\succ_{R^N}$ of some profile of preferences R^N. Our Lemma 6.1 strengthens this result by insisting that R^N be tight.[7]

The second idea of the construction is to force the Mas-Colell bargaining set to be empty by inserting above every would-be element of it, i.e., every nonenhancing position $\vec{\alpha}$, a new alternative $\vec{\alpha}^*$ that will render $\vec{\alpha}$ (or more precisely, the corresponding payoff vector) nonweakly Pareto optimal. Of course, there is a danger that by doing this we introduce new candidates for belonging to $\mathcal{MB}$. It turns out that we can avoid this if in the original profile (before the new alternatives are inserted) the following holds true: For every nonenhancing position $\vec{\alpha} = (\alpha_i)_{i \in N}$ and every alternative α, a majority of voters $i \in N$ prefer α to α_i.

Most of the work is devoted to constructing the original profile so as to guarantee this property. First, we choose the tournament. Lemma 6.2 asserts that we can choose it so that every alternative beats exactly half of the other alternatives, and it never happens that all the alternatives that beat a given alternative are in turn beaten by (or equal to) another alternative. Its proof uses a known construction of quadratic residue tournaments.[8] Lemma 6.4 asserts, essentially, that if the tournament is chosen as in Lemma 6.2, and if every linear order appears in the profile approximately $n/m!$ times, then the profile has the desired property. The proof of Lemma 6.4 is based on some calculations of the relative frequency of linear orders that display certain patterns, which we carry out in advance of the lemma. These calculations, and the lemma itself, are conveniently expressed in terms of the uniform probability measure on $L(A)$, but the method of proof is not probabilistic.

Finally, we have to construct the profile so that it will satisfy the premises of Lemma 6.4. It needs to have a prescribed associated tournament, and at the same time be approximately uniform in terms of the number of appearances of each linear order. We achieve this by building the profile in two parts with disjoint sets of voters. The first part, produced by Lemma 6.1, realizes the prescribed tournament. The second part is uniform (and hence does not affect the tournament), and is chosen large enough so that the overall profile is sufficiently close to being uniform.

We proceed with the details.

[7] Incidentally, the smallest number of voters n that is needed for McGarvey's theorem (as a function of m) has been studied in the literature. McGarvey's original proof (which also permitted us to prescribe ties between pairs of alternatives) required $n = m(m-1)$, and subsequent research (see Stearns [20] and Erdős and Moser [10]) has shown that $n = O(m/\log m)$ suffices and is the right order of magnitude. Our proof requires $n = 2m - 3$.

[8] Quadratic residue tournaments have been used in the combinatorial literature as examples of explicitly constructed tournaments that display randomlike properties. See, e.g., Graham and Spencer [12] and Alon and Spencer [1].

Table 6. Sketch of a profile R^N.

R^1	$\cdots$	$R^{(n_0+1)/2}$	$R^{(n_0+3)/2}$	$\cdots$	R^{n_0}	R^{n_0+1}	R^n
α_0	$\cdots$	α_0	$t_1(R_0^{(n_0+3)/2})$	$\cdots$	$t_1(R_0^{n_0})$	$t_1(R_0)$	$t_{m-1}(R_0)$
$t_1(R_0^1)$	$\cdots$	$t_1(R_0^{(n_0+1)/2})$	$t_2(R_0^{(n_0+3)/2})$	$\cdots$	$t_2(R_0^{n_0})$	$t_2(R_0)$	$t_{m-2}(R_0)$
$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
$t_{k_0-1}(R_0^1)$	$\cdots$	$t_{k_0-1}(R_0^{(n_0+1)/2})$	$t_{k_0}(R_0^{(n_0+3)/2})$	$\cdots$	$t_{k_0}(R_0^{n_0})$	$t_{k_0}(R_0)$	$t_{m-k_0}(R_0)$
$t_{k_0}(R_0^1)$	$\cdots$	$t_{k_0}(R_0^{(n_0+1)/2})$	$t_{k_0+1}(R_0^{(n_0+3)/2})$	$\cdots$	$t_{k_0+1}(R_0^{n_0})$	$t_{k_0+1}(R_0)$	α_0
$t_{k_0+1}(R_0^1)$	$\cdots$	$t_{k_0+1}(R_0^{(n_0+1)/2})$	$t_{k_0+2}(R_0^{(n_0+3)/2})$	$\cdots$	$t_{k_0+2}(R_0^{n_0})$	$t_{k_0+2}(R_0)$	$t_{m-k_0-1}(R_0)$
$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
$t_{m-2}(R_0^1)$	$\cdots$	$t_{m-2}(R_0^{(n_0+1)/2})$	$t_{m-1}(R_0^{(n_0+3)/2})$	$\cdots$	$t_{m-1}(R_0^{n_0})$	$t_{m-1}(R_0)$	$t_2(R_0)$
$t_{m-1}(R_0^1)$	$\cdots$	$t_{m-1}(R_0^{(n_0+1)/2})$	α_0	$\cdots$	α_0	α_0	$t_1(R_0)$

Lemma 6.1. *For every tournament $T = (A, \succ)$ there exists a finite set N of voters and a preference profile $R^N \in L(A)^N$ such that n is odd and for all $\alpha, \beta \in A$,*

$$\alpha \succ \beta \;\Rightarrow\; |\{i \in N \mid \alpha\ R^i\ \beta\}| = \frac{n+1}{2}. \tag{12}$$

Proof. We argue by induction on $m = |A|$. If $m = 2$, then $\succ$ is a linear order and the statement is true (with $n = 1$ and $R^1 = \succ$). If $m > 2$, then select $\alpha_0 \in A$, define $A_0 = A \backslash \{\alpha_0\}$ and let $\succ_0$ be the restriction of $\succ$ to A_0. By the inductive hypothesis there is a set N_0 with an odd number of elements and $R_0^{N_0} \in L(A_0)^{N_0}$ such that

$$\alpha, \beta \in A_0, \quad \alpha \succ_0 \beta \;\Rightarrow\; |\{i \in N_0 \mid \alpha\ R_0^i\ \beta\}| = \frac{n_0+1}{2}.$$

Let $n = n_0 + 2$, $B = \{\beta \in A_0 \mid \alpha_0 \succ \beta\}$, and let $R_0 \in L(A_0)$ such that, for all $\alpha \in B$ and all $\beta \in A_0 \backslash B$, $\alpha\ R_0\ \beta$. Put $k_0 = |A_0 \backslash B|$. Moreover, let R_0^* be the reverse linear order of R_0. Now, define $R^i \in L(A)$ for all $i \in N$ as follows (see Table 6).

