ZOR – Mathematical Methods of Operations Research (1995) 42:189–216

# Directed and Weighted Majority Games

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*Abstract:* In this paper we deal with several classes of simple games; the first class is the one of ordered simple games (i.e. they admit of a complete desirability relation). The second class consists of all zero-sum games in the first one.

First of all we introduce a "natural" partial order on both classes respectively and prove that this order relation admits a rank function. Also the first class turns out to be a rank symmetric lattice. These order relations induce fast algorithms to generate both classes of ordered games.

Next we focus on the class of weighted majority games with $n$ persons, which can be mapped onto the class of weighted majority zero-sum games with $n + 1$ persons.

To this end, we use in addition methods of linear programming, styling them for the special structure of ordered games. Thus, finally, we obtain algorithms, by combining *LP*-methods and the partial order relation structure. These fast algorithms serve to test any ordered game for the weighted majority property. They provide a (frequently minimal) representation in case the answer to the test is affirmative.

## 0 Introduction

In this paper we consider four subclasses of simple games. A simple game is a cooperative multiperson game in which each coalition either wins, i.e. is all powerful ("obtains a fixed positive payoff"), or loses, i.e. is completely ineffective ("obtains no payoff"). Frequently it is possible to specify the winning coalitions of a simple game by assigning nonnegative weights to the players such that the aggregated weight of each winning coalition exceeds a positive level or coincides with it, whereas the weight of each losing coalition is less than this level. The vector which consists of both, the level and the weights of the players, is known as a representation of the game, which is in this case a weighted majority game. The set of these games is the first of the considered subclasses. For the explicit definitions we refer to Section 1.

The terms "simple" and "weighted majority" were introduced by *v.* Neumann and Morgenstern (1944). Both simple and weighted majority games appear in many applications of game theory (see e.g. Shapley (1962)).

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Up to now no direct procedure of generating the class of weighted majority $n$-person games is known. An indirect method consists of checking the representability of each simple $n$-person game. It is well-known – see Maschler and Peleg (1966) – that weighted majority games have a complete desirability relation, i.e. are (up to a permutation of the set of players) directed games. Thus only directed games have to be tested as soon as a procedure to generate them is known. The second class of considered simple games is the one of directed games. The remaining two subsets consist of all zero-sum games in the classes just mentioned.

In Section 1 we introduce our basic notation and recall explicitely the strong relationship between $n$-person weighted majority games and $n+1$-person weighted majority zero-sum games. Therefore it is sufficient to consider directed zero-sum games in order to test representability.

In Section 2 it turns out that both, the class of directed games and the one of directed zero-sum games with a fixed number of players, are ranked partially ordered sets in a natural way. This fact leads to the important consequence that the maximal number of shift minimal coalitions of a directed game can be computed recursively with respect to the number of players – Proposition 2.3. Moreover the structure of these partially ordered sets allows to construct algorithms which generate these games.

Section 3 presents these generating procedures. The main result of this section is Proposition 3.4 applied to directed zero-sum games.

Section 4 shows methods to compute a representation of a directed game if this game is a weighted majority game. These algorithms can be used to test an arbitrary directed game for the weighted majority property. They are based on the Simplex Method and can easily be rewritten in such a way that the eventual representation is an integer one, as shown in the Concluding Remarks of this section. Moreover it turns out that the knowledge of the completeness of the desirability relation generically reduces the sizes of the considered tableaux – compared with methods not using the completeness of this relation – and thus the number of necessary pivot steps. To be more precise, Lemma 4.5, which is the main theoretical result of Section 4, can be applied to replace the incidence matrix by the shift minimal matrix with a few additional rows in the directed case.

Finally some illustrating examples and figures are given in the Appendix. Particularly the two tables, obtained with the help of a computer, demonstrate the efficiency of the algorithms presented in the main part of this paper.

## 1 Basic Notation and Preliminary Results

Duing this paper let $n$ be a natural number, $\Omega = \Omega_n = \{1, \ldots, n\}$ be the set of *players*, and $\mathscr{P}(\Omega) = \{S \mid S \subseteq \Omega\}$ the set of *coalitions*. If $v$: $\mathscr{P}(\Omega) \to \{0, 1\}$ is a mapping – the characteristic function – then $(\Omega, \mathscr{P}(\Omega), v)$ is called *simple n-person game*. Note that we do not assume, as usual, $v(\emptyset) = 0$ for symmetry reasons (see Section 2). Since the nature of $\Omega$ and of $\mathscr{P}(\Omega)$ is determined by the characteristic function, we call $v$ a simple game as well. A coalition $S$ is often identified with the indicator function $1_S$, considered as $n$-vector. A coalition $S$ is *winning*, if $v(S) = 1$, and *losing* otherwise. The set of winning coalitions is abbreviated by $W_v$.

In a *monotone* simple game all subcoalitions of the losing coalitions are losing. If each proper subcoalition of a winning coalition is losing, this coalition is a *minimal* winning coalition. It should be noted that a monotone simple game is completely determined by the set of its minimal winning coalitions, denoted by $W^m$ or $W_v^m$, if the dependence of the game is to be stressed.

For each coalition $S$ let $D(S) = \sum_{i \in S} 2^{n-i}$ denote the corresponding number in the decimal system. Also, let $\tilde{S} = (\tilde{S}_1, \ldots, \tilde{S}_n)$ be the $n$-vector defined by $\tilde{S}_j = |S \cap \Omega_j|$. That is, $\tilde{S}$ counts the number of players in $S$ having indices less or equal to $j$, for all $j \in \Omega = \Omega_n$.

From now on all considered simple games are assumed to be monotone.

The matrix with $n$ columns

$$I := I(v) := (1_S)_{S \in W_v^m}$$

is called *incidence matrix* of $v$. Denote by $I_{j.}$ the $j$-th row of $I$. We assume that the rows of $I$ are ordered by means of $D(\cdot)$, i.e., $D(I_{j.}) > D(I_{k.})$ whenever $j < k$.

Two simple $n$-person games $v$ and $v'$ are *equivalent*, if there is a permutation $\pi$ of $\Omega$ such that $v \circ \pi = v'$. For most of our purposes it suffices to focus our interest on equivalence classes. Therefore, we will choose a canonical representative of each class. The formal notation is given as follows. For any subsystem $W \subseteq \mathscr{P}(\Omega)$ of coalitions, consider the numbers $(2^n - D(S))_{S \in W}$ which completely determine $W$. The binary expansion

$$B(W) = \sum_{S \in W} 2^{2^n - D(S)}$$

can be attached to $W$; actually $B(\cdot)$ induces the "natural" lexicographic order on subsystems of $\mathscr{P}(\Omega)$. This motivates

*Definition 1.1:* The equivalence class of a simple game $v$ (with respect to permutations of the set of players) is denoted $[v]$. The game $v^c$ is defined to be the representative of $[v]$, such that

$$B(W_{v^c}) = \min\{B(W_{v'}) | v' \in [v]\}$$

is satisfied. Thus the canonical representative $v^c$ of $[v]$ is the first in the natural lexicographical ordering of the sets $W_{v'}$ ($v' \in [v]$), considered as subsets of $\mathscr{P}(\Omega)$.

Let $v$ be a simple game. The relation $\preceq \subseteq \Omega^2$, defined by $i \preceq j$, if $v(\{i\} \cup S) \leq v(\{j\} \cup S)$, for all coalitions $S$ satisfying $\{i,j\} \cap S = \emptyset$, is called *desirability relation* of $v$ (see Maschler and Peleg (1966)). For a more general definition of desirability with respect to coalitions we refer to Einy (1985).

The simple game $v$ is called an *ordered game* if its desirability relation is complete and a *directed game* if additionally $1 \succeq 2 \succeq \cdots \succeq n$ is valid. Concerning this notation we also refer to Ostmann (1987, 1989) and Sudhölter (1989).

Two players $i$ and $j$ are *interchangeable* or *of the same type*, iff $i \succeq j$ and $j \succeq i$, which is abbreviated by $i \sim j$. Besides we recall that $i$ is a *null player* or *dummy*, if $v(S \cup \{i\}) = v(S)$ for all $S \in \mathscr{P}(\Omega)$.

