Int J Game Theory (2001) 29:597–623

International Journal of Game Theory

Axiomatizations of the core on the universal domain and other natural domains*

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Received September 1999/Final version December 2000

Abstract. We prove that the core on the set of all transferable utility games with players contained in a universe of at least five members can be axiomatized by the zero inessential game property, covariance under strategic equivalence, anonymity, boundedness, the weak reduced game property, the converse reduced game property, and the reconfirmation property. These properties also characterize the core on certain subsets of games, e.g., on the set of totally balanced games, on the set of balanced games, and on the set of superadditive games. Suitable extensions of these properties yield an axiomatization of the core on sets of nontransferable utility games.

Key words: TU game, core, kernel; NTU game.

1. Introduction

The core on balanced cooperative transferable utility (TU) games and on some subclasses can be axiomatized (see, e.g., Peleg (1986,1989)). However, in the well-known axiomatizations either nonemptiness or the property of "coincidence with the core on two-person games" are employed. The assumption that every game under consideration has a nonempty core, is crucial within this context. The characterization of the core presented in this paper does neither refer to balanced games nor does it use one of the axioms just mentioned. That may be regarded as an advantage over the axiomatizations that are known from literature. With the exception of the "zero inessential game property", which requires the solution to be nonempty when applied to

* The authors are indebted to Chih Chang and Bezalel Peleg for several helpful conversations and they are grateful to two anonymous referees for some helpful remarks.

a two-person "flat" game, the axioms employed in the present results have been used to characterize the core or the prenucleolus. As a byproduct the result can be seen as an implicit characterization of the class of balanced TU games. Peleg (1985) provided an axiomatization of the core on the class of all nontransferable utility (NTU) games with a nonempty core. The results of the present paper show that the core can be axiomatized by intuitive properties on the class of all NTU games. Thus Theorem 7.4 yields an implicit characterization of the class of NTU games that have a nonempty core. Moreover, the axiomatizations are "robust" in the sense that they can be applied to many subclasses of games. The classes of all balanced or totally balanced TU games and the class of all NTU games with a nonempty core, which have already been considered in literature, constitute examples of this kind. Further examples are the classes of totally balanced or balanced NTU games and the class of superadditive TU games. As far as we know, the core has not been characterized on any of the latter classes up to now.

The paper is organized as follows: In Section 2 the notation is presented and some definitions of well-known properties of solutions are recalled. An axiom which has not been frequently used up to now is the zero inessential game property (ZIG). For the detailed description see Definition 2.1. This property requires the solution to be nonempty, when applied to a two-person "flat" game, and is, thus, much weaker than nonemptiness.

Section 3 recalls the definition of some versions of the reduced game property and the converse reduced game property and some well-known characterizations of the prenucleolus and the core which are relevant in our context. Moreover, a further version of the reduced game property, which we call reconfirmation property (RCP), is introduced (see Definition 3.5) and discussed. This property is, alike the reduced game property, a set-valued generalization of the reduced game property for single-valued solutions as introduced by Sobolev (1975).

In Section 4 it is shown that on the set of games with player sets contained in a universe of at least five members, the core is the unique solution that satisfies ZIG, covariance under strategic equivalence, anonymity (AN), the (weak) reduced game property and its converse, RCP, and individual rationality (IR). Especially the last property can be weakened. Boundedness can be used to replace IR.

In Section 5 it is shown that Theorem 4.2 is also valid for every subset of games that contains every totally balanced game and does not contain nonbalanced two-person games. The considered set of games is, thus, "closed under weak reduction" with respect to members of the core (meaning that every two-person reduced game with respect to a member of the core belongs to the considered set of games). Among others the subset of all superadditive (balanced) games has the required properties. On such sets of games AN can be dropped as a condition.

In Section 6 it turns out that boundedness can be weakened and it can even be replaced by AN in the second theorem, if the converse reduced game property is defined with respect to feasible payoffs instead of preimputations. Moreover, the impact of ZIG is discussed.

The main results of the "TU part", namely Theorem 4.2 and Theorem 5.1 can be generalized to NTU games (see Theorems 7.4 and 7.5) as shown in Section 7. Of course, some of the axioms for TU games have to be suitably extended for NTU games. Moreover, boundedness is replaced by "reasonableness from below" (REAS). This intuitively justified property does, unlike in the TU case, not imply boundedness. The axiomatization does not only hold for the class of all NTU games but also, e.g., for the class of totally balanced games and certain superclasses. In view of the fact that an NTU game with a nonempty core is not necessarily balanced, the class of NTU games with a nonempty core cannot be characterized with the help of balancedness. Theorem 7.4, however, yields an implicit characterization of the class of NTU games with a nonempty core.

Section 8 presents examples of solutions which show some aspects of the impact of the axioms used in the characterizations. Furthermore, these solutions can be used to show that the properties that occur in the main results (Theorems 4.2, 5.1, 7.4, and 7.5) are logically independent. Finally, a table, which "summarizes" the main results of this paper, as well as some remarks are included in this section.

Some results of Sections 6 and 7 are proved in Section 9.

2. Notation and definitions

Let $\mathcal{U}$ be the set of *players*. A *cooperative transferable utility game* – a *TU game* – is a pair (N, v), where N is a finite nonvoid subset of $\mathcal{U}$ and $v : 2^N \to \mathbb{R}, v(\emptyset) = 0$ is a mapping (the *coalitional function*). Here $2^N = \{S \subseteq N\}$ is the set of *coalitions* of (N, v). If $\emptyset \neq S \subseteq N$, then (S, v) denotes the *subgame* of (N, v) w.r.t. the coalition S. (The coalitional function of the subgame w.r.t. S is the restriction of v to subsets of S.) Let $\Gamma_{\mathcal{U}}$ denote the set of all TU games.

The set of *feasible payoff vectors* of a TU game (N, v) is denoted by

$$X^*(N, v) = \{x \in \mathbb{R}^N \mid x(N) \leq v(N)\},$$

whereas

$$X(N, v) = \{x \in \mathbb{R}^N \mid x(N) = v(N)\}$$

is the set of *preimputations* of (N, v) (also called set of *Pareto optimal* feasible payoffs of (N, v)). Here

$$x(S) = \Sigma_{i \in S}\, x_i \;\; (x(\emptyset) = 0)$$

for each $x \in \mathbb{R}^N$ and $S \subseteq N$. Additionally, let x_S denote the restriction of x to S, i.e.

$$x_S = (x_i)_{i \in S} \in \mathbb{R}^S.$$

For disjoint coalitions $S, T \subseteq N$ and $x \in \mathbb{R}^N$ let $(x_S, x_T) = x_{S \cup T}$.

A *solution* σ on a set Γ of TU games is a mapping that associates with every game $(N, v) \in \Gamma$ a set $\sigma(N, v) \subseteq X^*(N, v)$.

If $\overline{\Gamma}$ is a subset of Γ, then the canonical restriction of a solution σ on Γ is a solution on $\overline{\Gamma}$. We say that σ is a solution on $\overline{\Gamma}$, too. If Γ is not specified, then σ is a solution on $\Gamma_{\mathcal{U}}$.

Some convenient and well-known properties of a solution σ on a set Γ of TU games are as follows.

(1) σ is *anonymous* (satisfies **AN**), if for each $(N, v) \in \Gamma$ and each bijective mapping $\tau : N \to N'$ with $(N', \tau v) \in \Gamma$

$$\sigma(N', \tau v) = \tau(\sigma(N, v))$$

holds (where $(\tau v)(T) = v(\tau^{-1}(T))$, $\tau_j(x) = x_{\tau^{-1}j}(x \in \mathbb{R}^N, j \in N', T \subseteq N')$). In this case (N, v) and $(N', \tau v)$ are *isomorphic* games.

(2) σ is *covariant under strategic equivalence* (satisfies **COV**), if for (N, v), $(N, w) \in \Gamma$ with $w = \alpha v + \beta$ for some $\alpha > 0, \beta \in \mathbb{R}^N$

$$\sigma(N, w) = \alpha\sigma(N, v) + \beta$$

holds. The games v and w are called *strategically equivalent*.

(3) σ satisfies *nonemptiness* (**NE**), if $\sigma(N, v) \neq \emptyset$ for $(N, v) \in \Gamma$.

(4) σ is *Pareto optimal* (satisfies **PO**), if $\sigma(N, v) \subseteq X(N, v)$ for $(N, v) \in \Gamma$.

(5) σ satisfies *individual rationality* (**IR**), if $x_i \geq v(\{i\})$ holds true for every $i \in N, (N, v) \in \Gamma$, and $x \in \sigma(N, v)$.

(6) σ satisfies *reasonableness* from below (**REAS**), if $x_i \geq \min_{S \subseteq N \setminus \{i\}} v(S \cup \{i\}) - v(S)$ holds true for every $i \in N, (N, v) \in \Gamma$, and $x \in \sigma(N, v)$.

Milnor (1952) introduced reasonableness (from above) for TU games. Recently Sudhölter and Peleg (2000) used reasonableness (on both sides) to characterize a new variant of the prekernel on TU games. Note that individual rationality implies REAS[1], thus the core satisfies REAS on any set of TU games. Moreover, it should be remarked that every game has feasible payoff vectors that are reasonable.

Some more notation will be needed. Let (N, v) be a TU game and $x \in \mathbb{R}^N$. The *excess of* a coalition $S \subseteq N$ at x is the real number $e(S, x, v) = v(S) - x(S)$. For different players $i, j \in N$ let $s_{ij}(x, v) = \max\{e(S, x, v) \mid i \in S \subseteq N \setminus \{j\}\}$ denote the *maximum surplus of i over j at x*.

The *core* of (N, v) is the set

$$\mathscr{C}(N, v) = \{x \in X^*(N, v) \mid e(S, x, v) \leq 0 \ \forall S \subseteq N\}$$

of feasible payoff vectors which generate nonpositive excesses. The *prekernel* of (N, v) is the set

$$\{x \in X(N, v) \mid s_{ij}(x, v) = s_{ji}(x, v) \ \forall i, j \in N \text{ with } i \neq j\}$$

of preimputations that *balance* the maximum surplus of the pairs of players. The *prenucleolus* of (N, v) is the set of preimputations that lexicographically minimize the nonincreasingly ordered vector of excesses of the coalitions. The prenucleolus of a game is a singleton.

[1] For 0-monotonic games (A game (N, v) is 0-monotonic, if the minimal marginal contribution $\min_{S \subseteq N \setminus \{i\}} v(S \cup \{i\}) - v(S)$ of every player $i \in N$ is attained by $v(\{i\})$.) individual rationality and reasonableness from below are equivalent.

The prekernel as well as the prenucleolus satisfy all properties mentioned so far except IR. The core satisfies NE on the subset of *balanced* games. For these notations and assertions see Davis and Maschler (1965), Schmeidler (1969), Bondareva (1963), and Shapley (1967).

Axiomatizations of the prenucleolus and the prekernel on $\Gamma_{\mathcal{U}}$ are due to Sobolev (1975) and Peleg (1986). The core can be axiomatized (see Peleg (1986, 1989)) on the subsets of balanced or totally balanced TU games[2]. In Section 3 the precise formulations of some of these characterizations as well as the relevant definitions will be recalled. The core satisfies all axioms except NE. We shall present an axiomatization of the core (see Theorem 4.2) which can be compared with the mentioned results. We cannot employ NE, because our attention is **not** restricted to balanced games. Therefore we demand less than NE.