If $i \le (n_0+1)/2$, then let R^i be the linear order that coincides with R_0^i on A_0 and ranks α_0 first, that is, $t_1(R^i) = \alpha_0$ and $t_{k+1}(R^i) = t_k(R_0^i)$ for $k = 1, \ldots, m-1$. If $(n_0+1)/2 < i \le n_0$, then let R^i be the linear order that coincides with R_0^i on A_0 and ranks α_0 last, that is, $t_k(R^i) = t_k(R_0^i)$ for $k = 1, \ldots, m-1$ and $t_m(R^i) = \alpha_0$. Also, let R^{n_0+1} be the order that coincides with R_0 on A_0 and ranks α_0 last, that is, $t_k(R^{n_0+1}) = t_k(R_0)$ for $k = 1, \ldots, m-1$ and $t_m(R^{n_0+1}) = \alpha_0$. Finally, let R^n be the ranking that coincides with R_0^* on A_0 and ranks α_0 between the elements of $A_0 \backslash B$ and the members of B, that is, $t_i(R^n) = t_i(R_0^*)$ for $i = 1, \ldots, k_0$, $t_{k_0+1}(R^n) = \alpha_0$, and $t_{j+1}(R^n) = t_j(R_0^*)$ for $j = k_0+1, \ldots, m-1$. The pair (N, R^N) satisfies the desired properties. □

Notation 6.1. Let $(A, \succ)$ be a tournament and $\beta \in A$. Denote

$$A_\succ^+(\beta) = A^+(\beta) = \{\alpha \in A \mid \beta \succ \alpha\}, \qquad A_\succ^-(\beta) = A^-(\beta) = \{\alpha \in A \mid \alpha \succ \beta\}.$$

Lemma 6.2. *There exist infinitely many positive integers m such that there exists a tournament $T = (A, \succ)$ with $|A| = m$ that satisfies the following properties*:

$$|A^+(\alpha)| = |A^-(\alpha)| = \frac{m-1}{2} \quad \textit{for all } \alpha \in A. \tag{13}$$

$$A^-(\alpha) \ne A^+(\beta) \quad \textit{for all } \alpha,\ \beta \in A. \tag{14}$$

$$\textit{For all } \alpha \in A \textit{ and } \beta \in A^-(\alpha) \textit{ there exists } \gamma \in A^-(\alpha)\backslash\{\beta\} \textit{ such that } \gamma \succ \beta. \tag{15}$$

Proof. The set $Q = \{p \in \mathbb{N} \mid p$ is a prime such that $p \equiv 3 \bmod 4\}$ is infinite. Let $p \in Q$, $p > 3$. Let $\mathbb{Z}_p = \{0, \ldots, p-1\}$ denote the field of residue classes modulo p. Recall that an element $t \in \mathbb{Z}_p \backslash \{0\}$ is called a quadratic residue modulo p if there exists $a \in \mathbb{Z}_p$ such that $a^2 \equiv t \bmod p$ (for the basic properties of quadratic residues that we use below, see, e.g., Hardy and Wright [13, Chapter VI]). Let $A = \mathbb{Z}_p$ and let $\succ$ on A be defined by $\alpha \succ \beta$ iff $\alpha, \beta \in A$ and $\alpha - \beta$ is a quadratic residue modulo p. It suffices to prove that $(A, \succ)$ satisfies the desired properties.

The fact that $(A, \succ)$ is a tournament that satisfies property (13) is an immediate consequence of the following claim.

CLAIM 1. *The set of quadratic residues* mod p *contains exactly one element of every set* $\{t, p-t\}$ *for every* $t \in A\backslash\{0\}$.

Assume the contrary. Because there are $(p-1)/2$ quadratic residues mod p, there exists $t \in A\backslash\{0\}$ such that t and $p-t$ are both quadratic residues. Therefore, there are $a, b \in A$ such that $a^2 \equiv t \bmod p$ and $b^2 \equiv -t \bmod p$. Thus, $a^2 \equiv -b^2 \bmod p$. Let $c \in A$ be the inverse of b, that is, $bc \equiv 1 \bmod p$. Then $(ac)^2 \equiv -1 \bmod p$. We conclude that $(ac)^{p-1} \equiv (-1)^{(p-1)/2} \bmod p$. Because $p \equiv 3 \bmod 4$, $(p-1)/2$ is odd and, hence, $(ac)^{p-1} \equiv -1 \bmod p$. On the other hand, by Fermat's theorem, $(ac)^{p-1} \equiv 1 \bmod p$ and the desired contradiction has been obtained.

The following claim enables us to show that (14) and (15) are satisfied.

CLAIM 2. *The prime p divides the sum of all quadratic residues* mod p.

If s denotes this sum, then because every quadratic residue is the square of two residues modulo p, $2s \equiv \sum_{a\in\mathbb{Z}_p} a^2 \bmod p$. Because $\mathbb{Z}_p$ is a field and $p \neq 2$,

$$4\sum_{a\in\mathbb{Z}_p} a^2 = \sum_{a\in\mathbb{Z}_p} (2a)^2 \equiv \sum_{a\in\mathbb{Z}_p} a^2 \mod p.$$

We conclude that $3s \equiv 0 \bmod p$. As $p > 3$, $s \equiv 0 \bmod p$.

In order to show (14) we assume, on the contrary, that $A^-(\alpha) = A^+(\beta)$. By Claim 2,

$$\sum_{\gamma\in A^-(\alpha)} (\gamma-\alpha) = \sum_{\gamma\in A^-(\alpha)} \gamma - \frac{p-1}{2}\alpha \equiv 0 \mod p$$

and

$$\sum_{\gamma\in A^+(\beta)} (\beta-\gamma) = \frac{p-1}{2}\beta - \sum_{\gamma\in A^+(\beta)} \gamma \equiv 0 \mod p.$$

By the assumption, $((p-1)/2)(\beta-\alpha) \equiv 0 \bmod p$, which is impossible.