*Lemma 1.2:* Let $v$ be an ordered game. Then $v^c$ is the unique directed game in the equivalence class $[v]$ of $v$.

*Proof:* The fact that $[v]$ contains a unique directed game is a straightforward consequence of the definitions.

It remains to show: if $v$ is directed, then $v^c = v$.

Consider the game $v^c$. We show for all $1 \leq k \leq n-1$ that $i \succeq j$ for all $i \in \Omega_k$, $j \in \Omega_n \backslash \Omega_k$. Assume on the contrary that there is a $k$ and $i \in \Omega_k$, $j \notin \Omega_k$, such that $i \preceq j$, $j \not\preceq i$. Let $\pi$ be the transposition on $\Omega$ defined by $\pi(i) = j$. Then $D(S) \geq D(\pi(S))$ ($i \in S \subseteq \Omega$), thus

$$B(W_{v^c}) = \sum_{S \in W_{v^c}} 2^{2^n - D(S)} > \sum_{S \in W_{v^c \circ \pi}} 2^{2^n - D(S)} = B(W_{v^c \circ \pi}) ,$$

a contradiction. q.e.d.

A *weighted majority game* $v$ is a simple game having a *representation* $(\lambda; m)$, i.e. a *level* $\lambda \in \mathbb{N}_0$ and a vector of *weights* $m \in \mathbb{N}_0^n$ such that

$$v(S) = \begin{cases} 1 , & \text{if } m(S) \geq \lambda \\ 0 , & \text{if } m(S) < \lambda \end{cases} .$$

Here, we use $m(S) := \sum_{i \in S} m_i$ ($S \in \mathscr{P}(\Omega)$) and call $m(S)$ the weight of coalition $S$.

A representation is called *minimal*, if it is minimal w.r.t. the weight of the grand coalition $\Omega$. Each weighted majority game is ordered; thus it is directed, iff it has a representation satisfying $m_1 \geq m_2 \geq \cdots \geq m_n$. Note that $m_i \geq m_j$ implies $i \succeq j$. For these definitions and assertions we refer to Maschler and Peleg (1966).

A simple game is a *constant-sum* game (we use the expression *zero-sum* game synonymously for historical reasons), if either $S$ or $\Omega\backslash S$ is winning (and not both are winning) and is a *superadditive* game, if at most $S$ or $\Omega\backslash S$ is winning for each coalition $S$. The *dual* game $v^*$ is defined by $v^*(S) = 1$, iff $v(\Omega\backslash S) = 0$ (see e.g. Shapley (1962)). The game $v$ is *dual superadditive* iff $v^*$ is superadditive. Note that both the classes of weighted majority games and of directed games are closed under duality. To verify the first assertion, observe that if $(\lambda; m)$ represents a weighted majority game and $\lambda$ is chosen minimally, then $(m(\Omega) + 1 - \lambda; m)$ represents $v^*$. Moreover, $v^* = v$, iff $v$ is a zero-sum game.

Note that each weighted majority game is dual superadditive or superadditive. Finally observe that * is an involution, i.e. $v^{**} = v$. Using Lemma 1.2 and some of the preceding assertions we obtain that $[v] = [v^*]$ enforces $v$ to be a zero-sum game in the case of directed games. This is no longer true in general, if $v$ is only monotonous (see e.g. Dubey an Shapley (1978)).

In the rest of this section we show that a list of all $(n + 1)$-person zero-sum weighted majority games is, in some sense, sufficient and necessary in order to generate a list of all $n$-person weighted majority games. This fact is suggested, e.g., by Wolsey (1976). First we need some notation.

*Definition 1.3:* Let $v$ be a directed superadditive $n$-person game. Define $\hat{v}$ to be the $n + 1$-person game, given by

$$\hat{v}(S) = \begin{cases} 1 \ , & \text{if } (S \in W_v) \text{ or } (n + 1 \in S \text{ and } S\backslash\{n + 1\} \in W_{v^*}) \\ 0 \ , & \text{otherwise} \end{cases} .$$

Then $v^0 := \hat{v}^c$ is called the *zero-sum extension* of $v$.

For a more general definition we refer to e.g. Einy and Lehrer (1989). The next two lemmata are used to formulate the proclaimed result explicitely.

*Lemma 1.4:* Let $v$ be a superadditive directed $n$-person game. Then

(i) $v^0$ is a monotone simple $n + 1$-person zero-sum game, not necessarily ordered.

(ii) $v$ is a weighted majority game, if and only if $v^0$ is a weighted majority game. In this case both of the following assertions are valid:

(a) If $(\lambda; m)$ represents $v$ and

$$i_0 = \max\{0\} \cup \{i | m_i > 2\lambda - m(\Omega) - 1\} \ ,$$

then

$$(\lambda; m_1, \ldots, m_{i_0}, 2\lambda - m(\Omega) - 1, m_{i_0+1}, \ldots, m_n)$$

is a representation of $v^0$.

(b) If $(\lambda; m_1, \ldots, m_{n+1})$ represents $v^0$, then

$$(\lambda; m_1, \ldots, m_j, m_{j+2}, \ldots, m_{n+1})$$

represents $v$ for some $j \in \{0, 1, \ldots, n\}$.

The first part of assertion (i) is a straightforward consequence of the definition of $v^0$, Einy (1985) gave an example of a zero-sum extension of a directed game not being ordered, and Aumann, Peleg and Rabinowitz (1965) showed assertion (ii). Therefore the proof of this lemma is skipped.

There is a converse statement to Lemma 1.4 in the case of a weighted majority game, which is formulated in Proposition 1.7 with the help of

*Definition 1.5:* Let $v$ be a directed $(n + 1)$-person zero-sum game. Consider the nonvoid sets $T_1, \ldots, T_{t(v)} \subseteq \Omega_{n+1}$ which are uniquely determined by the requirements:

(a) $\bigcup_{k=1}^{t(v)} T_k = \Omega_{n+1}$,

(b) $i, j \in T_k$ implies $i$ and $j$ are of the same type for all $1 \leq k \leq t(v)$,

(c) $i \in T_k, j \in T_{k+1}$ implies $j \precsim i, j \not\succsim i$ for all $1 \leq k < t(v)$.

Then the sets $T_k$ are the *types* of the game. Let $\tilde{t}(v)$ be the number of non dummy types, i.e. $\tilde{t}(v) = t(v)$, if $n + 1$ is not a dummy, and $\tilde{t}(v) = t(v) - 1$ otherwise. For each $k \in \Omega_{t(v)}$ we define the $k$-th *underlying game* of $v$ to be an $n$-person game, denoted $v^{(k)}$ via

$$v^{(k)}(S) = v(\{i \in \Omega_{n+1} | (i < i_0 \text{ and } i \in S) \text{ or } (i > i_0 \text{ and } i - 1 \in S)\}) ,$$

$S \in \Omega$, where $i_0 \in T_k$ is chosen arbitrarily. This game $v^{(k)}$ is well defined, because all players of $T_k$ are of the same type.

It should be noted that the $k$-th underlying game of $v$ is the game which arises from $v$ as follows. At first an arbitrary player of the $k$-th type $T_k$ has to be dropped. Then only the winning coalitions, which do not contain this player, are considered to be the winning coalitions of the new game.

*Lemma 1.6:* If $v$ is a directed $(n + 1)$-person zero-sum game and $k, \bar{k} \in \Omega_{t(v)}$, then

(i) $v^{(k)}$ is a superadditive directed game,
(ii) $v^{(k)} = v^{(\bar{k})}$ if and only if $k = \bar{k}$,
(iii) $(v^{(k)})^0 = v$,
(iv) if $v^{(k)}$ is a zero-sum game then $k = t(v) > \tilde{t}(v)$, i.e. player $n + 1$ is a dummy of $v$.

A proof is skipped, as all necessary arguments are straightforward and almost trivial. Using the last two lemmata we obtain our proclaimed result.

*Proposition 1.7:* The set of directed superadditive $n$-person weighted majority games is the set of all underlying games of the directed $(n + 1)$-person zero-sum weighted majority games.