Definition 2.1. *A solution on a set Γ of TU games satisfies the* **zero inessential** *(two-person) game property (***ZIG***), if for every* **zero inessential** *(synonymously called* **flat***) two-person game $(N, v) \in \Gamma$, i.e. $|N| = 2, v(S) = 0$ for $S \subseteq N$, the solution yields a nonempty set, i.e., $\sigma(N, v) \neq \emptyset$ holds true.*

Note that we do not force a solution which satisfies ZIG to yield $(0,0) \in \mathbb{R}^N$ for every flat two-person game; we only require nonemptiness. An idea of this property is as follows. Suppose two players bargain about how to share the worth of the grand coalition w.r.t. their two-person game. If this worth is positive or negative, it may happen that they do not reach any agreement. If the game is flat they should be indifferent between leaving the game, thus obtaining zero (0), and sharing the worth (0) of the grand coalition equally. ZIG only requests that the two players do not leave the flat game without an agreement.

A TU game (N, v) is called *inessential*, if it is additive, i.e., if there exists a vector $x \in \mathbb{R}^N$ satisfying $v(S) = x(S)$ for every coalition $S \subseteq N$, and it is called zero inessential, if additionally $x = 0$ holds true. It should be noted that ZIG does not imply that the solution is nonempty when applied to a zero inessential game of more than two players.

Note that the core satisfies the zero inessential game property on every class of games, because the core of an inessential game (N, v) defined by $v(S) = x(S)$ for some vector $x \in \mathbb{R}^N$ consists of the vector x.

3. Reduced game properties

It is the aim of this section to recall some axiomatizations of the prenucleolus and the core (see Sobolev (1975) and Peleg (1989)) and to introduce a new variant of the reduced game property which enables us to characterize the core on many classes of games. First we recall the definition of the "reduced game" and of some classical "reduced game properties".

Definition 3.1. *Let (N, v) be a TU game, let $\emptyset \neq S \subseteq N$, and $x \in X^*(N, v)$. The* **reduced game** *w.r.t. S and x is the game $(S, v^{S,x})$ defined by*

[2] A TU game is *totally balanced*, if each of its subgames is balanced.

$$v^{S,x}(T) = \begin{cases} 0, & \textit{if } T = \emptyset \\ v(N) - x(N \backslash S), & \textit{if } T = S \\ \max\{v(T \cup Q) - x(Q) \mid Q \subseteq N \backslash S\}, & \textit{otherwise} \end{cases}.$$

Definition 3.2. *Let σ be a solution on a set Γ of* TU *games. Then σ satisfies the*

(1) **reduced game property** *(***RGP***), if the following condition holds: If $(N, v) \in \Gamma, \emptyset \neq S \subseteq N$, and $x \in \sigma(N, v)$, then $(S, v^{S,x}) \in \Gamma$ and $x_S \in \sigma(S, v^{S,x})$.*
(2) **weak reduced game property** *(***WRGP***), if the following condition holds: If $(N, v) \in \Gamma, \emptyset \neq S \subseteq N, |S| \leq 2$, and $x \in \sigma(N, v)$, then $(S, v^{S,x}) \in \Gamma$ and $x_S \in \sigma(S, v^{S,x})$.*
(3) **converse reduced game property** *(***CRGP***), if the following condition holds: If $(N, v) \in \Gamma$ satisfies $|N| \geq 2$, if $x \in X(N, v)$, and if for every $S \subseteq N$ with two members $(S, v^{S,x}) \in \Gamma$ and $x_S \in \sigma(S, v^{S,x})$, then $x \in \sigma(N, v)$.*

Note that Definition 3.2(2) is due to Peleg (1989) and that RGP implies WRGP. Furthermore, note that the prekernel and the core satisfy CRGP and RGP, if the set Γ of TU games is rich enough. The two mentioned characterizations of the prenucleolus and the core are summarized in the following theorems.

Theorem 3.3 *(Sobolev (1975)). If the universe $\mathcal{U}$ of players is infinite, then the prenucleolus is the unique solution on $\Gamma_{\mathcal{U}}$ that satisfies single-valuedness,* COV, AN, *and* RGP.

A solution on a set Γ of TU games satisfies *superadditivity* (**SUPA**), if $x^1 + x^2 \in \sigma(N, v^1 + v^2)$, whenever $(N, v^1), (N, v^2), (N, v^1 + v^2) \in \Gamma, x^1 \in \sigma(N, v^1)$ and $x^2 \in \sigma(N, v^2)$.

Theorem 3.4 *(Peleg (1989)). If the universe $\mathcal{U}$ contains at least four players, then the core is the unique solution on the set of totally balanced games in $\Gamma_{\mathcal{U}}$ that satisfies* NE, SUPA, WRGP, CRGP, *and* IR.

In fact Peleg (1993) proved a slightly weaker version of this theorem. Recently, Peleg and the second author showed the theorem in its present form (a proof is available from the authors).

Let $\Gamma_{\mathcal{U}}^b$ and $\Gamma_{\mathcal{U}}^{tb}$ respectively denote the set of all **b**alanced and **t**otally **b**alanced games in $\Gamma_{\mathcal{U}}$. As soon as the class of games under consideration contains a nonbalanced game, the core does not satisfy NE and, thus, this axiom cannot be employed in a characterization of the core in this case. In what follows a further variant of the reduced game property is presented, which, together with ZIG, is strong enough (see Section 4) to replace NE.

Definition 3.5. *The solution σ on a set Γ of* TU *games satisfies the*

(1) **reconfirmation property** *(***RCP***), if the following condition is satisfied for every $(N, v) \in \Gamma$, every $x \in \sigma(N, v)$ and every $\emptyset \neq S \subseteq N$: If $(S, v^{S,x}) \in \Gamma$ and $y \in \sigma(S, v^{S,x})$, then $(y, x_{N \backslash S}) \in \sigma(N, v)$ holds true.*
(2) **weak reconfirmation property** *(***WRCP***), if the following condition is satisfied for every $(N, v) \in \Gamma$, every $x \in \sigma(N, v)$, and every $\emptyset \neq S \subseteq N$:*

If $(S, v^{S,x}) \in \Gamma$ *and* $y \in \sigma(S, v^{S,x})$ *satisfies* $y(S) \geq x(S)$, *then* $(y, x_{N \setminus S}) \in \sigma(N, v)$ *holds true.*

For interpretations of the notion of the reduced game, the reduced game property, and the converse reduced game property see, e.g., Peleg (1986) or Maschler (1992). RCP occurs in Balinsky and Young (1982) as one condition of a property they call "uniformity". Shimomura (1992) uses the term "flexibility". An interpretation of RCP is as follows. If a coalition of the TU game chooses any proposal of the solution when applied to the corresponding reduced game, then the combination of this proposal and the initial proposal restricted to the complement coalition, may or may not constitute a member of the solution of the original game. If any coalition is able to adjust its proposal in the aforementioned way without leaving the solution, then this kind of "stability" is called reconfirmation property. In this case the "passive" members of the complement coalition are "reconfirmed". If, in addition, any coalition is only allowed to choose adjustments that provide a payoff which is at least the payoff it started with, then the arising "stability" property is the weak reconfirmation property.

In some sense RGP is a "reduced game property from above". Indeed, if a solution satisfies RGP, then the restriction of any member of the solution of a game belongs to the solution of the corresponding reduced game. RCP reflects, in some sense, the opposite direction. Every member of the solution of a reduced game yields an element of the solution of the game, whenever it is combined with the corresponding restriction of the initial element of the solution. More precisely, on $\Gamma_{\mathscr{U}}$ the reduced game properties can be described as follows. A solution σ satisfies RGP or RCP respectively, if for every game $(N, v) \in \Gamma_{\mathscr{U}}$, every $x \in \sigma(N, v)$, and every nonempty coalition $S \subseteq N$

$$\{y \in \mathbb{R}^S \mid (y, x_{N \setminus S}) \in \sigma(N, v)\} \subseteq \sigma(S, v^{S,x})$$

or

$$\{y \in \mathbb{R}^S \mid (y, x_{N \setminus S}) \in \sigma(N, v)\} \supseteq \sigma(S, v^{S,x})$$

holds true respectively.

Remark 3.6:

(1) *The properties* RGP *and* RCP *are equivalent for single-valued solutions on* $\Gamma_{\mathscr{U}}$.
(2) *A solution that satisfies* RCP *satisfies* WRCP *as well.*
(3) *The core satisfies* RCP *on every set* Γ *of games.*

Proof: The first two assertions are consequences of the corresponding definitions. In order to show the third assertion, let $(N, v) \in \Gamma, x \in \mathscr{C}(N, v), \emptyset \neq S \subseteq N, u = v^{S,x}$, and $y \in \mathscr{C}(S, u)$. With $z = (y, x_{N \setminus S})$ it remains to show that $z \in \mathscr{C}(N, v)$ holds. Let $T \subseteq N$ and distinguish the following cases. If $T \cap S = \emptyset$ or if $T \cap S = S$, then $v(T) - z(T) = v(T) - x(T)$ by Pareto optimality of z. Thus $v(T) - z(T) \leq 0$, because $x \in \mathscr{C}(N, v)$. In the remaining case, i.e., $\emptyset \neq S \cap T \neq S$, the observation that

$$\begin{aligned} v(T) - z(T) &= v((T \cap S) \cup (T \setminus S)) - x(T \setminus S) - y(T \cap S) \\ &\leq v^{S,x}(T) - y(T) \end{aligned}$$

shows that $v(T) - z(T) \leq 0$ holds true. **q.e.d.**

In Section 4 we show that WRGP, WRCP, and CRGP can be used to characterize the core on the set of all games. The following remark is useful for some proofs (of Lemma 4.6 and Theorems 4.2, 6.3) of the next sections.

Remark 3.7: *Let σ^1 and σ^2 be solutions on a set Γ of games. If σ^1 satisfies* PO *and* WRGP*, if σ^2 satisfies* CRGP*, and if $\sigma^1(N,v) \subseteq \sigma^2(N,v)$ for every game $(N,v) \in \Gamma$ with at most two persons, then σ^1 is a subsolution of σ^2.*

Proof: It suffices to show $\sigma^1(N,v) \subseteq \sigma^2(N,v)\ \forall (N,v) \in \Gamma$ with $|N| \geq 3$. If $x \in \sigma^1(N,v)$, then $x_S \in \sigma^1(S, v^{S,x})$ for every coalition $\emptyset \neq S \subseteq N$ with $|S| \leq 2$ by WRGP of σ^1. Therefore $x_S \in \sigma^2(S, v^{S,x})$ for these coalitions by the assumption, thus $x \in \sigma^2(N,v)$ by CRGP of σ^2 and PO of σ^1. **q.e.d.**

4. A Characterization of the core

This section is devoted to a characterization of the core on the set of all games. Though the core satisfies IR, we do not employ this property in the main result (Theorem 4.2) of this section. The reason is that any individually rational solution automatically specifies the empty set when applied to any nonbalanced two-person game or to any game which does not possess individually rational and feasible payoffs. In our opinion this kind of "emptiness" condition is hard to justify for a solution on the set of **all** TU games. Though the weaker property of reasonableness from below seems to be adequate in this context, we shall use the following property which is implied by REAS.