In order to show (15) we assume, on the contrary, that there exists $\beta \in A^-(\alpha)$ such that $\beta \succ \gamma$ for all $\gamma \in A^-(\alpha)\backslash\{\beta\}$. Hence, $A^-(\alpha)\backslash\{\beta\} = A^+(\beta)\backslash\{\alpha\}$. Claim 2 yields

$$\sum_{\gamma\in A^-(\alpha)\backslash\{\beta\}} (\gamma-\alpha) = \sum_{\gamma\in A^-(\alpha)\backslash\{\beta\}} \gamma - \frac{p-3}{2}\alpha \equiv (\alpha-\beta) \mod p$$

and

$$\sum_{\gamma\in A^+(\beta)\backslash\{\alpha\}} (\beta-\gamma) = \frac{p-3}{2}\beta - \sum_{\gamma\in A^+(\beta)\backslash\{\alpha\}} \gamma \equiv (\alpha-\beta) \mod p.$$

By the assumption, $((p+1)/2)(\beta-\alpha) \equiv 0 \bmod p$, which is impossible. $\square$

For any set A of m alternatives, let prob_A be the uniform probability measure on $L(A)$, that is, $\text{prob}_A: 2^{L(A)} \to \mathbb{R}$ is defined by $\text{prob}_A(T) = |T|/m!$ for all $T \subseteq L(A)$.

REMARK 6.1. Let $\alpha, \gamma \in A$, $\alpha \neq \gamma$, and let $Z \subseteq A\backslash\{\alpha, \gamma\}$. Then

$$\text{prob}_A(\{R \in L(A) \mid \exists \zeta \in Z \text{ such that } \alpha\ R\ \zeta \text{ and } \gamma\ R\ \zeta\}) = |Z|/(|Z|+2). \tag{16}$$

Indeed, we may assume that $A = Z \cup \{\alpha, \gamma\}$. Let $z = |Z|$. There are $(m-1)!$ elements R of $L(A)$ such that $t_m(R) = \alpha$. A similar statement is valid for γ. We conclude that

$$|\{R \in L(A) \mid t_m(R) \in Z\}| = m! - 2(m-1)! = (m-2)(m-1)! = z(m-1)!$$

and, hence, (16) is true.

LEMMA 6.3. *Let $t \in \mathbb{Z}$ such that $t \geq 0$ and $2t+1 \leq m$. Let $\alpha, \beta_r, \gamma_r \in A$, $r = 1, \ldots, t$, be $2t+1$ distinct elements and define for any $r = 0, \ldots, t$,*

$$c_r = \text{prob}_A(\{R \in L(A) \mid \exists k \in \{1, \ldots, r\} \textit{ such that } \alpha\ R\ \gamma_k\ R\ \beta_k\}).$$

Then $c_0 = 0$, and

$$c_r = \frac{1}{2r+1}\left(\frac{2^r-1}{2^r} + 2rc_{r-1}\right) \quad \textit{for all } r = 1, \ldots, t. \tag{17}$$

Proof. Clearly $c_0 = 0$. Let $r \in \{1, \ldots, t\}$. We may assume that $m = 2r+1$. There are $((2^r-1)/2^r)(m-1)!$ preferences $R \in L(A)$ with the properties that $t_1(R) = \alpha$ and that $\gamma_k \; R \; \beta_k$ for some $k = 1, \ldots, r$. Also, for every $k = 1, \ldots, r$, there are $(m-1)c_{r-1}(m-2)!$ preferences $R \in L(A)$ such that $t_1(R) = \beta_k$ and $\alpha \; R \; \gamma_l \; R \; \beta_l$ for some $l \in \{1, \ldots, r\} \setminus \{k\}$, because the rank of γ_k is any element of $2, \ldots, m$. The same number of preferences occurs if γ_k is ranked first. We conclude that there are

$$d_r = \frac{2^r - 1}{2^r}(m-1)! + 2rc_{r-1}(m-1)!$$

preferences $R \in L(A)$ such that $\alpha \; R \; \gamma_k \; R \; \beta_k$ for some $k = \{1, \ldots, r\}$. Equation (17) follows, because $c_r = d_r/m!$. □

Remark 6.2. Let $c_0 = 0$. Successive computation of $c_1, \ldots, c_6$ via (17) yields that $c_6 > \frac{1}{2}$.

Lemma 6.4. *For any tournament $T = (A, \succ)$ with $m \geq 453$ that satisfies (13)–(15), the following holds true: For every $\alpha \in A$ and every mapping h: $A^-(\alpha) \to A$ such that $h(\beta) \succ \beta$ for all $\beta \in A^-(\alpha)$,*

$$\mathrm{prob}_A(\{R \in L(A) \mid \exists \beta \in A^-(\alpha) \text{ such that } \alpha \; R \; h(\beta) \; R \; \beta\}) > \frac{1}{2}. \tag{18}$$

Proof. Two cases may be distinguished.

Case 1. There exists $\gamma \in A$ such that $|h^{-1}(\gamma)| \geq 23$. By Lemma 6.2 there exists $\beta \in A^-(\alpha)$ such that $\gamma \notin \{\beta, h(\beta)\}$. Let $Z = h^{-1}(\gamma)$. Let

$$L_1 = \{R \in L(A) \mid \exists \zeta \in Z \text{ such that } \alpha \; R \; \gamma \; R \; \zeta\} \qquad \text{and} \qquad L_2 = \{R \in L(A) \mid \gamma \; R \; \alpha \; R \; h(\beta) \; R \; \beta\}.$$

Then $L_1 \cap L_2 = \varnothing$. As $Z \subseteq A^-(\alpha)$ and as $\beta \in A^-(\alpha)$, it suffices to show that $\mathrm{prob}_A(L_1) + \mathrm{prob}_A(L_2) > 1/2$. Now, $\mathrm{prob}_A(L_2) = 1/4!$ and, by Remark 6.1,

$$\mathrm{prob}_A(L_1) = \frac{1}{2}\frac{|Z|}{|Z|+2} = \frac{1}{2} - \frac{1}{|Z|+2} \geq \frac{1}{2} - \frac{1}{25} > \frac{1}{2} - \frac{1}{4!},$$

where $|Z| \geq 23$ implies the weak inequality.