In Proposition 1.7 only superadditive $n$-person games are considered. The missing assertion concerning dual superadditive $n$-person weighted majority games follows especially from Lemma 1.6 (iv) by looking at dual games and is therefore not stated in detail. We only formulate the exact result concerning the cardinalities of these sets of games in Corollary 1.8.

Let $Z_n$ and $Z_n^r$ denote the set of directed $n$-person zero-sum games and those having a representation respectively. That is,

$$Z_n = \{v | v \text{ is a directed } n\text{-person zero sum-game}\} \text{ and}$$

$$Z_n^r = \{v \in Z_n | v \text{ is a weighted majority game}\} .$$

Moreover let $R_n$ be the set of directed $n$-person weighted majority games. From the fact that $R_n$ can be partitioned into its superadditive games and dual superadditive non zero-sum games, formally written

$$R_n = \{v \in R_n | v \text{ superadditive}\} \cup \{v \in R_n | v^* \in R_n, v^* \text{ superadditive}, v^* \notin Z_n^r\} ,$$

we obtain the following result, concerning the cardinality of $R_n$.

*Corollary 1.8:*

(i) $|R_n \cap \{v | v \text{ is superadditive}\}| = \sum_{v \in Z_{n+1}^r} t(v) = |R_n \cap \{v| \text{ is dual superadditive}\}|$
(ii) $|Z_n^r| = \sum_{v \in Z_{n+1}^r} t(v) - \tilde{t}(v)$
(iii) $|R_n| = \sum_{v \in Z_{n+1}^r} t(v) + \tilde{t}(v) = 2 \cdot \sum_{v \in Z_{n+1}^r} t(v) - |Z_n^r|$.

In the next section it turns out that the classes of directed games and of directed zero-sum games together with certain natural relations form partially ordered sets; the first being additionally a lattice. This gives some insight into the structure of these classes of games. Moreover the partial order relations will enable us to formulate algorithms which generate both, the directed $n$-person and the directed $n$-person zero-sum games. These procedures are presented in Section 3.

## 2 The Partially Ordered Sets

This section is organized as follows. At first it is recalled that each directed game is completely determined by a subset of its minimal winning coalitions, the shift minimal coalitions (see Ostmann (1987)). Moreover the definition of the span of a coalition – Definition 2.1 – induces a relation on the set of coalitions. The arising partially ordered set is isomorphic to the well-known lattice of "partitions".

In what follows it turns out that the directed games can be considered as the filters of this lattice. These filters ordered by inclusion form again a lattice. Finally a relation on the directed zero-sum games is defined very similarly, such that this class again is a partially ordered set.

At first some notation is needed.

*Definition 2.1:* The *span of a coalition* $S$ is the set $\langle S\rangle = \{T \subseteq \Omega \mid \tilde{T} \geq \tilde{S}\}$. Moreover, define the *span of a subset* $A \subseteq \mathscr{P}(\Omega)$ by $\langle A\rangle = \bigcup_{S \in A} \langle S\rangle$.

It is known (see e.g. Ostmann (1987)) that $v$ is a directed game, iff $\langle W_v\rangle = W_v$. Moreover, in this case there is a unique minimal subset (i.e. minimal w.r.t. inclusion) $W_v^s \subseteq W_v$ such that $\langle W_v^s\rangle = W_v$. The elements of $W_v^s$ are the *shift minimal coalitions* of $v$, which are automatically minimal winning coalitions. The directed game $v$ is completely determined by $W_v^s$. The corresponding submatrix of the incidence matrix is the *shift minimal matrix* of $v$, abbreviated

$$I^s := I^s(v) = (1_S)_{S \in W_v^s} \,.$$

For this notation we again refer to Ostmann (1987).

*Definition 2.2:* Two coalitions $S$, $T \in \mathscr{P}(\Omega)$ are defined to satisfy $S \preceq T$, if $\tilde{S} \leq \tilde{T}$; and $S \prec\!\cdot\, T$ if $S \neq T$, $S \preceq T$ and additionally $S \preceq R \preceq T$ implies $R \in \{S, T\}$. The relations $\preceq$ and $\prec\!\cdot$ are called *order relation* and *cover relation* respectively.

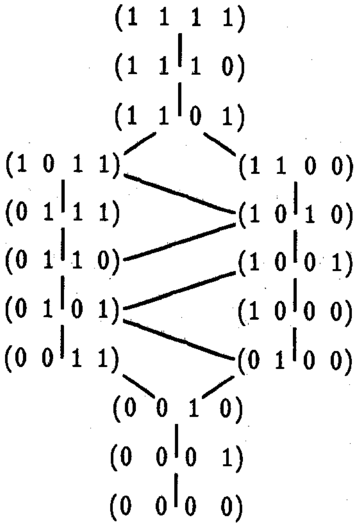


**Fig. 1**

With this notation $(\mathscr{P}(\Omega), \preceq)$ is a partially ordered set and the order relation is the reflexive and transitive closure of the cover relation. This partially ordered set can be illustrated by its Hasse diagram, i.e. by the directed graph, whose vertex set is $\mathscr{P}(\Omega)$ and whose edge set consists of all pairs $(S, T)$ with $S \prec\!\cdot\, T$. In Fig. 1 $1_S$ and $1_T$ are joined by an edge and $1_T$ lies above $1_S$, iff $S \prec\!\cdot\, T$ $(n = 4)$.

The partially ordered set $(\mathscr{P}(\Omega), \preceq)$ is isomorphic to the famous partially ordered set of "partitions" $(M(n), \leq)$, where

$$M(n) = \{a = (a_1, \ldots, a_n) \in \mathbb{N}_0^n \,|\, 0 = a_1 = a_2 = \cdots = a_h < a_{h+1} < \cdots < a_n \leq n$$

$$\text{for some } h \in \{0\} \cup \Omega_n\}$$

The isomorphism is obviously induced by the bijective mapping on the corresponding vertex sets

$$\mathscr{P}(\Omega) \to M(n) \ , \qquad S \mapsto (0, \ldots, 0, n+1-i_1, n+1-i_2, \ldots, n+1-i_{|S|}) \ ,$$

where

$$i_1 > \cdots > i_{|S|} \qquad \text{and} \quad S = \{i_1, \ldots, i_{|S|}\} \ .$$

This partially ordered set $(M(n), \leq)$ was introduced by Euler (1750) and it can easily be seen that it has a unique rank function $\left(\text{given by } a \mapsto \sum_{i=1}^{n} a_i\right)$ with maximal rank $\binom{n+1}{2}$. Here a rank function $r$ of a partially ordered set $(P, \preceq)$ is a mapping from $P$ to $\mathbb{N} \cup \{0\}$ with $r(\bar{x}) = 0$ for some minimal element $\bar{x} \in P$ such that $x \prec\cdot\, y (x, y \in P)$ implies $r(y) = r(x) + 1$.

Moreover it is a lattice, i.e., to each two elements $a$, $b$ there is a unique minimal element covering both and a unique maximal element covered by both $a$ and $b$ (which can be seen in $\mathscr{P}(\Omega)$ by observing that $R = \max(\min)\{S, T\}$, where $\tilde{R}_j = \max(\min)\{\tilde{S}_j, \tilde{T}_j\}$ $(S, T \in \mathscr{P}(\Omega))$ componentwise).

Finally $(M(n), \leq)$ is rank symmetric (which is seen in $(\mathscr{P}(\Omega), \preceq)$ using the mapping $S \mapsto \Omega \backslash S$ for all $S \in \mathscr{P}(\Omega)$). Here a partially ordered set $(P, \preceq)$ with rank function $r$ is rank symmetric, if $r(P) = \max\{r(x) | x \in P\}$ exists and the cardinalities of the $k$-th and the $(r(P) - k)$-th level of $P$ (where the $k$-th level of $P$ is the set $\{x \in P | r(x) = k\}$) coincide for $0 \leq k \leq r(P)$.