Definition 4.1. *A solution σ on a set Γ satisfies* **boundedness** *(*__BOUND__*), if $\sigma(N,v)$ is bounded (from below[3]) for every game $(N,v) \in \Gamma$.*

Of course, IR implies REAS and REAS implies BOUND. The "natural" generalization of REAS to NTU games plays an important role in Section 7. Moreover, Corollary 4.8 shows that IR can as well be used in a characterization of the core. The main result of this section is the following theorem.

Theorem 4.2. *If the universe $\mathcal{U}$ contains at least* 5 *players, then the core is the unique solution on $\Gamma_{\mathcal{U}}$ that satisfies* ZIG, AN, COV, WRGP, WRCP, CRGP, *and* BOUND.

This result shows that the two variants of the reduced game property, WRGP and WRCP, together with the converse reduced game property, are strong enough to rule out any other solution, if additionally the well-accepted anonymity and covariance properties are assumed and a very mild nonemptiness condition (ZIG) and the "technical" condition BOUND are employed. On the one hand this theorem may be regarded as a negative result, because it uniquely characterizes a solution, which specifies the empty set for many games under consideration. On the other hand the theorem yields a new characterization of the set of games with a nonempty core.

[3] There is a "lower bound vector" $b \in \mathbb{R}^N$ (which may depend on (N,v)) satisfying $b \leq x\ \forall x \in \sigma(N,v)$.

We postpone the proof of Theorem 4.2 and shall now prove five useful lemmata globally assuming that σ is a solution on some set Γ satisfying $\Gamma_{\mathscr{U}}^{tb} \subseteq \Gamma \subseteq \Gamma_{\mathscr{U}}$. The **standard** solution of a two-person game (N, v) is denoted by $x^{(N,v)}$ (i.e., $x_i^{(N,v)} = (v(\{i\}) - v(\{j\}) + v(N))/2$, where $N = \{i,j\}$). We start with the following simple result.

Lemma 4.3. *If σ satisfies* COV, WRGP *and* BOUND, *then σ satisfies* PO.

Proof: Let $(N, v) \in \Gamma$. If $|N| = 1$, then COV and BOUND imply that every member of $\sigma(N, v)$ is Pareto optimal. If $|N| \geq 2$, then WRGP applied to an arbitrary coalition $S \subseteq N$ of size 1 implies Pareto optimality of $\sigma(N, v)$. **q.e.d.**

For the remainder of this section we assume that the universe $\mathscr{U}$ of players contains at least three members, let us say 1, 2 and 3.

Lemma 4.4. *If σ satisfies* ZIG, COV, WRGP, CRGP, *and* BOUND *and if $(N, v) \in \Gamma$ is the inessential game given by $v(S) = x(S)$ for some $x \in \mathbb{R}^N$, then $x \in \sigma(N, v)$.*

Proof: In the case $|N| = 2$ ZIG, COV, and BOUND show that $\{x\} = \sigma(N, v)$. The fact that Γ contains every inessential two-person game in $\Gamma_{\mathscr{U}}$ together with WRGP implies the assertion for $|N| = 1$. If $|N| \geq 3$, then CRGP shows the assertion. **q.e.d.**

Let EXT A denote the set of extreme points of the convex subset A of some Euclidean space.

Lemma 4.5. *If $|\mathscr{U}| \geq 5$ and σ satisfies* ZIG, COV, WRGP, WRCP, CRGP, *and* BOUND, *then EXT$\mathscr{C}(N, v) \subseteq \sigma(N, v)$ for every $(N, v) \in \Gamma$ with $|N| = 2$.*

Proof: Let $(N, v) \in \Gamma$ satisfy $|N| = 2$, let us say $N = \{1, 5\}$, and $\mathscr{C}(N, v) \neq \emptyset$. It suffices to show that $(v(N) - v(\{5\}), v(\{5\}))$[4] $\in \sigma(N, v)$. Without loss of generality we assume $v(\{1\}) = v(\{5\}) = 0$ (by COV) and $v(N) = 1$ (by Lemma 4.4 and COV). Let $\tilde{N} \subseteq \mathscr{U}$ be a superset of N of cardinality 5, let us say $\tilde{N} = \{1, 2, 3, 4, 5\}$, and define for every $\alpha \in \mathbb{R}$ the game $(\tilde{N}, w_\alpha)$, by

$$w_\alpha(S) = \begin{cases} \alpha - 4 \cdot |S|, & \text{if } \begin{cases} |S| \leq 2 \text{ and } S \notin \{\{2,4\},\{3,4\}\} \\ \text{or } S \in \{\{2,3,5\},\{1,4,5\},\{1,2,3\}\} \end{cases} \\ \alpha - 4, & \text{if } S = \{2,3,4\} \\ \alpha - 1, & \text{if } |S| = 4, \\ 0, & \text{if } S \in \{\emptyset, \tilde{N}\} \\ \alpha, & \text{otherwise} \end{cases} .$$

We abbreviate w_0 by w. (In the present proof only $w = w_0$ is needed, but different values of the parameter α will be used in two other proofs.) Let $x = (0, 0, 0, 0, 0) \in \mathbb{R}^{\tilde{N}}$ and $u = w^{\{1,2,3,4\},x}$.

[4] If $N \subseteq \mathbb{N}$, i.e., $N = \{i_1, \ldots, i_n\}$, where $i_1 < \cdots < i_n$, then we identify $x \in \mathbb{R}^N$ with $(x_{i_1}, \ldots, x_{i_n})$.

Claim 1: $(\tilde{N}, w)$ is totally balanced.

Let $\emptyset \neq S \subseteq \tilde{N}, S \neq \tilde{N}$. It remains to show that the subgame (S, w) is balanced. We distinguish the following cases:

(1) $|S| \leq 2$: The fact that $w(\{i\}) = -4$ for $i \in \tilde{N}$ and $v(S) \geq -4|S|$ shows balancedness in this case.
(2) $w(S) = 0$: The fact that $w(T) \leq 0$ for $S \subseteq \tilde{N}$ shows balancedness in this case.
(3) $S \in \{\{2,3,5\},\{1,4,5\},\{1,2,3\}\}$: Then (S, w) is inessential, thus the core is non-empty.
(4) $S = \{2,3,4\}$: Then $(-4,-4,4) \in \mathscr{C}(S, w)$ can easily be checked.
(5) $S = \{1,2,3,4\}$: Then $(1,-2,-2,2) \in \mathscr{C}(S, w)$ holds true.
(6) $S = \{1,2,3,5\}$: Then $(1,-1,-1,0) \in \mathscr{C}(S, w)$ holds true.
(7) $S = \{1,2,4,5\}$: Then $(-1,2,-1,-1) \in \mathscr{C}(S, w)$ holds true.
(8) $S = \{1,3,4,5\}$: Then $(-1,2,-1,-1) \in \mathscr{C}(S, w)$ holds true.
(9) $S = \{2,3,4,5\}$: Then $(-1,-1,1,0) \in \mathscr{C}(S, w)$ holds true.

Claim 2: $(\{1,2,3,4\}, u)$ is totally balanced.

Indeed, u is balanced, because $u(S) \leq 0$ for every $S \subseteq \{1,2,3,4\}$ and $u(\{1,2,3,4\}) = 0$. Moreover, $u(S) \geq -4 \cdot |S|, u(\{i\}) = -4$ shows balancedness of one- and two-person subgames. If $S = \{1,2,4\},\{1,3,4\}$, then $u(S) = 0$ and the subgame (S, u) is balanced. Finally, if $S = \{1,2,3\}$ or $S = \{2,3,4\}$, then $u(S) = -1$ and the vector $(1,-1,-1)$ or $(-1,-1,1)$ respectively belongs to the core.

Now the proof can be completed. Claims 1 and 2 show that both, $(\tilde{N}, w)$ and the reduced game $(\{1,2,3,4\}, u)$, belong to Γ. We come up with $s_{ij}(x, w) = 0$ for $i, j \in \tilde{N}$ with $i \neq j$, thus $x \in \sigma(\tilde{N}, u)$ by CRGP and Lemma 4.4. Let $y \in \mathbb{R}^{\{1,2,3,4\}}$ be given by $y = (1,-1,-1,1)$. Then

$$s_{ij}(y, u) = 0 \quad \forall i, j \in \{1,2,3,4\}, \tag{4.1}$$

thus $y \in \sigma(\{1,2,3,4\}, u)$ by CRGP and Lemma 4.4. By WRCP $z = (y, 0) \in \sigma(\tilde{N}, w)$. The fact that $s_{51}(z, w) = 0 > -1 = s_{15}(z, w)$ finishes the proof, because it shows that the reduced game $(\{1,5\}, w^{\{1,5\},z})$ is (N, v) and that $(1,0) \in \sigma(N, v)$ by WRGP. **q.e.d.**

Lemma 4.6. *If* $|\mathscr{U}| \geq 5$ *and* σ *satisfies* ZIG, COV, WRGP, WRCP, CRGP, *and* BOUND, *then* $\mathscr{C}$ *is a subsolution of* σ.

Proof: In view of Lemma 4.3 and Remark 3.7 it suffices to show that $\mathscr{C}(N, v) \subseteq \sigma(N, v)$ for every $(N, v) \in \Gamma$ satisfying $|N| = 2$. Let (N, v) be a balanced two-person game in Γ and let $x \in \mathscr{C}(N, v)$, let us say $N = \{1,2\}$. By COV we can assume that $v(\{1\}) = v(\{2\}) = 0$ and without loss of generality we can assume $x_1 \leq x_2$. If $v(N) = 0$, then Lemma 4.4 finishes the proof. Therefore $v(N) = 1$ can be assumed by COV. Therefore $0 \leq x_1 \leq 1 - x_1 = x_2 \leq 1$ holds true. Take any player $i \in \mathscr{U} \backslash N$, let us say $i = 3$, and define $(\tilde{N}, u)$ with $\tilde{N} = \{1,2,3\}$ by

$$u(S) = \begin{cases} 1 - x_1, & \text{if } S = N, \\ 1, & \text{if } S = \tilde{N} \\ 0, & \text{otherwise} \end{cases}.$$

Then $(\tilde{N}, u)$ is totally balanced. With $y = (0, 1 - x_1, x_1) \in \mathbb{R}^{\tilde{N}}$ we come up with

$$u^{N,y}(\{1\}) = u^{N,y}(\{2\}) = 0, \quad u^{N,y}(N) = 1 - x_1, \tag{4.2}$$

$$u^{\{1,3\},y}(\{1\}) = u^{\{1,3\},y}(\{3\}) = 0, \quad u^{\{1,3\},y}(\{1,3\}) = x_1, \quad \text{and} \tag{4.3}$$

$$u^{\{2,3\},y}(\{2\}) = 1 - x_1, \quad u^{\{2,3\},y}(\{3\}) = 0, \quad u^{\{2,3\},y}(\{2,3\}) = 1, \tag{4.4}$$

thus $y \in \sigma(\tilde{N}, u)$ by Lemma 4.4, Lemma 4.5, COV, and CRGP. Again by Lemma 4.5 (or Lemma 4.4 in the trivial case that $x_1 = 0$) the preimutation which arises from $x_{\{1,3\}}$ by exchanging the components belongs to $\sigma(\{1,3\}, u^{\{1,3\},y})$. WRCP shows that $z = (x_1, 1 - x_1, 0) \in \sigma(\tilde{N}, u)$ holds true, thus WRGP applied to the reduced game $(N, u^{N,z})$ shows that $x \in \sigma(N, u^{N,z})$. However, this reduced game is (N, v). **q.e.d.**

Lemmata 4.3, ..., 4.6 will be used in the proof of Theorem 4.2 as well as in the next section. The following result only applies in the current section.