Case 2. For all $\gamma \in A$, $|h^{-1}(\gamma)| \leq 22$. In this case, we may choose pairwise distinct alternatives $\beta_1, \gamma_1, \beta_2, \gamma_2, \ldots, \beta_6, \gamma_6$ so that $\beta_k \in A^-(\alpha)$ and $\gamma_k = h(\beta_k)$ for $k = 1, \ldots, 6$. Such a choice may be achieved inductively, by selecting

$$\beta_k \in A^-(\alpha) \setminus \bigcup_{i=1}^{k-1} [\{\gamma_i\} \cup h^{-1}(\{\beta_i, \gamma_i\})] \tag{19}$$

and letting $\gamma_k = h(\beta_k)$. By the assumption of this case, the set appearing in square brackets in (19) has at most 45 elements, and therefore the union in (19) has at most 225 elements. Because we are assuming that $m \geq 453$, we have $|A^-(\alpha)| = (m-1)/2 \geq 226$, and therefore the choice indicated in (19) is feasible. Now the probability in question is at least

$$\mathrm{prob}_A(\{R \in L(A) \mid \exists k \in \{1, \ldots, 6\} \text{ such that } \alpha \; R \; \gamma_k \; R \; \beta_k\}).$$

The proof is complete by Lemma 6.3 and Remark 6.2. □

Now we are able to construct simple majority voting games with an odd number of players, whose Mas-Colell bargaining sets are empty. Let $T = (A, \succ)$ be a tournament with $m \geq 453$ that satisfies (13)–(15). Lemma 6.2 guarantees the existence of T. Let N_0, n_0 odd, and $Q_0^{N_0} \in L(A)^{N_0}$ be such that (12) is satisfied (for $N = N_0$ and $R^N = Q_0^{N_0}$). Lemma 6.1 guarantees the existence of N_0 and $Q_0^{N_0}$. Let N be obtained from N_0 by adding $k \cdot m!$ new voters and let Q^N be obtained from $Q_0^{N_0}$ by assigning each preference of $L(A)$ to k of the new voters. Note that (12) remains valid for $R^N = Q^N$. Moreover, we assume that k is sufficiently large such that the following condition is satisfied. The empirical distribution of preferences in Q^N is close enough to the uniform distribution that the conclusion of Lemma 6.4 holds true when prob_A is replaced by this empirical distribution—that is, by the probability measure prob on $L(A)$ that is determined by $\mathrm{prob}(\{R\}) = |\{i \in N \mid R = Q^i\}|/n$ for all $R \in L(A)$. Lemma 6.4 guarantees the existence of k.

A vector $\vec{\alpha} = (\alpha_i)_{i \in N}$, $\alpha_i \in A$ for all $i \in N$, is a *position*. Let $\vec{\alpha}$ be a position and $\beta \in A$. We say that $\vec{\alpha}$ *enhances* β (at Q^N) if for every $\gamma \in A$ such that $\gamma \succ \beta$ there exists $i \in N$ such that $\alpha_i \; Q^i \; \gamma \; Q^i \; \beta$ and $\alpha_i \neq \gamma$. Note that the set of positions is partially ordered. Indeed, let $\vec{\alpha}$ and $\vec{\beta}$ be positions. Then define $\vec{\alpha} \geq \vec{\beta}$ iff $\alpha_i \; Q^i \; \beta_i$ for all $i \in N$. Note that if $\vec{\alpha} \geq \vec{\beta}$ and $\vec{\beta}$ enhances an alternative γ, then $\vec{\alpha}$ enhances γ as well.

We call a position $\vec{\alpha}$ *nonenhancing* (at Q^N) if it does not enhance any $\beta \in A$. If, in addition, every position $\vec{\beta} \geq \vec{\alpha}$, $\vec{\beta} \neq \vec{\alpha}$, enhances some $\gamma \in A$, then we call $\vec{\alpha}$ *maximal* nonenhancing (MNE). Note that for any nonenhancing position $\vec{\alpha}$ there exists an MNE position $\vec{\beta}$ such that $\vec{\beta} \geq \vec{\alpha}$.

Lemma 6.5. *If $\vec{\alpha}$ is a nonenhancing position and if $\alpha \in A$, then* $|\{i \in N \mid \alpha_i\, Q^i\, \alpha\}| < n/2$.

Proof. Let $S = \{i \in N \mid \alpha_i\, Q^i\, \alpha\}$ and $\beta \in A^-(\alpha)$. Because $\vec{\alpha}$ does not enhance β, there exists $h(\beta) \in A$ such that $h(\beta) \succ \beta$ and, for all $i \in N$, if $\alpha_i\, Q^i\, h(\beta)\, Q^i\, \beta$, then $\alpha_i = h(\beta)$. As $h(\beta) \neq \alpha$, $\{i \in N \mid \alpha\, Q^i\, h(\beta)\, Q^i\, \beta\} \subseteq N \backslash S$. Therefore, h: $A^-(\alpha) \to A$ is a function as in Lemma 6.4 and $\{i \in N \mid \exists \beta \in A^-(\alpha)$ such that $\alpha\, Q^i\, h(\beta)\, Q^i\, \beta\} \subseteq N \backslash S$. By Lemma 6.4 and construction, $|N \backslash S| > n/2$, and the proof is complete. □

Construction (cont.): Let $A^* = \{\vec{\alpha}^* \mid \vec{\alpha}$ is an MNE position of $Q^N\}$ be a set whose cardinality is the number of MNE positions, of alternatives such that $A \cap A^* = \varnothing$. For every voter $i \in N$, let $R^i \in L(A \cup A^*)$ be a preference that arises from Q^i by inserting every alternative in A^* into Q^i in such a way that

$$\vec{\alpha}^*\; R^i\; \alpha \;\Leftrightarrow\; \alpha_i\; Q^i\; \alpha \quad \text{for all } \alpha \in A \text{ and all } \vec{\alpha}^* \in A^*. \tag{20}$$

In other words, the new alternative that corresponds to the position $\vec{\alpha}$ is inserted just above $\vec{\alpha}$. The internal order between new alternatives that are inserted in the same slot is immaterial. Note that, by Lemma 6.5, in the tournament associated with R^N, $\succ_{R^N}$ (see Notation 2.1), every $\alpha \in A$ beats any $\vec{\alpha}^* \in A^*$, i.e.,

$$\alpha \succ_{R^N} \vec{\alpha}^* \quad \text{for all } \alpha \in A,\; \vec{\alpha}^* \in A^*. \tag{21}$$

Let $u^N \in \mathcal{U}^{R^N}$ and $V = V_{u^N}$.