Let $\alpha_k^n$ denote the cardinality of the $k$-th level of $(M(n), \leq)$, i.e.

$$\alpha_k^n = \left| \left\{ a \in M(n) \,\middle|\, \sum_{i=1}^{n} a_i = k \right\} \right| .$$

A subset of pairwise comparable elements of a partially ordered set is a chain. A subset containing no chain of cardinality $k + 1$ is a $k$-family. Stanley (1980) proved that $(M(n), \leq)$ has the strong Sperner property. That is, the maximal cardinality of a $k$-family is the largest sum of cardinalities of $k$ levels, formally written

$$\max\{|F| \,|\, F \text{ is a } k\text{-family of } (M(n), \leq)\}$$
$$= \max\left\{ \sum_{j=1}^{k} \alpha_{i_j}^n \,\middle|\, 0 \leq i_1 < \cdots < i_k \leq \binom{n+1}{2} \right\} .$$

Moreover he showed that $(M(n), \leq)$ is rank unimodal. That means $\alpha_1^n \leq \cdots \leq \alpha_k^n \geq \alpha_{k+1}^n \geq \cdots \geq \alpha_{\binom{n+1}{2}}^n$ for some $k$. For another proof of the strong Sperner property and the rank unimodality, only using methods of linear algebra, we refer to Proctor (1982a, b). These proofs – besides – show a famous conjecture of Erdös and Moser (1965). For the notations we refer to Engel and Gronau (1985). Now it is clear by the preceding definitions that the directed $n$-person games are exactly the filters of $(\mathscr{P}(\Omega), \preceq)$, i.e. if $v$ is directed, then $W_v$ is a filter and vice vesa. Here a filter of a partially ordered set $(P, \preceq)$ is a subset $F \subseteq P$ such that $x \in F$ and $y \in P$ with $x \preceq y$ implies $y \in F$. Moreover each filter is spanned by its minimal elements, which are exactly the shift minimal coalitions of the corresponding game.

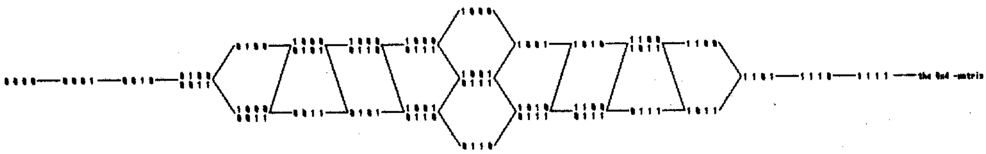

**Fig. 2**

Here it should be noted that the filter $\mathscr{P}(\Omega)$ must not be excluded, because we allow the empty coalition to be a winning one (see Section 1).

With the help of these results the maximal cardinality of the set of shift minimal coalitions of a directed $n$-person game can be given. It is well-known that $\alpha_k^{n+1} = \alpha_k^n + \alpha_{k-n-1}^n$ holds true. Thus especially the number $\alpha^n_{\lceil \binom{n+1}{2}/2 \rceil}$ can easily be computed recursively. Using the Sperner property, the rank unimodality and symmetry of the lattice $(M(n), \leq)$, we easily obtain the following interesting result.

*Proposition 2.3:* $\max\{|W_v^s| \,|\, v \text{ is a directed } n\text{-person game}\} = \alpha^n_{\lceil \binom{n+1}{2}/2 \rceil}$.

Now we come back to the directed games, considered as filters of $(\mathscr{P}(\Omega), \preceq)$ or $(M(n), \leq)$. These filters are ordered by inclusion and it easily turns out that $(\{W_v | v \text{ is a directed } n\text{-person game}\}, \supseteq)$ again is a ranked partially ordered set with rank function $r$, defined by $r(W_v) = 2^n - |W_v|$, and total rank $2^n$. The case $n = 4$ is illustrated in Fig. 2, where $I^s(v)$ is written instead of $W_v$. In order to distinguish these partially ordered sets from the sets $(\mathscr{P}(\Omega), \preceq)$, we sketch the corresponding Hasse diagrams in such a way that the larger elements are on the right hand side of the smaller elements (not as above in the sketches of the Hasse Diagrams of the sets $(\mathscr{P}(\Omega), \preceq)$).

This partially ordered set is a lattice, since $\langle W_v \cup W_{v'} \rangle = W_v \cup W_{v'}$ and $\langle W_v \cap W_{v'} \rangle = W_v \cap W_{v'}$ for each pair of directed games $(v, v')$. Moreover the rank symmetry is easily checked by applying the mapping $v \mapsto v^*$ and observing that the restriction on the $k$-th level is bijective on the $2^n - k$-th level. Additionally we conjecture that it is rank unimodal, although the linear algebra methods used by Proctor (1982a, b) cannot solve this problem. The set of filters ordered by inclusion is indeed uniquely modular but there is no edge labeling for general $n$ as it exist in the lattice $(M(n), \leq)$, sketched in Fig. 4 of the Appendix for $n = 4, 5, 6$.

In the end of this section it is shown that the set of directed $n$-person zero-sum games $Z_n$ ordered with respect to a certain relation form a partially ordered set. At first we define the relation on $Z_n$, where $n \geq 2$ is assumed for the rest of this section.

*Definition 2.4:* For games $v_1, v_2 \in Z_n$ define $v_1 \preceq v_2$, iff $W_{v_1^{(1)}} \subseteq W_{v_2^{(1)}}$. (For the definition of these underlying games $v_i^{(1)}$ see Definition 1.5.) Moreover, let $T^{\max}(v)$ be the lexicographically maximal losing coalition of $v$ for each directed game $v$.

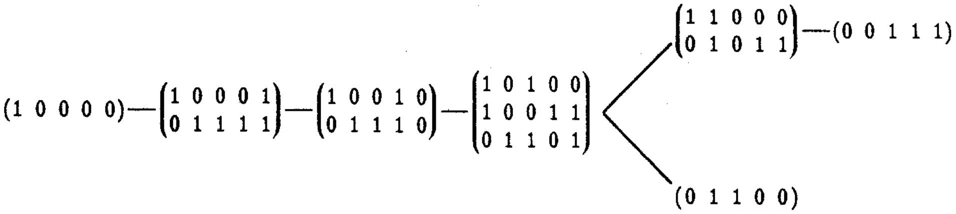


**Fig. 3**

Besides, notice that there is a canonical bijection from $W_{v(1)}$ to $W_v \cap \{S \subseteq \Omega | 1 \notin S\}$, given by $S \mapsto 1 + S$. It should be remarked that $(Z_n, \preceq)$ is a ranked partially ordered set, where the rank function $Z_n \to \mathbb{N}_0$ is given by $v \mapsto |W_{v(1)}|$. Let the $i$-th level of this partially ordered set be denoted by $Z_{n,i}$, i.e.

$$Z_{n,i} = \{v \in Z_n | i = |W_{v(1)}|\} \ .$$

Fig. 3 sketches the corresponding Hasse diagram in the case $n = 5$ and shows that $(Z_n, \preceq)$ is no lattice in general, since e.g. (0 0 1 1 1) and (0 1 1 0 0) have no supremum.

It is clear that $Z_{n,0}$ contains the unique game $v$ characterized by $W_v^s = \{\{1\}\}$. Moreover $Z_{n,i}$ vanishes for $i > 2^{n-2}$, because the mapping from $\{S \subseteq \Omega | 1 \notin S\}$ to $\{S \subseteq \Omega | 1 \in S\}$, $S \mapsto S \cup \{1\}$ is injective and the monotonicity of each member $v$ of $Z_n$ implies that $|W_{v(1)}| \leq |W_v \backslash W_{v(1)}|$, but $|W_v| = 2^{n-1}$ by definition.

*Remark 2.5:* Note that $Z_{n,i} \neq \emptyset$ for $0 \leq i \leq r_n := \max\{i \in \mathbb{N}_0 | Z_{n,i} \neq \emptyset\}$.

Indeed, $Z_{n,r_n}$ is nonempty by definition. Let $v$ be an element of $Z_{n,i+1}$ $(i < r_n)$ and $S$ be the lexicographically minimal winning coalition of $v$, thus $1 \notin S$ and $S$ is shift minimal (winning). Moreover $W_v \backslash \{S\}$ characterizes a directed game $v'$ with $2^{n-1} - 1$ winning coalitions. Obviously the coalition $\Omega \backslash S$ is the lexicographically maximal losing coalition of $v'$. Therefore $W_{v'} \cup \{\Omega \backslash S\}$ characterizes a zero-sum game $\tilde{v} \in Z_{n,i}$, i.e. $W_{\tilde{v}} = W_v \cup \{T^{\max}(v)\} \backslash \{\Omega \backslash T^{\max}(v)\}$.