Lemma 4.7. *If* $|\mathscr{U}| \geq 5, \Gamma = \Gamma_{\mathscr{U}}$, *and* σ *satisfies* AN, COV, WRGP, WRCP, CRGP, *and* BOUND, *then*

$$\sigma(N, v) = \emptyset$$

for every two-person game $(N, v) \in \Gamma_{\mathscr{U}}$ *satisfying* $\mathscr{C}(N, v) = \emptyset$.

Proof: Assume, on the contrary, that there is $\bar{x} \in \sigma(N, v)$. Without loss of generality $N = \{1, 2\}$ can be assumed. Moreover, by COV, we can assume that $v(\{1\}) = v(\{2\}) = 0$ and $v(N) = -1$ hold true. Without loss of generality $\bar{x}_1 \leq \bar{x}_2 = -1 - \bar{x}_1$ (by PO).

Step 1: We prove that the standard solution $x^{(N,v)}$ belongs to $\sigma(N, v)$.

With $\tilde{N} = \{1, 2, 3\}$ we define a game $(\tilde{N}, w)$ by

$$w(S) = \begin{cases} -\bar{x}_1, & \text{if } S = \{1\}, N \\ -\bar{x}_2, & \text{if } |S| = 2 \text{ and } S \neq N \\ \bar{x}_1 - 2\bar{x}_2, & \text{if } |S| = 1 \text{ and } S \neq \{1\} \\ 0, & \text{if } S = \emptyset, \tilde{N} \end{cases}.$$

COV, AN, and CRGP imply that $y^0 = (0, 0, 0) \in \sigma(\tilde{N}, w)$. Indeed, a straightforward computation shows that

$$s_{12}(y^0, w) = s_{13}(y^0, w) = s_{23}(y^0, w) = s_{12}(\bar{x}, v) = -\bar{x}_1$$

and

$$s_{21}(y^0, w) = s_{31}(y^0, w) = s_{32}(y^0, w) = s_{21}(\bar{x}, v) = -\bar{x}_2$$

hold true.

With $a = \bar{x}_2 - \bar{x}_1$ and using AN and COV we come up with $(a, -a) \in \sigma(N, w^{N,y^0})$. Putting $y^1 = (a, -a, 0)$ we get $y^1 \in \sigma(\tilde{N}, w)$ by WRCP.

Note that

$$s_{13}(y^1, w) = s_{31}(y^1, w) = -\bar{x}_1. \tag{4.5}$$

Equation (4.5) (together with WRGP, AN and COV) shows our claim.

Step 2: Now the proof can be finished. Let $\tilde{N}, w_\alpha, x, y, z$ be defined as in the proof of Lemma 4.5 and put $\bar{\alpha} = 1/2$. Then (compare with (4.1))

$$s_{ij}(x, w_{\bar{\alpha}}) = s_{kl}(y, u_{\bar{\alpha}}) = 1/2 \quad \forall i, j \in \tilde{N},\ k, l \in \{1, 2, 3, 4\} \text{ with } i \neq j, k \neq l,$$

where $u_{\bar{\alpha}} = w_{\bar{\alpha}}^{\{1,2,3,4\},x}$. CRGP and Step 1 imply $x \in \sigma(\tilde{N}, w_{\bar{\alpha}})$ and $y \in \sigma(\{1, 2, 3, 4\}, u_{\bar{\alpha}})$, thus $z \in \sigma(\tilde{N}, w_{\bar{\alpha}})$ by WRCP. The fact that

$$s_{51}(z, w_{\bar{\alpha}}) = 1/2 > -1/2 = s_{15}(z, w_{\bar{\alpha}})$$

shows that $(\{1, 5\}, w_{\bar{\alpha}}^{\{1,5\},z})$ is inessential. By WRGP the restriction $z_{\{1,5\}}$ is a member of $\sigma(\{1, 5\}, w_{\bar{\alpha}}^{\{1,5\},z})$, thus, for every $\beta > 0$,

$$(1 + \beta, 1 - \beta)/2 \in \sigma(\{1, 5\}, w_{\bar{\alpha}}^{\{1,5\},z})$$

by COV. This observation contradicts BOUND. **q.e.d.**

Proof of Theorem 4.2: As shown in Sections 2 and 3 the core satisfies the desired properties. In order to show the uniqueness part, let σ be a solution on $\Gamma_{\mathscr{U}}$ which satisfies all properties. In view of Remark 3.7 and Lemma 4.3 it suffices to show that σ coincides with the core on two-person games. Let $(N, v) \in \Gamma_{\mathscr{U}}$ satisfy $|N| = 2$. In view of Lemma 4.6 it remains to show that $\sigma(N, v) \subseteq \mathscr{C}(N, v)$ is true. Assume the contrary. If $\mathscr{C}(N, v) = \emptyset$, then Lemma 4.7 completes the proof. If $\mathscr{C}(N, v) \neq \emptyset$, then we can assume that $N = \{1, 2\}$. With $\tilde{N} = \{1, 2, 3\}$ we define $(\tilde{N}, u)$ by $u(S) = v(S \cap N)$. Choose $x \in \sigma(N, v) \backslash \mathscr{C}(N, v)$ and observe that $(x^{(N,v)}, 0) \in \sigma(\tilde{N}, u)$ by CRGP and Lemma 4.6. Therefore $y = (x, 0) \in \sigma(\tilde{N}, u)$ by WRCP. By WRGP $y_{\{1,3\}} \in \sigma(\{1, 3\}, u^{\{1,3\},y})$. However, the fact that this reduced game is **not** balanced directly leads to a contradiction to Lemma 4.7. **q.e.d.**

Now a further characterization of the core on $\Gamma_{\mathscr{U}}$ can be presented. This characterization employs individual rationality instead of boundedness and anonymity.

Corollary 4.8. *If the universe $\mathcal{U}$ contains at least five players, then the core is the unique solution on $\Gamma_{\mathcal{U}}$ that satisfies* ZIG, COV, WRGP, WRCP, CRGP, *and* IR.

Proof: Individual rationality implies BOUND. In the proof of Theorem 4.2 anonymity is only used to prove Lemma 4.7. If the core of a two-person TU game is empty, then this game does not possess any individually rational and feasible payoff vector. Thus Corollary 4.8 is an immediate consequence of the proof of Theorem 4.2. **q.e.d.**

5. Several classes of games

This section shows that Theorem 4.2 remains valid, even without the anonymity assumption, for certain subsets of $\Gamma_{\mathcal{U}}$, e.g., for the set of totally balanced games, for the set of balanced games, and for the set of superadditive games. This "robustness" of the result may be regarded as an advantage over the "classical" axiomatizations which are crucially based on the assumption that every game under consideration has a nonempty core.

Theorem 5.1. *If the universe $\mathcal{U}$ contains at least five players and Γ is a set of TU games with $\Gamma_{\mathcal{U}}^{tb} \subseteq \Gamma \subseteq \Gamma_{\mathcal{U}}$ which does not contain any nonbalanced two-person TU game, then the core on Γ is the unique solution that satisfies* ZIG, COV, WRGP, WRCP, CRGP, *and* BOUND.

Proof: In order to prove this theorem, the proof of Theorem 4.2 can be literally copied with the exception of the two sentences referring to Lemma 4.7. The first sentence can be dropped, because every game under consideration has a nonempty core. In the last sentence "contradiction to Lemma 4.7" has to be replaced by "contradiction to WRGP, because every two-person game under consideration has a nonempty core". **q.e.d.**

Remark 5.2:

(1) Note that anonymity, which is used in Lemma 4.6 to prove Theorem 4.2, is not needed in Theorem 5.1, because it is not necessary to apply Lemma 4.7.

(2) Theorem 5.1 applies to any set of games containing all totally balanced games and not containing nonbalanced two-person games. Examples of sets of this kind are

(a) the set of all totally balanced games,

(b) the set of all balanced games,

(c) the set of all balanced superadditive games, and

(d) the set of all superadditive games.

In view of the fact that a reduced game of, e.g., a totally balanced game or a superadditive game w.r.t. a core element is not necessarily a totally balanced game or a superadditive game respectively, the core does not satisfy the stronger version RGP *of* WRGP *in these cases. Note that, different from Theorem 5.1, Theorem 4.2 remains valid, if* WRGP *is replaced by the stronger property* RGP.

6. Modifications of the axioms

It is the purpose of this section to discuss some aspects of the impact of several axioms used in Theorem 4.2 and Theorem 5.1. As this section is independent of what follows, some readers may prefer to turn directly to Section 7, which presents the analogous characterization results of the core in the context of NTU games.

It is possible to relax BOUND. Indeed, this property is only used in three proofs. The first occurrence can be located in the proof of Lemma 4.3. In fact, only BOUND^1, i.e., boundedness for one-person games, is needed here. BOUND secondly occurs in the proof of Lemma 4.4 and it is used to show that the standard solution belongs to the solution when applied to any two-person inessential game. BOUND is thirdly used in the proof of Lemma 4.7, actually in the form of BOUND^{2f}, i.e., boundedness, if the solution is restricted to two-person flat games. If $|\mathscr{U}| \geq 2$, then Lemma 4.3 remains true, if BOUND^1 is replaced by BOUND^{2f} and ZIG and RCP are added. Thus BOUND can be replaced by BOUND^{2f} in both Theorems, if WRCP is replaced by RCP. We shall refer to these modified results (see Section 8) by adding a "$*$", i.e., Theorems 4.2* and 5.1* differ from Theorems 4.2 and 5.1 only inasmuch as BOUND and WRCP are replaced by BOUND^{2f} and RCP.

Moreover, the following three results will be proved in Section 9.

Theorem 6.1. *Theorem 5.1 is valid, if* BOUND *is replaced by* AN *and* BOUND^1.

However, it is possible to replace BOUND by AN, if a stronger version of CRGP is used. A solution σ on a set Γ of games satisfies **CRGP***, if the phrase "$x \in X(N,v)$" in the definition of CRGP (see Definition 3.2(3)) is replaced by "$x \in X^*(N,v)$". Moreover, RCP is needed instead of WRCP.

Theorem 6.2. *If the universe $\mathscr{U}$ contains at least five players and if Γ is a set of games with $\Gamma_{\mathscr{U}}^{tb} \subseteq \Gamma \subseteq \Gamma_{\mathscr{U}}$ which does not contain any nonbalanced two-person game, then the core is the unique solution on Γ that satisfies* ZIG, COV, WRGP, RCP, CRGP*, *and* AN.