Proposition 6.1. $\mathcal{MB}(N, V) = \varnothing$.

Proof. Assume, on the contrary, that there is some $x \in \mathcal{MB}(N, V)$. Let $y \in \mathbb{R}^N$ be defined by $y^i = \min\{u^i(\alpha) \mid \alpha \in A \cup A^*, u^i(\alpha) \geq x^i\}$ for all $i \in N$. Then $y \in \mathcal{MB}(N, V)$ as well. Moreover, there is a position $\vec{\alpha}$ of R^N such that $y^i = u^i(\alpha_i)$ for all $i \in N$. As $y \in \mathcal{MB}(N, V)$, the position $\vec{\alpha}$ has the following properties:

$$\exists \alpha \in A \cup A^* \text{ such that } \alpha\; R^i\; \alpha_i\; \forall\, i \in N. \tag{22}$$

$$\nexists \beta \in A \cup A^* \text{ such that } \beta\; R^i\; \alpha_i \text{ and } \beta \neq \alpha_i\; \forall\, i \in N. \tag{23}$$

$$\nexists \beta \in A \text{ such that } |\{i \in N \mid \beta\; R^i\; \alpha_i, \beta \neq \alpha_i\}| > n/2 \text{ and } \vec{\alpha} \text{ enhances } \beta \text{ at } R^N. \tag{24}$$

Indeed, (22) and (23) are true, because $y \in V(N)$, and y is weakly Pareto optimal. In order to show (24), let $\beta \in A$ satisfy $|S| > n/2$, where $S = \{i \in N \mid \beta\; R^i\; \alpha_i, \beta \neq \alpha_i\}$. Then $(S, u^S(\beta))$ is an objection against y. Hence, there exist $\gamma \in A \backslash \{\beta\}$ and $T \subseteq N$, $|T| > n/2$ such that $u^i(\gamma) \geq \max\{y^i, u^i(\beta)\}$ for all $i \in T$ (note that γ must be in A, rather than A^*, due to (21)). By Lemma 6.1, $T = \{i \in N \mid \gamma\; R^i\; \beta\}$ and $|T| = (n+1)/2$. Hence, $\vec{\alpha}$ does not enhance β.

Claim 1. *The position $\vec{\alpha}$ does not enhance any $\beta \in A$.*

In view of (24) we may assume that $|\{i \in N \mid \alpha_i\; R^i\; \beta\}| > n/2$. Let $\alpha \in A \cup A^*$ satisfy (22). Then $|\{i \in N \mid \alpha\; R^i\; \beta\}| > n/2$. Therefore, either $\alpha = \beta$ or $\alpha \succ_{R^N} \beta$. If $\alpha = \beta$, then $\alpha_i\; R^i\; \gamma\; R^i\; \beta$ for some $i \in N$ implies that $\alpha_i = \gamma$. If $\alpha \succ_{R^N} \beta$, then $\alpha \in A$ by (21), and $\alpha_i\; R^i\; \alpha\; R^i\; \beta$ for some $i \in N$ implies $\alpha_i = \alpha$. Therefore, in both cases $\vec{\alpha}$ does not enhance β.

Claim 2. *There exists a position $\vec{\beta}$ satisfying $\beta_i \in A$ and $\beta_i\; R^i\; \alpha_i$ for all $i \in N$ such that $\vec{\beta}$ does not enhance any member of A (that is, $\vec{\beta}$ is nonenhancing at Q^N).*

Let $i \in N$ and let $\delta_i = t_1(Q^i)$ (that is, i's best alternative in A). We now show that $\delta_i\; R^i\; \alpha_i$. Assume, on the contrary, $\alpha_i\; R^i\; \delta_i$, $\alpha_i \neq \delta_i$. Let γ_i be i's lowest alternative in A, that is, $\gamma_i = t_m(Q^i)$. If $\delta \in A \cup A^*$ satisfies $\delta \succ_{R^N} \gamma_i$, then $\delta \in A$ by (21). Moreover, $\alpha_i\; R^i\; \delta\; R^i\; \gamma_i$ and $\alpha_i \neq \delta$. Hence, $\vec{\alpha}$ enhances γ_i and a contradiction to Claim 1 is established. Let β_i be i's lowest alternative in A weakly above α_i, that is, $\beta_i \in A$, $\beta_i\; R^i\; \alpha_i$, and $\beta'_i\; R^i\; \alpha_i$ implies $\beta'_i\; Q^i\; \beta_i$ for all $\beta'_i \in A$. By construction, because $\vec{\alpha}$ does not enhance any $\beta \in A$, neither does $\vec{\beta}$. Claim 2 has been shown.

Select any MNE position $\vec{\beta}$ at Q^N that satisfies the conditions of Claim 2. By (20), $\vec{\beta}^*\; R^i\; \beta_i$ and $\vec{\beta}^* \neq \beta_i$ for all $i \in N$. Combined with the fact that $\beta_i\; R^i\; \alpha_i$ holds for all $i \in N$, this contradicts (23). □

7. Two models with many voters. We here present two models in which special assumptions about the distribution of preferences in the population of voters lead to existence results when there are many voters.