## 3 Generating Procedures

First of all we present the theoretical foundations of two algorithms. The first algorithm is constructed with the aim to generate a subgraph of the lattice of directed $n$-person games. The second one generates a subgraph of the partially ordered set of directed $n$-person zero-sum games. Both subgraphs contain all vertices of the corresponding partially ordered sets and, additionally, are trees. The necessary assertions concerning directed games and directed zero-sum games are formulated simultaneously because of the similarity of both generating methods.

Finally, we show that both algorithms can easily be extended to procedures which generate the remaining edges in addition.

In this section we assume $n \geq 3$ for all assertions concerning zero-sum games and start with

*Definition 3.1:*

(i) Let $D_n$ denote the set of directed $n$-person games and $D_{n,i}$ denote the $i$-th level of this set, i.e.,

$$D_{n,i} = \{v \in D_n | r(v) = i\}$$

for $0 \leq i \leq 2^n$ (See Section 2 for the definition of the rank function $r$).

(ii) Let $v$ be a directed $n$-person game. Then

$$W_v^l = \{S \in W_v^s | T \in W_v \text{ for each } T \text{ with } D(T) \geq D(S)\}$$

is the set of *large coalitions* of $v$.

With the help of the next lemma we define two mappings (see Definition 3.3). By applying these mappings to a game of the corresponding partially ordered set we obtain an edge or a set of edges respectively.

*Lemma 3.2:* Let $v$ and $v^0$ be a game in $D_{n,j}$ and $Z_{n,i}$ for some $0 \leq j \leq 2^n$ and $0 \leq i \leq r_n$ respectively.

(i) If $i > 0 < j$ then the games $\tilde{v}$ and $\tilde{v}^0$ characterized by

$$W_{\tilde{v}} = W_v \cup \{T^{\max}(v)\} \qquad \text{and} \qquad W_{\tilde{v}^0} = (W_{v^0} \backslash \{\Omega \backslash T^{\max}(v^0)\}) \cup \{T^{\max}(v^0)\}$$

are elements of $D_{n,j-1}$ and $Z_{n,i-1}$ respectively.

(ii) If $S \in W_v^l$ or $S \in W_{v^0}^l$ respectively then $\bar{v} \in D_{n,j+1}$ and $\bar{v}^0 \in Z_{n,i+1}$ where $\bar{v}$ and $\bar{v}^0$ are the games characterized by

$$W_{\bar{v}} = W_v \backslash \{S\} \qquad \text{and} \qquad W_{\bar{v}^0} = (W_{v^0} \backslash \{S\}) \cup \{\Omega \backslash S\}$$

respectively.

*Proof:* All assertions concerning $v$ are straightforward consequences of the corresponding definitions. Therefore, only the zero-sum case has to be considered.

(i) Observing that $D(T^{\max}(v^0)) \geq 2^{n-1}$ assertion (i) is directly implied by Remark 2.5.

(ii) In order to verify (ii) it suffices to show that $\bar{v}^0$ is directed, since this game clearly has the zero-sum property.

Assume on the contrary that $\bar{v}$ is not directed and put $T := \Omega\backslash S$, thus $\langle T\rangle\backslash(W_{v^0}\backslash\{S\} \cup \{T\}) \neq \emptyset$. Take a coalition $T^1$ of this nonvoid subset of all coalitions, then $\tilde{T} \leq \tilde{T}^1$, $T^1 \notin W_v$, thus $\widetilde{\Omega\backslash T^1} \leq \widetilde{\Omega\backslash T} = \tilde{S}$ and $\Omega\backslash T^1 \in W_{v^0}$. Therefore $T^1$ must coincide with $S$, because $S$ is shift minimal.

If $|T| = 1$, then $\Omega\backslash\{1\}$ is no winning coalition, since $\langle\Omega\backslash\{1\}\rangle \ni S \neq \Omega\backslash\{1\}$ (note that $1 \in S$ holds true because $S$ is a large coalition) and $\langle\{1\}\rangle \ni S \neq \{1\}$, since $n \geq 3$ holds true. Thus $\{1\}$ is no winning coalition. The union of these last two coalitions is $\Omega$, a contradiction to the zero-sum property of $v$. Therefore define:

$$t_1 = \min T \qquad \text{and} \quad t_2 = \min T\backslash\{t_1\} \ .$$

If $t_2 = t_1 + 1$, then

$$\langle T\rangle \ni T^2 := T \cup \{1\}\backslash\{t_1\} \neq T^3 := T \cup \{1\}\backslash\{t_2\} \in \langle T\rangle$$

and clearly $S$ covers both of these coalitions, i.e. $S \in \langle T^2\rangle \cap \langle T^3\rangle$, which contradicts the shift minimality of $S$. In the remaining case, i.e. $t_2 > t_1 + 1$, $T^3$ can be substituted by $T \cup \{t_2 - 1\}\backslash\{t_2\}$ and the same arguments lead to a contradiction. q.e.d.

The assertions of this lemma concerning the directed game $v$ can be reformulated as follows. The games $v$ and $\tilde{v}$ are joined by an edge. Moreover, $v$ and $\bar{v}$, for each $\bar{v}$, are connected by an edge in the lattice of directed games. Clearly, an analogon is true in case of the directed zero-sum game $v^0$. This motivates the following

*Definition 3.3:*

(i) Let $\varphi^0\colon Z_n\backslash Z_{n,0} \to Z_n$ and $\varphi\colon D_n\backslash D_{n,0} \to D_n$ be defined by $\varphi(v) = \tilde{v}$ and $\varphi(v^0) = \tilde{v}^0$ according to Lemma 3.2.

(ii) If $S \in W_v^l$ or $S \in W_{v^0}^l$ ($v \in D_n$, $v^0 \in Z_n$), then let $v_S$ and $v_S^0$ be the games $\bar{v}$ and $\bar{v}^0$ of Lemma 3.2, part (ii), respectively and define $\rho\colon D_n \to \mathscr{P}(D_n)$ and $\rho^0\colon Z_n \to \mathscr{P}(Z_n)$ by $\rho(v) = \{v_S | S \in W_v^l\}$ and $\rho^0(v^0) = \{v_S^0 | S \in W_{v^0}^l\}$.

Combining the last definitions we obtain the following important result.

*Proposition 3.4:*

(i) $\varphi(\rho(v)) = \{v\}$ and $\varphi^0(\rho^0(v^0)) = \{v^0\}$ for all $v \in D_n$ and $v^0 \in Z_n$ with $\rho(v) \neq \emptyset$ and $\rho^0(v^0) \neq \emptyset$.
(ii) $\rho(\varphi(v)) \ni v$ and $\rho^0(\varphi^0(v^0) \ni v^0$ for all $v \in D_n \backslash D_{n,0}$ and $v^0 \in Z_n \backslash Z_{n,0}$.
(iii) $|\rho(v)| = |W_v^l|$ and $|\rho^0(v^0)| = |W_{v^0}^l|$ for all $v \in D_n$ and $v^0 \in Z_n$.
(iv) $D_{n,j+1} = \dot{\bigcup}_{v \in D_{n,j}} \rho(v)$ and $Z_{n,i+1} = \dot{\bigcup}_{v^0 \in Z_{n,1}} \rho^0(v^0)$ for all $0 \leq i < r_n$, $0 \leq j < 2^n$.

Now the arising algorithms to generate all directed $n$-person games and all those zero-sum games can be formulated. We restrict ourselves to the zero-sum case since the other one is completely analogous.

Starting with the unique game of $Z_{n,0}$ and applying $\rho^0$ yields $Z_{n,1}$. If $Z_{n,i}$ is constructed and $i < r_n$, then $Z_{n,i+1}$ is obtained by applying $\rho^0$ to each element of $Z_{n,i}$.