In order to show the impact of ZIG we now describe **all** solutions that satisfy AN, COV, WRGP, RCP, CRGP, and IR. For proofs see Section 9. Let int$\mathscr{C}$ denote the **interior of the core**, i.e.,

$$\operatorname{int}\mathscr{C}(N,v) = \{x \in \mathscr{C}(N,v) \mid e(S,x,v) < 0 \ \forall \emptyset \neq S \neq N\}.$$

Note that int$\mathscr{C}$ satisfies all properties of the theorems except ZIG. Let Φ denote the *empty solution*, defined by $\Phi(N,v) = \emptyset$ and let Φ^1 be the solution which specifies the empty set for every game with at least two persons and selects the unique Pareto optimal element for every one-person game.

Theorem 6.3. *If the universe $\mathscr{U}$ contains at least five players and if $\Gamma = \Gamma_{\mathscr{U}}$ or if Γ is a set of games with $\Gamma_{\mathscr{U}}^{tb} \subseteq \Gamma \subseteq \Gamma_{\mathscr{U}}$ which does not contain any nonbalanced two-person game, then the unique solutions that satisfy* AN, COV, WRGP, WRCP, CRGP, *and* BOUND *on $\Gamma_{\mathscr{U}}$ or $\Gamma_{\mathscr{U}}^{tb}$ respectively, are $\Phi, \Phi^1, \operatorname{int}\mathscr{C}$, and $\mathscr{C}$.*

Note that AN cannot be dropped as a condition of Theorem 6.3 even in the case of totally balanced games as the following example shows.

Example 6.4. Choose distinct players, let us say 1 and 2, of $\mathscr{U}$. The solution σ, defined by

$$\sigma(N,v) = \begin{cases} \mathscr{C}(N,v), & \text{if } N \subseteq \{1,2\} \\ \emptyset, & \text{otherwise} \end{cases},$$

satisfies all axioms of Theorem 6.3 except AN.

7. The NTU case

A *cooperative nontransferable utility game* – an **NTU** game – is a pair (N, V), where $N \subseteq \mathscr{U}$ is a finite nonvoid set and V is a mapping that assigns to each coalition $\emptyset \neq S \subseteq N$ a subset $V(S)$ of $\mathbb{R}^S$ such that

(1) $V(S)$ is nonempty and comprehensive,
(2) $V(S) \cap (x_S + \mathbb{R}^S_+)$ is bounded for every $x_S \in \mathbb{R}^S$,
(3) $V(S)$ is closed,
(4) if $x_S, y_S \in \partial V(S)$ and $x_S \geq y_S$, then $x_S = y_S$

For interpretations and discussions of the properties (1)–(4) of an NTU game see Peleg (1985). Let $\Delta_{\mathscr{U}}$ denote the set of NTU games. Note that a TU game (N,v) can be regarded as an NTU game (N, V^v) in a canonical way. Indeed, the game (N, V^v) is defined by

$$V^v(S) = X^*(S,v) = \{x \in \mathbb{R}^S \mid x(S) \leq v(S)\}.$$

Hence $\Gamma_{\mathscr{U}}$ can be embedded into $\Delta_{\mathscr{U}}$.

Some definitions are recalled. The feasible payoff $x \in V(N)$ belongs to the *core* $\mathscr{C}(N, V)$ of the NTU game (N, v), if no coalition $S \neq \emptyset$ can improve upon x. Recall that S can *improve* upon x, if there is some $y \in V(S)$ such that $y \gg x_S$ (where "$\gg$" means the strict inequality for every component). A *solution* σ on a set Δ of NTU games is a mapping that associates with every game $(N, V) \in \Delta$ a set $\sigma(N, V) \subseteq V(N)$.

If $x \in V(N)$ and $\emptyset \neq S \subseteq N$, then the *reduced game* $(S, V^{S,x})$ w.r.t. S and x is defined by

$$V^{S,x}(T) = \begin{cases} \{y_S \in \mathbb{R}^S \mid (y_S, x_{N\setminus S}) \in V(N)\}, & \text{if } T = S \\ \bigcup_{Q \subseteq N\setminus S}\{y_T \in \mathbb{R}^T \mid (y_T, x_Q) \in V(T \cup Q)\}, & \text{otherwise} \end{cases}.$$

Note that a reduced game is an NTU game (see Lemma 3.3 of Peleg (1985)).

The straightforward generalizations of NE, RGP, WRGP, CRGP, RCP, IR, BOUND, and AN to a solution σ on a set Δ of NTU games are skipped. We recall the following result which is valid for the set $\Delta^c_{\mathscr{U}}$ of NTU games with a nonempty core.

Theorem 7.1 *(Peleg (1985)).*

(1) If the universe $\mathcal{U}$ *contains at least three players, then the core is the unique solution on* $\Delta_{\mathcal{U}}^c$ *that satisfies* NE, WRGP, CRGP, *and* IR.
(2) If the universe $\mathcal{U}$ *contains infinitely many players, then the core is the unique solution on* $\Delta_{\mathcal{U}}^c$ *that satisfies* NE, RGP, *and* IR.

For every element of the core of an NTU game Peleg constructed an NTU game such that (a) its player set contains the player set of the original game, (b) its core is a singleton, and (c) its reduced game w.r.t. this unique core element coincides with the original game and the restriction of the unique core element is the core element of the original game. Therefore this construction shows that the core does not satisfy RCP in general. However, a suitable extension of WRCP will be defined which, together with some other axioms, allows to characterize the core on $\Delta_{\mathcal{U}}$ and on many subsets. In order to obtain such a "robust" axiomatization of the core, some other properties of a solution on a set of TU games have to be extended to a solution on a set of NTU games. A solution σ on a set Δ of NTU games is said to satisfy

(1) **COV**, if for $(N, V), (N, W) \in \Delta$ with $W = \alpha * V + \beta$ for some[5] $\alpha, \beta \in \mathbb{R}^N$ with $\alpha \gg 0$

$$\sigma(N, W) = \alpha * \sigma(N, V) + \beta$$

holds,

(2) **ZIG**, if for every 0-inessential[6] two-person game $(N, V) \in \Delta$,

$$\sigma(N, V) \neq \emptyset,$$

(3) **WRCP**, if the following condition is satisfied: If $(N, V) \in \Delta, x \in \sigma(N, v)$, $S \subseteq N$ such that $(S, V^{S,x}) \in \Delta$, and $y \in \sigma(S, V^{S,x})$ satisfies

$$\bigcup_{Q \subseteq N \setminus S} \{z \in \mathbb{R}^S \mid z \gg 0, (y + z, x_Q) \in V(S \cup Q)\}$$
$$\subseteq \bigcup_{Q \subseteq N \setminus S} \{z \in \mathbb{R}^S \mid z \gg 0, (x_S + z, x_Q) \in V(S \cup Q)\},$$

then $(y, x_{N \setminus S}) \in \sigma(N, v)$ holds true.

Note that ZIG is an extension of the corresponding property on any set of TU games. A similar property is used in Hart (1985). Note that the core satisfies COV and ZIG on any set of NTU games. Moreover, WRCP is an extension of the corresponding property for solutions on TU games. It is required that the grand coalition of the reduced game is only allowed to make an adjustment which (1) is a member of the solution of the reduced game and (2) which has the property that if the coalition can improve upon this proposal, then it can improve upon the original proposal in the same way. We shall show that the core satisfies this weak reconfirmation property.

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[5] Here $\alpha * x = (\alpha_i x_i)_{i \in N}$, whenever $x \in \mathbb{R}^N$.

[6] An NTU game (N, V) is *a-inessential*, if $a \in \mathbb{R}^N$ satisfies $a_S \in \partial V(S)$ for every coalition $\emptyset \neq S \subseteq N$.

Lemma 7.2. *Let Δ be a set of NTU games. The core satisfies* WRCP *on Δ.*

Proof: Let $(N, V) \in \Delta, x \in \mathscr{C}(N, V), \emptyset \neq S \subseteq N$, and $(S, U) \in \Delta$, where $U = V^{S,x}$. Let $y \in \mathscr{C}(S, U)$ be such that

$$\bigcup_{Q \subseteq N \setminus S} \{z \in \mathbb{R}^S \mid z \gg 0, (y + z, x_Q) \in V(S \cup Q)\}$$

$$\subseteq \bigcup_{Q \subseteq N \setminus S} \{z \in \mathbb{R}^S \mid z \gg 0, (x + z, x_Q) \in V(S \cup Q)\}.$$

Then there is no coalition $Q \subseteq N \setminus S$ such that $S \cup Q$ can improve upon $z = (y, x_{N \setminus S})$, because $x \in \mathscr{C}(N, V)$. By the definition of the reduced game there is no coalition $\emptyset \neq T \subseteq S$ such that $T \cup Q$ can improve upon z for any $Q \subseteq N \setminus S$, because $y \in \mathscr{C}(S, U)$. Finally no nonempty coalition $Q \subseteq N \setminus S$ can improve upon z, because $z_Q = x_Q$. **q.e.d.**

Though the core, when considered as a solution on the set of all NTU games, satisfies the extensions of all axioms employed in Theorem 4.2, we do not know whether the core is the unique solution that has these properties. In what follows we present a "natural" extension of REAS to NTU games.

Definition 7.3. *A solution σ on a set Δ of* NTU *games satisfies* **REAS**, *if for every $(N, V) \in \Delta, x \in \sigma(N, V)$*

$$x_i \geq \min_{S \subseteq N \setminus \{i\}} \max\{t \in \mathbb{R}^{\{i\}} \mid (t, y) \in V(S \cup \{i\}) \ \forall y \in V(S)\}$$

holds true for $i \in N$. Here $V(\emptyset) = \mathbb{R}^{\emptyset}$ and $\max \emptyset = -\infty$ *by convention.*

Note that the "minmax" is well-defined by the definition of an NTU game. Recall that, if a solution is restricted to a set of TU games, then REAS implies BOUND. This is no longer true in the general NTU context[7]. However, it remains true that IR implies both, REAS and BOUND, thus the core satisfies REAS on any set of NTU games. Finally, it should be remarked that every NTU game has feasible payoff vectors that are reasonable.

The results of this section, Theorems 7.4 and 7.5, resemble the assertions of Theorems 4.2 and 5.1. The only formal difference can be seen in the fact that REAS is employed for NTU games instead of BOUND, which was used for TU games.

Theorem 7.4. *If the universe $\mathscr{U}$ contains at least five players, then the core is the unique solution on $\Delta_{\mathscr{U}}$ that satisfies* ZIG, AN, COV, WRGP, WRCP, CRGP, *and* REAS.

Proof: The core satisfies the required properties by its definition, Theorem 7.1, and Lemma 7.2. In order to show the uniqueness part let σ be a solution that satisfies the axioms. We may assume $\{1, 2, 3\} \subseteq \mathscr{U}$. By WRGP and CRGP of

[7] For every two-person NTU game (N, v) such that $0 \in V(\{i\})$ and $V(N)$ is contained in the strictly negative orthant, i.e., $x \in V(N)$ implies $x_i < 0 \ \forall i \in N$, any member of $V(N)$ is reasonable from below.