The first model is probabilistic. Let A be a fixed set of m alternatives, and let $L = L(A)$. We assume that each $R \in L$ appears with positive probability $p_R > 0$ in the population of potential voters, where $\sum_{R\in L} p_R = 1$. Now let $(\mathcal{R}^i)_{i\in\mathbb{N}}$ be a sequence of independent and identically distributed random variables such that $\Pr(\mathcal{R}^i = R) = p_R$ for all $i \in \mathbb{N}$, $R \in L$. Let $\mathcal{R}^N = (\mathcal{R}^1, \ldots, \mathcal{R}^n)$ be the corresponding random profile of preferences for n voters, and let $(N, V(\mathcal{R}^N))$ be the random simple majority voting game that is associated via some utility representation $u^{N,R^N} = (u^{i,R^i})_{i\in N}$ for each realization R^N of $\mathcal{R}^N$.

We are going to prove that in this model the limiting probability, as $n \to \infty$, that the bargaining set and the Mas-Colell bargaining set are nonempty, equals one. We note that the analogous statement does not hold true for the core. In the case of the core, the limiting probability in question is that of the existence of a weak Condorcet winner. This has been studied quite a lot in the literature (see, e.g., Sen [19] and Gehrlein [11]). In the simplest setup, where $p_R = 1/m!$ for every $R \in L(A)$, it is known that the limiting probability that there exists a weak Condorcet winner is strictly less than one for every $m \geq 3$, and it tends to zero as $m \to \infty$. In the more general setup that we consider here, it is even possible to choose $p_R > 0$ so that this limiting probability will equal zero (see Example 7.1 below).

Define, for $j = 1, \ldots, m$,

$$\varepsilon_j(p) = \varepsilon_j = \min_{\alpha\in A} \sum_{R\in L:\, \alpha = t_j(R)} p_R.$$

Because $p_R > 0$ for all $R \in L$, $\varepsilon_j > 0$. Note that for any $\gamma < 1$,

$$\lim_{n\to\infty} \Pr\left(\mathcal{R}^N \in \left\{R^N \in L^N \,\middle|\, \min_{\alpha\in A} |\{i \in N \mid \alpha = t_j(R^i)\}| \geq \gamma\varepsilon_j n\right\}\right) = 1. \tag{25}$$

Theorem 7.1. $\lim_{n\to\infty} \Pr(\mathcal{M}(N, V(\mathcal{R}^N)) \neq \varnothing) = \lim_{n\to\infty} \Pr(\mathcal{MB}(N, V(\mathcal{R}^N)) \neq \varnothing) = 1.$

Proof. Call $R^N \in L^N$ *good* if for all $\alpha \in A$ there exists $i \in N$ such that $\alpha = t_m(R^i)$. If R^N is good, then $0 \in \mathcal{M}(N, V)$, where $V = V_{u^{N,R^N}}$. Regarding $\mathcal{MB}(N, V)$ when R^N is good, we distinguish two cases. If there is a weak Condorcet winner α, then $u^{N,R^N}(\alpha) \in \mathcal{MB}(N, V)$. If no such α exists, then $0 \in \mathcal{MB}(N, V)$. Thus, we see that in order to prove both parts of the theorem, it suffices to show that $\mathcal{R}^N$ is good with probability tending to one as n tends to infinity. This fact is implied by (25) applied to $j = m$. $\square$

As shown in the next theorem, the probability that a positive fraction of voters may receive maximal utility in the Mas-Colell bargaining set tends to one if n tends to infinity.

Theorem 7.2. *If $\varepsilon^* < \varepsilon_1(p)$, then*

$$\lim_{n\to\infty} \Pr(|\{i \in N \mid x^i = u^{i,R^i}(t_1(R^i)) \text{ for some } x \in \mathcal{MB}(N, V(\mathcal{R}^N))\}| \geq \varepsilon^* n) = 1.$$

The following lemma is used in the proof of Theorem 7.2.

Lemma 7.1. *Let (N, V) be a zero-normalized superadditive NTU game, $i \in N$, and $x \in \mathbb{R}^N_+$ such that x is weakly Pareto optimal in $V(N)$ and $x^j = 0$ for all $j \in N\setminus\{i\}$. If $\mathcal{C}(N, V) = \varnothing$ and $\mathcal{C}(N\setminus\{i\}, V) = \varnothing$,*[9] *then $x \in \mathcal{MB}(N, V)$.*

Proof. Let (P, y) be an objection such that P is maximal. By superadditivity, $N\setminus\{i\} \subseteq P$. If $i \in P$, then there exists a counterobjection, because $\mathcal{C}(N, V) = \varnothing$. If $P = N\setminus\{i\}$, then there exists a counterobjection, because $\mathcal{C}(N\setminus\{i\}, V) = \varnothing$. $\square$

Proof of Theorem 7.2. By (25), by (ii) of Remark 2.1, and by Lemma 7.1, it suffices to prove that

$$\lim_{n\to\infty} \Pr(\mathcal{C}(N, V(\mathcal{R}^N)) = \varnothing \text{ and } \mathcal{C}(N\setminus\{i\}, V(\mathcal{R}^N)) \neq \varnothing \text{ for some } i \in N) = 0. \tag{26}$$

Let R^N be a realization of $\mathcal{R}^N$ and let $u^N = u^{N,R^N}$, $V = V_{u^N}$. Define a binary relation $\rhd_{R^N} = \rhd$ on A as follows. For $\alpha, \beta \in A$, $\alpha \neq \beta$, define

$$\alpha \rhd \beta \iff |\{i \in N \mid \alpha\ R^i\ \beta\}| = 1 + [n/2].$$

[9] Here the set $N\setminus\{i\}$ is identified with $\{1, \ldots, n-1\}$ and $(N\setminus\{i\}, V)$ denotes the corresponding subgame.

Claim: If $\mathscr{C}(N, V) = \varnothing$ and $\mathscr{C}(N\backslash\{i\}, V) \neq \varnothing$ for some $i \in N$, then there exist $\alpha, \beta \in A$, $\alpha \neq \beta$, such that $\alpha \rhd \beta$. Indeed, if $\mathscr{C}(N\backslash\{i\}, V) \neq \varnothing$, then there exists $\beta \in A$ such that $u^{N\backslash\{i\}}(\beta) \in \mathscr{C}(N\backslash\{i\}, V)$. Because $u^N(\beta) \notin \mathscr{C}(N, V)$, there exists $\alpha \in A\backslash\{\beta\}$ such that $|\{j \in N \mid \alpha\ R^j\ \beta\}| > n/2$. However, $|\{j \in N\backslash\{i\} \mid \alpha\ R^j\ \beta\}| \leq n/2$, and therefore $\alpha \rhd \beta$.