It should be remarked that this algorithm can be modified in such a way that the arising procedure computes all edges of the partially ordered set $(Z_n, \preceq)$:

If $v \in Z_n$ and $v' \in \rho^0(v)$, then $v$ and $v'$ are joined by an edge. Here only the edges from $v \in Z_{n,i}$ to $v_S \in Z_{n,i+1}(S \in W_v^l)$ are considered, thus this procedure generates a tree.

But there is a canonical extension $\bar{\rho}^0$ of $\rho^0$, which considers all shift minimal coalitions $S$ of $v$ with $D(S) \geq 2^{n-1}$ instead of the large coalitions only, and we obtain that $v \preceq v'$and $v$, $v'$ are joined by an edge, iff $v' \in \bar{\rho}^0(v)$. The proof is an analogon to the one of Lemma 3.2. The 6-person case is sketched in Fig. 5 of the Appendix.

Compared to the algorithm which generates a tree this method clearly is much more slowly, since all edges of the partially ordered set are generated. It has an analogon in the case of directed games – induced by $\bar{\rho}$ which considers all shift minimal coalitions of each game.

*Remark 3.5:* There is another algorithm generating all directed $n$-person games. This procedure successively constructs the games with respect to their lexicographical position, does not reveal the order structure of the class of games, and is thus of less theoretical interest. Moreover, it has no analogon in the zero-sum case. Therefore we skip a detailed description of this method, though it is faster than the one mentioned above.

All methods have been implemented on a computer. For examples we refer to the Appendix.

The last section gives an answer to the question how the games of $Z_n$ can be tested for representability.

## 4 Testing of Directed Zero-Sum Games for Representability

It is the aim of this section to compute one element of the least core of a given directed zero-sum game using the Simplex Method. Peleg (1968) proved that in the weighted majority case each element of the least core corresponds to a representation of the game, thus this procedure yields the desired test of representability. For further connections between representations of weighted majority games (homogeneous games), the least core, and other solution concepts see Peleg and Rosenmüller (1992), Peleg, Rosenmüller and Sudhölter (1994), and Rosenmüller and Sudhölter (1994).

At first we demonstrate that the least core of each monotone simple zero-sum game coincides with the set of equilibria of a certain non-cooperative matrix game (see Corollary 4.3). Then we show that this matrix being the incidence matrix can be substituted by the shift minimal matrix with a few additional rows in the directed case (see Lemma 4.5). Finally two algorithms are formulated explicitly. Both methods compute an extreme point of the least core of an arbitrary directed zero-sum game.

We start recalling some definitions and properties concerning weighted majority zero-sum games.

If $(\lambda; m)$ is a representation of an $n$-person weighted majority zero-sum game, $m(T) < m(\Omega)/2 < m(S)$ for all coalitions $T \notin W_v$, $S \in W_v$. For the sake of brevity we will drop the level $\lambda$ in this case, i.e. $(\lambda; m)$ is identified with $m = (m_1, \ldots, m_n)$. Moreover $\overline{m} = \left(\frac{m_1}{m(\Omega)}, \ldots, \frac{m_n}{m(\Omega)}\right)$ is called a *normalized* representation of $v$. Conversely, a *payoff vector* $\tilde{m} = (\tilde{m}_1, \ldots, \tilde{m}_n)$, i.e. $\tilde{m}(\Omega) = 1$, $m_i \geq 0$ $(i \in \Omega)$, is a normalized representation of a weighted majority zero-sum game, if there is no coalition $S$ with $\tilde{m}(S) = \frac{1}{2}$.

In what follows we use an approach similar to the first step of the algorithm computing the nucleolus considered by Wolsey (1976) (we also refer to Kopelowitz (1967)), to compute a payoff vector for each directed zero-sum game, which is a representation in the case of a weighted majority game. For the definition of the nucleolus Schmeidler (1966) is referred to.

*Definition 4.1:* If $v \in Z_n$, then define

$$X_v := \{x \in \mathbb{R}^n | x \geq 0, x(\Omega) = 1 \text{ and}$$
$$x(S) \geq \max\{\min\{y(T) | T \in W_v\} | y \geq 0, y(\Omega) = 1\} \text{ for all } S \in W_v\} ,$$

$$q_v := \min\{x(S) | S \in W_v^m\} \qquad \text{for each } x \in X_v ,$$

$$\overline{X}_v := \{x \in X_v | x_1 \geq \cdots \geq x_n\} .$$

Note that the set $X_v$ remains unchanged, if $W_v$ is substituted by $W_v^m$ at all places, and that this set is the least core in the sense of Maschler, Peleg and Shapley (1979). Observe that $X_v$ and $\bar{X}_v$ are convex polyhedrons, containing the nucleolus of $v$ and being subsets of the set of normalized representations of $v$ in the weighted majority case. Indeed, Peleg (1968) only showed that the nucleolus is a normalized representation in this case. But his proof remains valid for each element of the least core.

We want to compute an extreme point of $X_v$ or $\bar{X}_v$ using the equilibrium concept of a non-cooperative matrix game which is characterized, roughly speaking, by $W_v^m$ or $W_v^s$ respectively.

Now we come to the detailed description of the matrix games. Let $\Gamma_v$ be the matrix game characterized by the transpose matrix of the incidence matrix of the directed $n$-person zero-sum game $v$, namely $A := I(v)^t$ (i.e. the sets of strategies $Y$ and $X$ for player $II$ and player $I$, respectively, are the sets of payoff $k$- and $n$-vectors, where $k$ is the number of minimal winning coalitions of $v$. A tupel of strategies $(x, y) \in X \times Y$ leads to the payoff $xAy$ for player $I$ and to $-xAy$ for player $II$). Moreover $\bar{x} \in X$ is an optimal strategy for player $I$, iff $\bar{x} \in X_v$. For this property we refer to e.g. Rosenmüller (1981), Section 1. The second matrix game $\Gamma_v^*$ is characterized by the matrix

$$A^* := E_{k,n} - I(v) \ , \qquad \text{where } E_{k,n} = \begin{bmatrix} 1 & \dots & 1 \\ \vdots & & \vdots \\ 1 & \dots & 1 \end{bmatrix}$$

is a $k \times n$ matrix. We conclude that $\tilde{x}$ is an optimal strategy for player $II$ w.r.t. $\Gamma_v^*$, iff $\tilde{x} \in X_v$. Let $e_n$ denote the $n$-vector $(1, \dots, 1)$.

An arbitrary $k \times n$ matrix $B$ is defined to have property $(P)$, if each entry of $B$ is nonnegative and $B$ has no all-zero column. If $\Gamma$ is the matrix game characterized by $B$, then the following lemma is well-known (see Brickmann (1989) and again Rosenmüller (1981)).

*Lemma 4.2:* Let $B$ be a $k \times n$ matrix with property $(P)$. Then the following holds: $\max\{y(\Omega) | y \geq 0 \text{ and } By \leq e_k\}$ and $\min\{x(\Omega_k) | x \geq 0 \text{ and } xB \geq e_n\}$ exist. If $\bar{y}$ and $\bar{x}$ is a maximizer and minimizer respectively, then $(\bar{x}, \bar{y})/\bar{y}(\Omega)$ is an optimal pair of strategies for $\Gamma$. Conversely, if $(\tilde{x}, \tilde{y})$ is an optimal pair of strategies for $\Gamma$, then there are such vectors $\bar{x}$ or $\bar{y}$ such that $\tilde{x} = \bar{x}/\bar{x}(\Omega_k)$ and $\tilde{y} = \bar{y}/\bar{y}(\Omega)$.

The maximization problem of Lemma 4.2 is the dual of the minimization problem, thus $\bar{x}(\Omega_k) = \bar{y}(\Omega)$.

The matrix $A$ (transpose of $I(v)$) trivially has property $(P)$, since no minimal winning coalition is empty. Moreover $A^*$ (consisting of all maximal losing coalitions) has property $(P)$, as long as $\{v\} \neq Z_{n,0}$ is valid. It is sufficient to test the elements of $Z_n \backslash Z_{n,0}$ for representability, because the only remaining game is trivially a weighted majority game ((1, 0, ..., 0) is a (normalized) representation). Therefore $v$ is assumed to be an element of $Z_n \backslash Z_{n,0}$ from now on.