$\mathscr{C}$ and σ it remains to prove that $\sigma(N, V)$ and $\mathscr{C}(N, V)$ coincide whenever $N \leq 2$.

Claim 1: If (N, W) is strategically equivalent to a TU game (N, v) (i.e., there are $\alpha, \beta \in \mathbb{R}^N, \alpha \gg 0$ such that $W = \alpha * V^v + \beta$), then $\sigma(N, W) = \mathscr{C}(N, W)$.

Indeed, by our extension of COV, this claim is directly implied by Theorem 4.2.

Claim 2: The solution σ is Pareto optimal (i.e., $\sigma(N, V) \subseteq \partial V(N)\ \forall (N, V) \in \Delta_{\mathscr{U}}$ (see Peleg (1985)).

Restricted to one-person games σ is Pareto optimal by REAS. WRGP implies PO in general.

Claim 3: If (N, V) is an inessential two-person game (i.e., if the game is strategically equivalent to a 0-inessential two-person game), then $\sigma(N, V) = \mathscr{C}(N, V)$.

By ZIG and REAS σ assigns the unique element of the core to every 0-inessential two-person game. Therefore the claim is a consequence of COV.

Claim 4: If $(N, V) \in \Delta_{\mathscr{U}}$ is a two-person game with a nonempty core and $i \in N$, then the unique Pareto optimal payoff vector $\bar{x}$ satisfying $\bar{x}_i = \max V(\{i\})$ (which is a member of the core) belongs to $\sigma(N, W)$.

In order to show Claim 4 we may assume without loss of generality that $N = \{2, 3\}$ and $i = 3$. Moreover, by COV, we may assume that $\max V(\{i\}) = 0 \in \mathbb{R}^{\{i\}}$ for $i = 2, 3$, hence $\bar{x} = (a, 0)$ for some $a \geq 0$.

Let $\tilde{N} = \{1, 2, 3\}$ and define $(\tilde{N}, W)$ by

$$W(S) = \begin{cases} \{y \in \mathbb{R}^{\tilde{N}} \mid (2y_1 + y_2 + 2a, y_3) \in V(N)\}, & \text{if } S = \tilde{N} \\ \{y \in \mathbb{R}^S \mid y(S) \leq -a\}, & \text{if } S = \{1, 2\} \\ \{y \in \mathbb{R}^S \mid y \leq 0\}, & \text{if } S = \{3\} \\ \{y \in \mathbb{R}^S \mid y(S) \leq -\mu\}, & \text{otherwise} \end{cases},$$

where $\mu = 1 + a$, and observe that $(\tilde{N}, W)$ is an NTU game. With $z = (0, -a, 0) \in \mathbb{R}^{\tilde{N}}$ we claim that $z \in \sigma(\tilde{N}, W)$. Indeed, for the coalitions S of the form $\{1, 3\}$ and $\{2, 3\}$ the games $(S, \tilde{V}^{S,z})$ are inessential games and z_S is the unique element of the core. Moreover, the coalitional function $U = \tilde{V}^{\{1,2\},z}$ can be computed as

$$U(T) = \begin{cases} \{y \in \mathbb{R}^T \mid 2y_1 + y_2 \leq -a\}, & \text{if } T = \{1, 2\} \\ \{y \in \mathbb{R}^T \mid y \leq -\mu\}, & \text{otherwise} \end{cases},$$

thus $(\{1, 2\}, U)$ is strategically equivalent to a TU game and $z_{\{1,2\}} \in \mathscr{C}(\{1, 2\}, U)$. Claims 1 and 3 together with CRGP show that $z \in \sigma(\tilde{N}, W)$. Moreover, the vector $(-a, a) \in \mathbb{R}^{\{1,2\}}$ belongs to the core of this reduced game, thus

$(-a, a) \in \mathscr{C}(\{1,2\}, U)$ holds true again by Claim 1. It is straightforward to verify that $y = (-a, a, 0) \in \mathbb{R}^{\tilde{N}}$ belongs to the core of $(\tilde{N}, W)$, thus $y \in \sigma(\tilde{N}, W)$ by WRCP. WRGP and the fact that $(N, W^{N,y}) = (N, V)$ shows the assertion.

Claim 5: If $(N, V) \in \Delta_{\mathscr{U}}^c$ is a two-person game, then $\mathscr{C}(N, V) = \sigma(N, V)$.

By REAS $\sigma(N, V) \subseteq \mathscr{C}(N, V)$. In order to prove the other inclusion we shall employ the same assumptions concerning (N, V) as in Claim 4. Take $(a, b) \in \mathscr{C}(N, V)$ and define $(\tilde{N}, W)$ as in Claim 4. Put $z = (0, -a, b)$ in this case and observe that $z_S \in \sigma(S, W^{S,z})$ by Claims 4 and 1. Moreover, $(-a, a) \in \mathbb{R}^{\{1,2\}}$ belongs to the corresponding reduced game by Claim 1 and $(-a, a, b)$ belongs to the core of the original game $(\tilde{N}, W)$, thus $(-a, a, b) \in \sigma(\tilde{N}, W)$ by WRCP. The proof is completed by the observation that $W^{N,(-a,a,b)} = V$ holds true.

Claim 6: If $(N, V) \in \Delta_{\mathscr{U}}$ is a two-person game that satisfies $\mathscr{C}(N, V) = \emptyset$, then $\sigma(N, V) = \emptyset$ is also valid.

Assume, on the contrary, $\sigma(N, V) \neq \emptyset$. Assume $N = \{1,2\}$, $0 = \max V(\{i\})$ and $(a, b) \in \sigma(N, V)$ satisfies $a \geq b$. By REAS (applied to player 2) $a \leq 0$, thus $b < 0$, because $(a, b) \in V(N)$. With $\tilde{N} = \{1,2,3\}$ define $(\tilde{N}, W)$ by

$$W(S) = \begin{cases} \{y \in \mathbb{R}^S \mid (y_1 + y_2 - a, y_3) \in V(N)\}, & \text{if } S = \tilde{N} \\ \{y \in \mathbb{R}^S \mid \max\{y_1 + \rho y_2, y_1 + y_2\} \leq a\}, & \text{if } S = \{1,2\} \\ \{y \in \mathbb{R}^S \mid y(S) \leq 0\}, & \text{if } S = \{3\} \\ \{y \in \mathbb{R}^S \mid y(S) \leq \mu\}, & \text{otherwise} \end{cases},$$

where $\mu = 2a + b - 1$ and $\rho = 1 - a - b$. Put $x = (a, a, b) \in \mathbb{R}^{\tilde{N}}$ and observe that for coalitions $S \in \{\{1,3\}, \{2,3\}\}$ the reduced games $(S, W^{S,x})$ are isomorphic to (N, V), thus $x_S \in \sigma(S, W^{S,x})$ by AN. For $S = \{1,2\}$ we obtain

$$W^{S,x}(T) = \begin{cases} \{y \in \mathbb{R}^T \mid y(T) \leq 2a - 1\}, & \text{if } |T| = 1 \\ \{y \in \mathbb{R}^T \mid y(T) \leq 2a\}, & \text{if } T = S \end{cases},$$

hence $x_S \in \sigma(S, W^{S,x})$ by Claim 1. The application of the CRGP yields $x \in \sigma(\tilde{N}, W)$. The vector $(2a - 1, 1) \in \mathbb{R}^N$ belongs to the core of the corresponding reduced game. Moreover, the coalition $\{1,2\}$ cannot improve upon $z = (2a - 1, 1, b) \in \mathbb{R}^{\tilde{N}}$, thus $z \in \sigma(\tilde{N}, W)$ holds true by WRCP. By WRGP $z_{\{1,3\}} = (2a - 1, b)$ belongs to $\sigma(\{1,3\}, W^{\{1,3\},z})$. This reduced game can be computed as

$$W^{\{1,3\},z}(T) = \begin{cases} \{y \in \mathbb{R}^T \mid y(T) \leq \mu\}, & \text{if } T = \{1\} \\ \{y \in \mathbb{R}^T \mid y(T) \leq 0\}, & \text{if } T = \{3\} \\ \{y \in \mathbb{R}^T \mid (y_1 + 1 - a, y_3) \in V(N)\}, & \text{if } T = \{1,3\} \end{cases}.$$

However, by REAS, we obtain

$$b = z_3 \geq \min\{0, \max\{t \in \mathbb{R}^{\{3\}} \mid (\mu + 1 - a, t) \in V(N)\}\}$$

$$= \min\{0, \max\{t \in \mathbb{R}^{\{3\}} \mid (a + b, t) \in N(N)\}\} > b$$

by condition (4) of the definition of an NTU game and, hence, the desired contradiction is established. **q.e.d.**

In order to prove an assertion which is the analogue of Theorem 5.1 the definition of "balancedness" in the NTU context is recalled. A collection $\mathcal{S} \subseteq 2^N \setminus \{\emptyset, N\}$ of coalitions of the finite set $\emptyset \neq N \subseteq \mathcal{U}$ is said to be *balanced*, if there are real numbers γ_S for $S \in \mathcal{S}$ which satisfy (a) $\gamma_S > 0$ $\forall S \in \mathcal{S}$ and (b) $\sum_{S \in \mathcal{S}} \gamma_S 1_S = 1_N$. The γ_S are *balancing coefficients* of $\mathcal{S}$. An NTU game (N, V) is *balanced*, if for every balanced collection $\mathcal{S}$ of N together with balancing coefficients $(\gamma_S)_{S \in \mathcal{S}}$ and every collection $(y^S)_{S \in \mathcal{S}}$ which satisfies $y^S \in \mathbb{R}^N, y^S_i = 0$ $\forall i \in N \setminus S$, and $y^S_S \in V(S)$ $\forall S \in \mathcal{S}$

$$\sum_{S \in \mathcal{S}} \gamma_S y^S \in V(N).$$

Note that balanced NTU games have a nonempty core but – unlike in the TU context – there are NTU games with a nonempty core that are not balanced (see Scarf (1967) who employed even a "weaker" version of balancedness). It should be remarked that a two-person NTU game is balanced, if and only if its core is nonempty. Let $\Delta^{tb}_{\mathcal{U}}$ denote the set of totally balanced NTU games.

Theorem 7.5. *If the universe $\mathcal{U}$ contains at least five players and Δ is a set of NTU games with $\Delta^{tb}_{\mathcal{U}} \subseteq \Delta \subseteq \Delta_{\mathcal{U}}$ which does not contain any nonbalanced two-person NTU game, then the core on Δ is the unique solution that satisfies* ZIG, COV, WRGP, WRCP, CRGP, *and* REAS

This theorem is proved in the Appendix.