Define $q^n_{\alpha,\beta} = \Pr(\mathscr{R}^N \in \{R^N \in L^N \mid \alpha \rhd_{R^N} \beta\})$.

In order to prove (26), it suffices to prove that $\lim_{n\to\infty} q^n_{\alpha,\beta} = 0$. Let $q = \sum_{R\in L:\, \alpha R \beta} p_R$. With $r = r(n) = 1 + [n/2]$ we obtain $q^n_{\alpha,\beta} = \binom{n}{r} q^r (1-q)^{n-r}$. We distinguish two cases. If n is even, then $r = (n/2) + 1$ and

$$q^n_{\alpha,\beta} = \binom{2r-2}{r} q^r (1-q)^{r-2} \leq \binom{2r}{r} q^2 (q(1-q))^{r-2} \leq \frac{(2r)!}{r!r!2^{2r-4}}.$$

Using $k! \approx \sqrt{2\pi k}(k/e)^k$ yields

$$\frac{(2r)!}{r!r!2^{2r-4}} \approx \frac{16}{\sqrt{\pi r}} \to_{r\to\infty} 0.$$

Similarly, we may approximate $q^n_{\alpha,\beta}$ if n is odd. $\square$

The next example shows that $\mathscr{MB}$ cannot be replaced by $\mathscr{M}$ in Theorem 7.2.

Example 7.1. Let $A = \{a, b, c\}$ and let p satisfy $p_{R^i} > 1/4$ for $i = 1, 2, 3$, where the R^i are defined by Table 2. Then

$$\lim_{n\to\infty} \Pr(\mathscr{R}^N \in \{R^N \mid a \succ_{R^N} b \succ_{R^N} c \succ_{R^N} a\}) = 1,$$

hence, the probability that the core is empty tends to one if n tends to infinity. Also, if (N, V) is a realization of $(N, V(\mathscr{R}^N))$ such that $\mathscr{C}(N, V) = \varnothing$, then $x \in \mathscr{M}(N, V)$ implies $x^i < u^{i, R^i}(t_1(R^i))$ for all $i \in N$. Indeed, this statement can be proved as follows: Let (N, V_{u^N}), $u^N \in \mathscr{U}^{R^N}$ for some $R^N \in L(A)^N$, be a majority voting game whose core is empty. Let $i \in N$ and let $x \in V(N) \cap \mathbb{R}^N_+$ satisfy $x^i = u^i(\alpha)$, where $\alpha = t_1(R^i)$. Hence, $x \leq u^N(\alpha)$ and there exists $\beta \in A$ such that $\beta \succ_{R^N} \alpha$. Let $P = \{j \in N \mid \beta\ R^j\ \alpha\}$ and let $y = u^P(\beta)$. Then (P, y) is a justified strong objection in the sense of $\mathscr{M}$ of any voter $j \in P$ against i so that $x \notin \mathscr{M}(N, V)$.

The second model involves replication. Let A be a fixed set of m alternatives, and let $L = L(A)$. Let $N = \{1, \ldots, n\}$, let $R^N \in L^N$, and let $u^N \in \mathscr{U}^{R^N}$. In order to replicate the simple majority voting game (N, V_{u^N}), let $k \in \mathbb{N}$ and denote

$$kN = \{(j, i) \mid i \in N,\ j = 1, \ldots, k\}.$$

Furthermore, let $R^{(j,i)} = R^i$ and $u^{(j,i)} = u^i$ for all $i \in N$ and $j = 1, \ldots, k$. Then $(kN, V_{u^{kN}})$ is the *k-fold replication* of (N, V_{u^N}).

Theorem 7.3. *If*

$$k \geq \begin{cases} n+2, & \text{if } n \text{ is odd,} \\ n/2+2, & \text{if } n \text{ is even,} \end{cases}$$

then $\mathscr{MB}(kN, V_{u^{kN}}) \neq \varnothing$.

Proof. If α is a weak Condorcet winner with respect to R^N, then $u^{kN}(\alpha) \in \mathscr{MB}(kN, V_{u^{kN}})$. Hence, we may assume that for every $\alpha \in A$ there exists $\beta(\alpha) \in A$ such that $\beta(\alpha) \succ_{R^N} \alpha$. Let $\tilde{x} \in \mathbb{R}^N_+$ be any weakly Pareto optimal element in $V_{u^N}(N)$. We define $x \in \mathbb{R}^{kN}$ by $x^{(1,i)} = \tilde{x}^i$ and $x^{(j,i)} = 0$ for all $i \in N$ and $j = 2, \ldots, k$ and claim that $x \in \mathscr{MB}(kN, V_{u^{kN}})$. Let (P, y) be an objection at x. Then there exists $\alpha \in A$ such that $y = u^P(\alpha)$. Let $\beta = \beta(\alpha)$ and let $T = \{i \in N \mid \beta\ R^i\ \alpha\}$. Then

$$|T| \geq \begin{cases} (n+1)/2, & \text{if } n \text{ is odd,} \\ n/2+1, & \text{if } n \text{ is even.} \end{cases} \tag{27}$$

Let $Q = \{(j, i) \mid i \in T, j = 2, \ldots, k\}$ and define $z \in \mathbb{R}^Q$ by $z^{(j,i)} = u^i(\beta)$ for all $i \in T$ and $j = 2, \ldots, k$. Then $|Q| = (k-1)|T|$ and $z > (y^{P\cap Q}, x^{Q\backslash P})$. By (27), $|Q| \geq (kn+1)/2$. Therefore, (Q, z) is a counterobjection to (P, y). $\square$

By means of an example, it may be shown that replication may not guarantee nonemptiness of the Aumann-Davis-Maschler bargaining set. Indeed, let $n = 4$, $A = \{a_1, \ldots, a_4, a_1^*, \ldots, a_4^*, b, c\}$, let R^N be given by Table 7, and let $u^N \in \mathscr{U}^{R^N}$.