*Corollary 4.3:* Let $v$ be a game of $Z_n \backslash Z_{n,0}$ and $k = |W_v^m|$. The following assertions are equivalent:

(i) $v$ is a weighted majority game.
(ii) $\max\{y(\Omega_k) | 0 \leq y \text{ is a } k\text{-vector and } y \cdot I(v) \leq e_n\} < 2$.
(iii) $\max\{x(\Omega_n) | 0 \leq x \text{ is an } n\text{-vector, } (E_{k,n} - I(v)) \cdot x \leq e_k\} > 2$.

The maximization problems (ii), (iii) of Corollary 4.3 can be solved by the Simplex Method. The resulting algorithms start with very simple initial tableaux consisting of either $I(v)^t$, $e_n$ (as last column) and $-e_k$ (as last row) or $E_{k,n} - I(v)$, $e_k$ (as last column) and $-e_n$ (as last row), where $k$ denotes the cardinality of $W_v^m$ (see Brickmann (1989)).

These algorithms work even in the case that the game started with is not directed (but still monotone). We proceed by constructing generically faster and quite similar algorithms, which can only be applied to directed zero-sum games. It is our aim to substitute the incidence matrix by the shift minimal matrix and a few additional rows.

*Definition 4.4:* For each $v \in Z_n$ define the $(n + k - 1) \times n$ matrices ($k = |W_v^s|$)

$$\tilde{I}(v) = \begin{pmatrix} & & & I^s(v) & & & \\ 1 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & -1 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & & -1 & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & -1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & -1 \end{pmatrix},$$

$$\bar{I}(v) = \begin{pmatrix} & & & E_{k,n} - I^s(v) & & & \\ -1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 1 & \dots & 0 & 0 & 0 \\ 0 & 0 & -1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & 0 & \dots & -1 & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & -1 & 1 \end{pmatrix}.$$

Now the direct analogon of Corollary 4.3 is the following

*Lemma 4.5:* Let $v \in Z_n \backslash Z_{n,0}$ and $k = |W_v^s|$. Then the following assertions are equivalent.

(i) $v$ is a weighted majority game.
(ii) $\max\{y(\Omega_k) | y \in \mathbb{R}^{k+n-1}, y \geq 0 \text{ and } y \cdot \tilde{I}(v) \leq e_n\} < 2$.
(iii) $\max\{x(\Omega_n) | x \in \mathbb{R}^n, x \geq 0 \text{ and } \bar{I}(v) \cdot x \leq (e_k, \underbrace{0, \ldots, 0}_{n-1 \text{ times}})\} > 2$.

*Proof:* Put $\bar{k} = |W_v^m|$. In view of Corollary 4.3 it suffices to show that

(1) $\delta_0 := \max\{y(\Omega_{\bar{k}}) | 0 \leq y \in \mathbb{R}^{\bar{k}} \text{ and } y \cdot I(v) \leq e_n\} = \max\{y(\Omega_k) | 0 \leq y \in \mathbb{R}^{k+n-1}$ and $y \cdot \tilde{I}(v) \leq e_n\} =: \delta_1$
and
(2) $\gamma_0 := \max\{x(\Omega) | 0 \leq x \in \mathbb{R}^n \text{ and } (E_{\bar{k},n} - I(v)) \cdot x \leq e_{\bar{k}}\} = \max\{x(\Omega) | 0 \leq x \in$ $\mathbb{R}^n$ and $\bar{I}(v) \cdot x \leq (e_k, 0, \ldots, 0)\} =: \gamma_1$.

ad (2): For each $x \in \mathbb{R}^n$ define

$$i(x) := \max\{i \in \{0\} \cup \Omega_n | x_1 \geq \cdots \geq x_i \geq \max\{x_j | i < j \leq n\}\} .$$

Take $\bar{x} \in \mathbb{R}^n$ such that

$$(\alpha) \quad \bar{x} \geq 0, (E_{\bar{k},n} - I(v)) \cdot \bar{x} \leq e_{\bar{k}} \quad \text{and} \quad \bar{x}(\Omega) = \gamma_0$$

is valid and $i(\bar{x})$ is maximal. Now $\bar{x}_1 \geq \cdots \geq \bar{x}_n$ is to be verified. Assume, on the contrary, $i(\bar{x}) < n$, let us say $\bar{x}_{i_0} = \max\{\bar{x}_i | i > i(\bar{x})\}$, thus $i_0 > i(\bar{x}) + 1$. Therefore $i(x) > i(\bar{x})$, where

$$x = (\bar{x}_1, \ldots, \bar{x}_{i(\bar{x})}, \bar{x}_{i_0}, \bar{x}_{i(\bar{x})+2}, \ldots, \bar{x}_{i_0-1}, \bar{x}_{i(\bar{x})+1}, \bar{x}_{i_0+1}, \ldots, \bar{x}_n) .$$

Moreover there is a maximal losing coalition $T$, i.e. $\Omega \backslash T \in W_v^m$, with $x(T) > 1$ (because of the maximality of $i(\bar{x})$). Thus $i_0 \notin T$, $i(\bar{x}) + 1 \in T$. Therefore $T' = T \cup \{i_0\} \backslash \{i(\bar{x}) + 1\}$ is a losing coalition, which satisfies $\bar{x}(T') = x(T) > 1$, a contradiction. These arguments directly imply $\gamma_1 \geq \gamma_0$.

Conversely take $x \in \{x \in \mathbb{R}^n | x \geq 0 \text{ and } \bar{I}(v) \cdot x \leq (e_k, 0, \ldots, 0)\}$, thus

$$(\beta) \quad x_1 \geq x_2 \geq \cdots \geq x_n \text{ by the definition of } \bar{I}(v) .$$

If $S$ is a minimal winning coalition of $v$, then there is a shift minimal coalition $S'$ such that $\tilde{S}' \leq \tilde{S}$. Let $T$ be a row of $E_{\bar{k},n} - I(v)$. Then $S = \Omega \backslash T$ is a minimal winning coalition, showing that $T' = \Omega \backslash S'$ is a row in $E_{k,n} - I^s(v)$ and $\tilde{T}' \geq \tilde{T}$. Thus $x(T) \leq x(T') \leq 1$ (by ($\beta$)), implying $\gamma_0 \geq \gamma_1$.

ad (1): Look at the dual problems:

Let $x \in \mathbb{R}^n$, $x \geq 0$, $I(v)x \leq e_{\bar{k}}$ and $x(\Omega) = \delta_0$. Then analog arguments as in ad (2) show that w.l.o.g. $x_1 \geq \cdots \geq x_n$, meaning $\tilde{I}(v)x \leq (e_k, 0, \ldots, 0)$, thus $\delta_1 \geq \delta_0$ by looking at the dual problems.

Conversely take $x \in \mathbb{R}^n$, $x \geq 0$ and $\tilde{I}(v) \cdot x \leq e_k$. Then $I(v)x \leq e_{\bar{k}}$, because of the fact $x_1 \geq \cdots \geq x_n$, thus $\delta_0 \geq \delta_1$. q.e.d.

Clearly the extreme points of $X_v$ and $\bar{X}_v$ are the normalized extreme points of the sets of maximizers of the problem (iii) of Corollary 4.3 and Lemma 4.5 respectively. In view of the proof of the last lemma we obtain the following

*Corollary 4.6:* $\bar{X}_v = \{x \in X_v | x_1 \geq \cdots \geq x_n\} = \{x \in \mathbb{R}^n | x_1 \geq \cdots \geq x_n$ and there is an $\bar{x} \in X_v$ and a permutation $\pi$ of $\Omega$ such that $x = \bar{x} \circ \pi\} \neq \emptyset$.

Now we formulate exlicitly the procedures:

*Algorithm I:* Let $v \in Z_n$, $A = \tilde{I}(v)^t$, $k = |W_v^s|$.