8. Examples, a summarizing diagram, and remarks

First of all seven examples of solutions are presented which satisfy many of the axioms discussed in this paper. These examples give some insight to the "sensitivity" of the main results. The seven solutions $\sigma^1, \ldots, \sigma^7$ are going to be defined for every NTU game $(N, V) \in \Delta_{\mathcal{U}}$ and can, thus, be restricted to any of the considered sets of games mentioned in this paper:

$$\sigma^1(N, V) = \begin{cases} V(N) \setminus \partial V(N), & \text{if } |N| \leq 2 \\ \emptyset, & \text{otherwise} \end{cases},$$

$$\sigma^2(N, V) = \begin{cases} \mathcal{C}(N, V), & \text{if } |N| \geq 2 \\ V(N), & \text{if } |N| = 1 \end{cases},$$

$$\sigma^3(N, V) = \begin{cases} \mathcal{C}(N, V), & \text{if } (N, V) \text{ is inessential} \\ \emptyset, & \text{otherwise} \end{cases},$$

$$\sigma^4(N, V) = \begin{cases} \sigma^3(N, V), & \text{if } |N| \le 2 \\ \{y \in \mathscr{C}(N, V) \mid y_S \in \sigma^3(S, V^{S,x}) \ \forall S \subseteq N \text{ with } |S| = 2\}, & \text{otherwise} \end{cases},$$

$$\sigma^5(N, V) = \{y \in \partial V(N) \mid y_i \ge \max V(\{i\}) \ \forall i \in N\},$$

$$\sigma^6(N, V) = \begin{cases} \{0\}, & \text{if } 0 \in \mathscr{C}(N, V) \\ \emptyset, & \text{otherwise} \end{cases}.$$

In order to define σ^7, choose $\tilde{N} \subseteq \mathscr{U}$ satisfying $|\tilde{N}| = 2$, let us say $\tilde{N} = \{1, 2\}$, and let $(\tilde{N}, U)$ be the NTU game which corresponds to the $(0 - (-1))$-normalized TU game $(\tilde{N}, u)$, defined by $u(i) = 0 \ \forall i \in \tilde{N}$ and $u(\tilde{N}) = -1$. Then define

$$\sigma^7(N, V) = \begin{cases} \mathscr{C}(N, V), & \text{if } V \text{ is not strategically equivalent to } U \\ \alpha * \{y \in \partial U(\tilde{N}) \mid y_1 = y_2\} + \beta, & \text{if } \exists \alpha, \beta \in \mathbb{R}^{\tilde{N}} \text{ with } \alpha \gg 0, \ V = \alpha * U + \beta \end{cases}.$$

Table 1 shows which axiom of the mentioned properties is satisfied by these solutions. A "–" means that the solution does not satisfy the axiom on any set Γ or Δ of games with $\Gamma^{tb}_{\mathscr{U}} \subseteq \Gamma$ or $\Delta^{tb}_{\mathscr{U}} \subseteq \Delta$, whenever the universe $\mathscr{U}$ contains at least five players. A "+" means that the property is satisfied. Finally "⊕" means that the axiom occurs in the corresponding characterization. Here "Theorem 4.2*" and "Theorem 5.1*" refers to the modification of Theorem 4.2 and Theorem 5.1 described in the second paragraph of Section 6.

Remark 8.1:

(1) In Section 9 we prove (see Lemmata 9.1 and 9.2) that σ^7 satisfies CRGP *and that σ^4 applied to any NTU game of at most three persons specifies a singleton or the empty set. The proofs of the other properties of the solutions $\sigma^1, \ldots, \sigma^7$ claimed in Table 1 are straightforward and not presented.*

(2) Table 1 shows that the properties used in the main results, Theorems 4.2, 5.1, 7.4, and 7.5, and in Corollary 4.8 are logically independent, as well as the properties used in the modifications of Theorems 4.2 and 5.1 presented in Section 6. Thus these characterizations of the core are, in fact, axiomatizations. Note that there are additional examples which show that the remaining Theorems 6.1 and 6.2 also constitute axiomatizations of the core.

(3) Note that the solution σ^4 satisfies the properties (see Table 1) that are used in the Theorems as long as the universe $\mathscr{U}$ contains at most four players. Thus the assumption $|\mathscr{U}| \ge 5$ is sharp in these results.

(4) Theorems 4.2 and 5.1 remain valid, if boundedness is replaced by reasonableness from below. Indeed REAS *implies* BOUND *for every solution which is defined on a set of TU games. A solution on a set of NTU games*

Table 1. Solutions and properties

	Φ	σ^7	σ^6	σ^5	σ^4	σ^3	σ^2	σ^1	$\mathscr{C}$	Theorem 4.2	4.2*	5.1	5.1*	7.4	7.5	Cor. 4.8
ZIG	−	+	+	+	+	+	+	+	+	⊕	⊕	⊕	⊕	⊕	⊕	⊕
AN	+	−[a]	+	+	+	+	+	+	+	⊕	⊕			⊕		
COV	+	+	−	+	+	+	+	+	+	⊕	⊕	⊕	⊕	⊕	⊕	⊕
WRGP	+	+	+	−	+	+	+	+	+	⊕	⊕	⊕	⊕	⊕	⊕	⊕
WRCP	+	+	+	+	−[b]	+	+	+	+	⊕		⊕		⊕	⊕	⊕
RCP	+	+[c]	+	+	−[b]	+	−	+	+[c]		⊕		⊕			
CRGP	+	+	+	+	+	−	+	+	+	⊕	⊕	⊕	⊕	⊕	⊕	⊕
BOUND	+	+	+	+	+	+	−	−	+	⊕		⊕				
BOUND[2f]	+	+	+	+	+	+	+	−	+		⊕		⊕			
IR	+	−[a]	+	+	+	+	−	−	+							⊕
REAS	+	+	+	+	+	+	−	−	+				⊕	⊕		

[a] If the class of games under consideration does not contain any game strategically equivalent to the $(0-(-1))$-normalized game $(\{1,2\}, U)$, then σ^7 satisfies this axiom.
[b] If $|\mathscr{U}| \leq 4$, then the axiom is satisfied.
[c] RCP is not satisfied on several classes of NTU games.

which satisfies REAS *does not necessarily satisfy* BOUND. *Note that we do not know whether it is possible to replace* REAS *by* BOUND *in Theorems 7.4 and 7.5.*

(5) Theorem 4.2 in the TU case and Theorem 7.1 in the NTU case can be regarded as implicit characterizations of the sets of TU and NTU games respectively which have a nonempty core. Indeed, a game has a nonempty core, if and only if the axiomatized solution applied to this game specifies a nonempty set. In the TU context a game with a nonempty core can be characterized (see Bondareva (1963) and Shapley (1967)) by balancedness. As far as we know, the class of NTU games possessing a nonempty core has not been characterized before.

9. Appendix

Proof of Theorem 6.1: Let σ satisfy the required axioms. In view of the second paragraph of Section 6 it suffices to show that $\sigma(N,v)$ is bounded, whenever $(N,v) \in \Gamma_{\mathscr{U}}$ is a flat two-person game. Assume, on the contrary, that $\sigma(N,v)$ is not bounded for the flat two-person game (N,v) and let us assume $N = \{1,2\}$ for simplicity. By AN $(x_1,x_2) \in \sigma(N,v)$ holds true, if and only if $(x_2,x_1) \in \sigma(N,v)$ holds true. By COV $\alpha \times (x_1,x_2) \in \sigma(N,v)$ holds, whenever $(x_1,x_2) \in \sigma(N,v)$ and $\alpha > 0$. Therefore we obtain

$$\sigma(N, v) \supseteq X(N, v) \backslash \{(0, 0)\}.$$

Choose $x \in \sigma(N, v)$ satisfying $x_1 < 0 < x_2 = -x_1$. Let $(\tilde{N}, w), y^0, y^1$ be defined as in the proof of Step 1 of Lemma 4.7. It is easy to check that this game is totally balanced, thus $y^0 \in \sigma(\tilde{N}, w)$ by COV, CRGP, and AN. Therefore $y^1 \in \sigma(\tilde{N}, w)$ by WRCP. However, the reduced game $(\{1, 3\}, w^{\{1,3\}, y^1})$ is not balanced. **q.e.d.**

Proof of Theorem 6.2: Let σ satisfy the required axioms. It suffices to show that $\sigma(N, v)$ is Pareto optimal for every flat two-person game $(N, v) \in \Gamma_{\mathscr{U}}$. Assume the contrary and let (N, v) be a flat game, let us say $N = \{1, 2\}$, and let $x \in \sigma(N, v)$ satisfy $x_2 \neq -x_1$, i.e., $x_2 < -x_1$. By AN we can assume $x_1 \leq x_2$. Take $i \in \mathscr{U} \backslash N$, let us say $i = 3$, and let $\tilde{N} = \{1, 2, 3\}$ and let $(\tilde{N}, w)$ be defined by

$$w(S) = \begin{cases} -x_1, & \text{if } S = \{1\}, N \\ -x_2, & \text{if } |S| = 2 \text{ and } S \neq N \\ \min\{x_1 - x_2, x_1 - 2x_2\}, & \text{if } S = \{2\}, \{3\} \\ -x_1 - x_2, & \text{if } S = \tilde{N} \\ 0, & \text{if } S = \emptyset \end{cases}.$$

Then $(\tilde{N}, w)$ is balanced, because $(-x_1, 0, -x_2) \in \mathscr{C}(\tilde{N}, w)$. Moreover, it is easy to check that this game is totally balanced. With $y = (0, 0, 0)$ the reduced two-person games are flat, thus AN, COV, CRGP*, and the equalities

$$s_{12}(y, w) = s_{13}(y, w) = s_{23}(y, w) = -x_1 = s_{12}(x, v)$$

and

$$s_{21}(y, w) = s_{31}(y, w) = s_{32}(y, w) = -x_2 = s_{21}(x, v)$$

show that $y \in \sigma(\tilde{N}, w)$. By WRGP $0 \in \sigma(\{3\}, w^{\{3\}, y})$. By COV $t \in \sigma(\{3\}, w^{\{3\}, y})$ is true for every $t < w^{\{3\}, y}(\{3\}) = -x_1 - x_2 > 0$, thus $y^t = (0, 0, t) \in \sigma(\tilde{N}, w)$ by RCP. Choose $t < x_1 - x_2$ and observe that (N, w^{N, y^t}) is **not** balanced. **q.e.d.**

Proof of Theorem 6.3: The four solutions satisfy the axioms. Let σ be a solution with the desired properties and let us assume that σ is neither the empty solution Φ, nor Φ^1, nor the core. By Theorem 4.2 and Theorem 5.1 respectively, σ does not satisfy ZIG. However, by Lemma 4.3, it satisfies PO, thus, by COV and WRGP, Φ^1 is a proper subsolution of σ. Let $(N, v) \in \Gamma$ be a two-person game.

Claim 1: $\sigma(N, v) \neq \emptyset$ holds true, if and only if $\operatorname{int} \mathscr{C}(N, v) \neq \emptyset$.

If (N, v) is flat or if $\mathscr{C}(N, v) = \emptyset$ respectively, then $\sigma(N, v) = \emptyset$ holds true by COV, because σ does not satisfy ZIG, or by Lemma 4.7 in case $\Gamma = \Gamma_{\mathscr{U}}$ respectively. WRGP, PO, and the fact that Φ^1 is a proper subsolution of σ imply this claim.

Claim 2: If $\operatorname{int}\mathscr{C}(N,v) \neq \emptyset$, then $\sigma(N,v)$ contains the standard solution $x^{(N,v)}$.