In Peleg and Sudhölter [17][10] it is shown that for any imputation there exists a strong objection that does not have a weak counterobjection. The proof of Theorem 3.1 in that paper proceeds by contradiction, and it

[10] Although in that paper the "=" in our (1) is replaced by ">," the relevant proof of Theorem 3.1 is not affected by this change.

TABLE 7. Preference profile on 10 alternatives.

R^1	R^2	R^3	R^4
a_1	a_4	a_3	a_2
a_2	a_1	a_4	a_3
a_2^*	a_1^*	a_4^*	a_3^*
a_1^*	c	a_3^*	a_2^*
c	a_4^*	c	b
b	b	b	a_4^*
a_3^*	a_2^*	a_1^*	a_4
a_3	a_2	a_1	c
a_4^*	a_3^*	a_2^*	a_1^*
a_4	a_3	a_2	a_1

may be modified in order to show that $\mathcal{M}(kN, V_{u^{kN}}) = \varnothing$ for every k. Assume that $x \in \mathcal{M}(kN, V_{u^{kN}})$. Then $x^{kN} \leq u^{kN}(\alpha)$ for some $\alpha \in A$. As in the proof of Theorem 3.1, we may distinguish 10 cases, because $|A| = 10$. The justified objections may be replaced by their k-fold replications; e.g., with $S_4 = \{2, 3, 4\}$, $(S_4, u^{S_4}(a_4))$ is a strong objection at any imputation $x \leq u^N(a_1)$. The k-fold replication of this objection is a strong objection at $x^{kN} \leq u^{kN}(a_1)$ of any copy $(j, 4)$ of player 4 against any copy $(l, 1)$ of player 1 and the existence of a weak counterobjection of $(l, 1)$ against $(j, 4)$ implies that $x^{(l,1)} \leq u^1(a_3)$. We may continue along the lines of the proof of Theorem 3.1 and show that $x^{kN} \ll u^{kN}(b)$ and all other cases lead to the same contradiction in a similar way.

Acknowledgments. The authors are grateful to the area editor of this journal and two referees for several remarks. The first author's work was done while he was a Fellow of the Institute for Advanced Studies at the Hebrew University of Jerusalem. The third author was supported by the Edmund Landau Center for Research in Mathematical Analysis and Related Areas, and by the Center for the Study of Rationality at the Hebrew University of Jerusalem. The article is an enhanced and modified version of Discussion paper # 376, Center for the Study of Rationality, The Hebrew University of Jerusalem.

References

[1] Alon, N., J. H. Spencer. 2000. *The Probabilistic Method*, 2nd ed. Wiley, New York.
[2] Asscher, N. 1976. An ordinal bargaining set for games without side payments. *Math. Oper. Res.* **1** 381–389.
[3] Aumann, R. J. 1961. The core of a cooperative game without side payments. *Trans. Amer. Math. Soc.* **98** 539–552.
[4] Aumann, R. J. 1967. A survey of cooperative games without side payments. M. Shubik, ed. *Essays in Mathematical Economics in Honor of Oskar Morgenstern*. Princeton University Press, Princeton, NJ, 3–27.
[5] Aumann, R. J., M. Maschler. 1964. The bargaining set for cooperative games. M. Dresher, L. S. Shapley, A. W. Tucker, eds. *Advances in Game Theory. Annals of Mathematical Studies*, Vol. 52. Princeton University Press, Princeton, NJ, 443–476.
[6] Aumann, R. J., B. Peleg. 1960. Von Neumann-Morgenstern solutions to cooperative games without side payments. *Bull. Amer. Math. Soc.* **66** 173–179.
[7] Banks, J. S. 2002. Strategic aspects of political systems. R. J. Aumann, S. Hart, eds. *Handbook of Game Theory*, Vol. 3, Ch. 59. Elsevier Science B. V., Amsterdam, The Netherlands, 2203–2228.
[8] Davis, M., M. Maschler. 1967. Existence of stable payoff configurations for cooperative games. M. Shubik, ed. *Essays in Mathematical Economics in Honor of Oskar Morgenstern*. Princeton University Press, Princeton, NJ, 39–52.
[9] Dutta, B., D. Ray, K. Sengupta, R. Vohra. 1989. A consistent bargaining set. *J. Econom. Theory* **49** 93–112.
[10] Erdős, P., L. Moser. 1964. On the representation of directed graphs as unions of orderings. *Magyar Tud. Akad. Mat. Kutató Int. Közl.* **9** 125–132.
[11] Gehrlein, W. V. 2002. Condorcet's paradox and the likelihood of its occurrence: Different perspectives on balanced preferences. *Theory Decision* **52** 171–199.
[12] Graham, R. L., J. H. Spencer. 1971. A constructive solution to a tournament problem. *Canad. Math. Bull.* **14** 45–48.
[13] Hardy, G. H., E. M. Wright. 1979. *An Introduction to the Theory of Numbers*, 5th ed. Oxford Science Publications, Oxford, UK.
[14] Holzman, R. 2001. The comparability of the classical and the Mas-Colell bargaining sets. *Internat. J. Game Theory* **29** 543–553.
[15] Mas-Colell, A. 1989. An equivalence theorem for a bargaining set. *J. Math. Econom.* **18** 129–139.
[16] McGarvey, D. C. 1953. A theorem on the construction of voting paradoxes. *Econometrica* **21** 608–610.
[17] Peleg, B., P. Sudhölter. 2005. On the non-emptiness of the Mas-Colell bargaining set. *J. Math. Econom.* **41** 1060–1068.
[18] Scarf, H. E. 1967. The core of an *n*-person game. *Econometrica* **35** 50–69.
[19] Sen, A. K. 1970. *Collective Choice and Social Welfare*. Holden-Day, San Francisco, CA.
[20] Stearns, R. E. 1959. The voting problem. *Amer. Math. Monthly* **66** 761–763.
[21] Vohra, R. 1991. An existence theorem for a bargaining set. *J. Math. Econom.* **20** 19–34.