We shall assume $N = \{1,2\}$ and, by COV, we may assume that (N,v) is the unanimity game. Choose $\bar{x} \in \sigma(N,v)$ satisfying $\bar{x}_1 \leq \bar{x}_2 = 1 - \bar{x}_1$, which is possible by AN, PO and Claim 1. As in Step 1 of the proof of Lemma 4.7 we define $(\tilde{N},w)$ and $y^0 = (0,0,0)$. Unlike in the mentioned proof, the game $(\tilde{N},w)$ is totally balanced here. Indeed, it can easily be checked that the two-person subgames are balanced and that, e.g., the vector $(-\bar{x}_1, 0, \bar{x}_1)$ belongs to the core. Hence $(\tilde{N},w) \in \Gamma$ is true. As in the mentioned proof COV, AN, and CRGP imply that $y^0 \in \sigma(\tilde{N},w)$. With $a = \bar{x}_2 - \bar{x}_1$ AN implies that $(a,-a) \in \sigma(\{1,3\}, w^{\{1,3\},y^0})$, thus $y^2 = (a,0,-a) \in \sigma(\tilde{N},w)$ by WRCP. The fact that $(0,a)$ is the standard solution of the game $(\{2,3\}, w^{\{2,3\},y^2})$ and that the core of this game has a nonempty interior, implies the claim by COV, AN, and WRGP.

Claim 3: σ is a subsolution of $\operatorname{int}\mathscr{C}$.

In view of Remark 3.7 and Claim 2 it suffices to show the desired inclusion for a two-person game (N,v) satisfying $\operatorname{int}\mathscr{C}(N,v) \neq \emptyset$. We proceed as in the proof of Theorem 4.2 by defining the game $(\tilde{N},u)$ which is totally balanced. Here the vector $(x^{(N,v)},0)$ is a member of $\sigma(\tilde{N},u)$ by CRGP and Claim 2. WRCP, WRGP, and Claim 1 imply the claim.

Claim 4: $\operatorname{int}\mathscr{C}(N,v) \subseteq \sigma(N,v)$.

Let $(\tilde{N},w_\alpha)$ and x,y be defined as in the proof of Lemma 4.5 and let $\bar{x} \in \operatorname{int}\mathscr{C}(N,v)$. By PO $\bar{x}_2 = 1 - \bar{x}_1$ and by AN we can assume that $\bar{x}_1 \leq \bar{x}_2$. Moreover, $\bar{x}_1 \geq 0$ holds true by Claim 3. Put $y^\beta = \beta \cdot y$ for $\beta \in \mathbb{R}$ and let $\tilde{\alpha} = -\bar{x}_1, \tilde{\beta} = 1 - 2\bar{x}_1 = \bar{x}_2 - \bar{x}_1$. Then $(\tilde{N},w_{\tilde{\alpha}})$ is totally balanced, because $\tilde{\alpha} \leq 0$. CRGP and Claim 2 imply $x \in \sigma(\tilde{N},w_{\tilde{\alpha}})$. With $u_{\tilde{\alpha}} = w_{\tilde{\alpha}}^{\{1,2,3,4\},x}$ we obtain that $(\{1,2,3,4\},u_{\tilde{\alpha}})$ is totally balanced, thus $y^{\tilde{\beta}} \in \sigma(\{1,2,3,4\},u_{\tilde{\alpha}})$ by the same reasons. Finally $z^{\tilde{\beta}} = (y^{\tilde{\beta}},0) \in \sigma(\tilde{N},w_{\tilde{\alpha}})$ holds true by WRCP. However,

$$s_{51}(\tilde{N},w_{\tilde{\alpha}}) = \tilde{\alpha}, \quad s_{15}(\tilde{N},w_{\tilde{\alpha}}) = \tilde{\alpha} - \tilde{\beta} = -x_2,$$

thus COV and WRGP imply Claim 4.

The proof is finished by CRGP. **q.e.d.**

Proof of Theorem 7.5: The core satisfies the required properties. In order to show the opposite implication, let σ be a solution on Δ which satisfies ZIG, COV, WRGP, WRCP, CRGP, and REAS. It is sufficient to repeat the relevant parts of the proof of Theorem 7.4. In that proof AN is only used to prove Claim 6, which is redundant in the present case, because every two-person game under consideration is assumed to be balanced. In view of Theorem 5.1 Claims 1, 2, and 3 are valid. In order to prove Claims 4 and 5 the games W have to be modified in such a way that they are totally balanced. This can be done simultaneously in both claims. Indeed, put $b = 0$ in Claim 4 and assume, by Claim 3, that V is not inessential. Thus $a > 0$ and $a \geq b$ can also be assumed. Observe that every NTU game $(\tilde{N},\tilde{W})$ which has the following properties, can be used to verify our claims.

(1) $\tilde{W}(\tilde{N}) = W(\tilde{N}) \cup \left\{ y \in \mathbb{R}^{\tilde{N}} \,\middle|\, \begin{array}{c} (y_1+y_2+a, y_3) \in V(N) \text{ and} \\ (2y_1+y_2+2a, y_3-(-2y_1-y_2-a)_+) \in V(N) \end{array} \right\}$,
where t_+ denotes the positive part of the real number t.
(2) $\tilde{W}(\{1,2\}) = W(\{1,2\}) \cap \{y \in \mathbb{R}^{\{1,2\}} | 2y_1 + y_2 \leq -a\}$.
(3) $(-\mu, 0) \in \tilde{W}(\{j,3\}) \subseteq W(\{j,3\}) \; \forall j = 1, 2$.
(4) $\tilde{W}(\{j\}) = W(\{j\}) \; \forall j \in \tilde{N}$.

Hence, there is a game $(\tilde{N}, \tilde{W})$ with the foregoing properties such that

$$\tilde{W}(\{j,3\}) \subseteq \{y \in \mathbb{R}^{\{j,3\}} \mid y \leq (-\mu + c, c)\} \quad \forall j = 1, 2,$$

where $1/2 \geq c > 0$ is chosen in such a way that $(0, c) \in V(N)$. Then all proper subgames are balanced. In order to show that the game is balanced observe that for every coalition S of the form $\{j\}$ or $\{j, 3\}$ with $j = 1, 2$

$$2y_1^S + y_2^S \leq -a - 1/2, \quad y_3^S \leq c \quad \forall y^S \in \mathbb{R}^{\tilde{N}} \text{ with } y_S^S \in V(S) \tag{9.1}$$

holds true and

$$2y_1 + y_2 \leq -a \quad \forall y \in V(\{1,2\}). \tag{9.2}$$

Moreover, for every $\emptyset \neq S \neq \tilde{N}, \{3\}$

$$y_1^S + y_2^S \leq -a \quad \forall y^S \in \mathbb{R}^S \text{ with } y_S^S \in V(S) \tag{9.3}$$

is valid. If $\mathscr{S}$ is a balanced collection with balancing coefficients γ_S and if $y^S \in \mathbb{R}^S$ satisfy $y_S^S \in V(S)$ for all $S \in \mathscr{S}$, then with $y = \sum_{S \in \mathscr{S}} \gamma_S y^S, \gamma_T = 0$, and $y^T = 0$ for every $T \notin \mathscr{S}$

$$\begin{aligned} 2y_1 + y_2 \leq & \sum_{S \in \mathscr{S} \text{ with } 1 \in S \text{ or } 2 \in S} \gamma_S(-a) \\ & + \sum_{S \in \mathscr{S} \text{ with } S \neq \{3\}, \{1,2\}} \gamma_S(-1/2) \text{ by (9.1) and (9.2)}, \end{aligned}$$

$$y_1 + y_2 \leq \sum_{S \in \mathscr{S} \text{ with } 1 \in S \text{ or } 2 \in S} \gamma_S(-a) \text{ by (9.3)},$$

and

$$y_3 \leq c(\gamma_{\{1,3\}} + \gamma_{\{2,3\}}) \text{ by (9.1)}$$

hold true. By balancedness

$$\sum_{S \in \mathscr{S} \text{ with } 1 \in S \text{ or } 2 \in S} \gamma_S \geq 1 \quad \text{and} \quad \gamma_{\{1,3\}} + \gamma_{\{2,3\}} \leq 1,$$

thus

$$2y_1 + y_2 \leq -a - 1/2(\gamma_{\{1,3\}} + \gamma_{\{2,3\}}) \quad \text{and} \quad y_1 + y_2 \leq -a.$$

We conclude that $(\tilde{N}, \tilde{W})$ is balanced. **q.e.d.**

Lemma 9.1. *If* (N, V) *is a three-person NTU game, then* $|\sigma^4(N, V)| \leq 1$.

Proof: Assume that $x, y \in \sigma^4(N, V)$ for some three-person game (N, V) with $N = \{1, 2, 3\}$. If $x_i = y_i$ for some $i \in N$, then $x = y$, because $V^{N\backslash\{i\},x} = V^{N\backslash\{i\},y}$. Assume, on the contrary $x \neq y$. Then we may assume without loss of generality that $y_3 > x_3$ and $x_i > y_i \geq \max V(\{i\})$ for $i = 1, 2$. Therefore, by definition of σ^4, $x_{\{1,2\}} \in \partial V(\{1, 2\})$, thus $y \notin \mathscr{C}(N, V)$. **q.e.d.**

We conclude that σ^4 satisfies RCP, if the universe $\mathscr{U}$ contains at most four players.

Lemma 9.2. σ^7 *satisfies CRGP.*

Proof: Let $N \subseteq \mathscr{U}$ with $|N| \geq 3$ and (N, V) be a game. Moreover, let $x \in \partial V(N)$ satisfy $x_S \in \sigma^7(S, V^{S,x})$, whenever $S \subseteq N$ with $|S| = 2$. If $\tilde{N} = \{1, 2\} \nsubseteq N$, then $x \in \mathscr{C}(N, V)$ by definition and CRGP of $\mathscr{C}$. If $S = \{1, 2\} \subseteq N$ and $(\tilde{N}, V^{\tilde{N},x})$ is not strategically equivalent to $(\tilde{N}, U)$, then, again by definition of σ^7 and CRGP of $\mathscr{C}$, the vector x belongs to the core. It suffices to show that the reduced game $(\tilde{N}, V^{\tilde{N},x})$ cannot be strategically equivalent to $(\tilde{N}, U)$. Assume the contrary. By COV we may assume that $V^{\{1,2\},x}$ coincides with U. Then, by definition of σ^7, we have $x_1 = x_2 < 0$. On the other hand $\max V^{\tilde{N},x}(\{i\}) = 0\ \forall i = 1, 2$, thus there is a coalition $T \subseteq N\backslash\{2\}$ with $1 \in T$ and $(0, x_{T\backslash\{1\}}) \in V(T)$. If $T = N\backslash\{2\}$, then for any $j \in T\backslash\{1\}$ there is some $\varepsilon > 0$ such that $(x_j + \varepsilon, x_{T\backslash\{j\}}) \in V(T)$ by condition (4) of the definition of an NTU game. This observation yields $x_{\{1,j\}} \notin \mathscr{C}(\{1, j\}, V^{\{1,j\},x})$ and, thus a contradiction. In the other case take $j \in N\backslash(T \cup \{2\})$ and observe that the fact $0 \in \{y | (y, x_{T\backslash\{1\}}) \in V(T)\} \subseteq V^{\{1,j\},x}(\{1\})$ yields a contradiction. **q.e.d.**

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