*First step:* Start with the initial tableau (see Brickmann (1989))

| 0 | n+1 ... n+k | n+k+1 ... 2n+k−1 | 0 |
|---|---|---|---|
| 1 | | | 1 |
| ⋮ | A | | ⋮ |
| n | | | 1 |
| 0 | −1 ... −1 | 0 ... 0 | 0 |

*Second step:* Apply the Simplex Method by choosing the pivot element according to e.g. Bland's Rule. If the entry $p$ in the last row and column is not smaller than 2, then continue with the fourth step. If no optimum is reached, take this new tableau and continue with the second step.

*Third step:* Define for each $i \in \Omega_n$

$$m_i = \begin{cases} 0, \text{ if } i \text{ is not contained in the first row of the tableau} \\ \text{the last entry of the column with first entry } i, \text{ otherwise} \end{cases}$$

and observe that $(m_1, \ldots, m_n)/p$ is a normalized representation of $v$. Now stop the algorithm.

*Fourth step:* Conclude that $v$ is no weighted majority game (by (ii) of Lemma 4.5).

*Algorithm II:* Let $v \in Z_n \backslash Z_{n,0}$ and $A^* = \bar{I}(v)$, where $k = |W_v^s|$.

(1) Start with the initial tableau

| 0 | 1 ... n | 0 |
|---|---|---|
| n+1 | | 1 |
| ⋮ | | ⋮ |
| n+k | $A^*$ | 1 |
| n+k+1 | | 0 |
| ⋮ | | ⋮ |
| 2n+k−1 | | 0 |
| 0 | −1 ... −1 | 0 |

(2) Apply the Simplex Method by choosing the pivot element according to Bland's Rule. If the entry $p$ in the last row and column exceeds 2, continue with (4). If no optimum is reached, take this new tableau and continue with (2).
(3) Conclude that $v$ is no weighted majority game (by (iii) of Lemma 4.5) and stop this algorithm.
(4) By (iii) of Lemma 4.5 $v$ is a weighted majority game.

This Algorithm II can be modified to

*Algorithm IIa:* Let the steps (1a) and (3a) be exactly the steps (1) and (3) from Algorithm II and introduce two further steps:

(2a) Apply the Simplex Method and compute $p$ as in (2). If no optimum is reached, take the new tableau and continue with (2a). If $p > 2$, continue with (4a).
(4a) Define

$$m_i = \begin{cases} 0, & \text{if } i \text{ is not contained in the first column} \\ \text{the last entry in the row with first entry } i, & \text{otherwise} \end{cases}$$

and conclude that $(m_1, \ldots, m_n)/p$ is a normalized representation of $v$.

It should be remarked that both algorithms, slightly modified, can be used to compute an extreme point of $\overline{X}_v$ even in the case $v$ being no weighted majority game: Apply the Simplex method until an optimum is reached. Now define the vector $m/p$ according to the third step or (4a) respectively and observe that this vector is an extreme point of $\overline{X}_v$ in any case.

*Concluding Remarks:*

(1) Let $v$ be an element of $Z_n$ or $Z_n \backslash Z_{n,0}$ respectively, which is a weighted majority game.

Then each of the algorithms I or IIa generates a normalized representation $\left(\frac{m_1}{p}, \ldots, \frac{m_n}{p}\right)$ respectively. A representation $(\overline{m}_1, \ldots, \overline{m}_n) \in \mathbb{N}_0^n$ is obtained by the following procedure:

$\overline{m}_i = m_i \cdot q$ $(i \in \Omega)$, where $q$ is the product of the pivot elements. Indeed the fact that $\overline{m}_i$ is a nonnegative integer can easily be verified by an inductive argument.

(2) In each case the vector $m = (m_1, \ldots, m_n)$ together with $p$ has the interesting property

$$\min\{m(S) | S \in W_v\} - \max\{m(T) | T \notin W_v\}$$

$$= \begin{cases} 2 - p \ , & \text{if Algorithm I is used} \\ p - 2 \ , & \text{if Algorithm IIa is used} \end{cases}$$

This fact is shown for Algorithm I by observing that $\min\{m(S) | S \in W_v\} = 1$ and $m(\Omega) = p$, thus $\max\{m(T) | T \notin W_v\} = p - 1$ ($v$ is a zero-sum game), and for Algorithm IIa analogously by interchanging the roles of $S$ and $T$.

Therefore $m/|2 - p|$ is a minimal representation in the weighted majority case, if $m_i/|2 - p| \in \mathbb{N}_0$ is satisfied. Surprisingly it turns out that this vector is indeed an integer vector in many cases. In fact, the least core of each weighted majority zero-sum game with less than 9 persons is a singleton, consisting of the unique normalized minimal representation of $v$. In the 9-person case the algorithms of the last section generate 319,124 directed zero-sum games, from which exactly 175,428 are weighted majority games. It can be verified that $\overline{X}_v$ and $X_v$ coincide and are singletons, consisting of the nucleolus, in the 9-person weighted majority zero-sum case with the exception of exactly 12 games. To be more precise all 9-person weighted majority zero-sum games have a unique minimal normalized representation which coincides with the least core and thus with the nucleolus with the exception of 14 games listed in the Appendix (Table 2), which have exactly two minimal representations differing only on one type of players.

Moreover, in 12 of these cases both representations are exactly the extreme points of the least core and the nucleolus is the midpoint of these representations. The sets $\overline{X}_v$ are the convex hulls of one of these representations and the nucleolus. In the remaining two cases no normalized minimal representation is contained in the least core though the set is a singleton. We discuss these games in what follows.

Here is the first game $v_1$:

This game is represented by $\overline{m} = (15\ 13\ 10\ 8\ 6\ 4\ 4\ 2\ 1)$, but the normalized representation $\overline{m}/63$ cannot be an element of $X_{v_1}$ or $\overline{X}_{v_1}$, since each of the preceding algorithms yields $\tilde{m} = (14.5\ 12.5\ 9.5\ 7.5\ 6\ 4\ 4\ 1.5\ 1.5)/61$ thus $\tilde{m}(S) \geq 31/61 > 32/63 = \tilde{m}(S_0)/63$ for all $S \in W_{v_1}$ and $S_0 = \{1, 2, 6\}$.

It remains to show that $\overline{m}$ is a minimal representation of $v$. Let $m$ be a minimal representation. Then $m_9 \geq 1$, since this game has no dummies. If $m_8 \geq 2$ is presumed, then we can prove 7 lemmata which successively show that $m_7 \geq 4$, $m_6 \geq 4$, $m_5 \geq 6$, $m_4 \geq 8$, $m_3 \geq 10$, $m_2 \geq 13$, $m_1 \geq 15$. We only have to exclude w.l.o.g. that $m_8 = m_9 = 1$. In this case each coalition $S \in W_{v_1}^m$ with $\{8, 9\} \cap S \neq \emptyset$ would satisfy $m(S) = \lambda := \min\{m(S) | S \in W_{v_1}\}$. Using these coalitions we successively obtain $m_7 = m_6$, $m_5 = 2m_7 - 1 = m_7 + 1$, thus $m_7 = 2$, $m_4 = 2m_7$, $m_3 = 3m_7 - 1$, $m_1 = m_2 + 1$; therefore $(m_3, \ldots, m_9) = (5\ \ 4\ \ 3\ \ 2\ \ 2\ \ 1\ \ 1)$. Since $(1\ \ 1\ \ 1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0)$ is a minimal winning coalition, we additionally obtain $9 \leq 2m_2 + 1 \leq 10$, thus $m_1 = 5$, $m_2 = 4$, $\lambda = 14$, but then $m(\{1, 2, 6\}) = 11 < 14$, a contradiction in view of the fact that this coalition is winning.

The second game $v_2$ is the one represented by $(17\ 15\ 11\ 9\ 7\ 5\ 4\ 2\ 1)$, this representation being minimal (this can be verified analogously to the first game), and $(16.5\ 14.5\ 10.5\ 8.5\ 7\ 5\ 4\ 1.5\ 1.5)/69 \in X_{v_2}$. We conclude again that no normalized minimal representation of $v_2$ is in $X_{v_2}$.

An example of a game with two extreme points of the least core is the zero-sum extension of the game considered by Dubey and Shapley (1978): $(13\ \ 7\ \ 6\ \ 6\ \ 4\ \ 4\ \ 4\ \ 3\ \ 2)/49$ is a normalized minimal representation of this game but the last two weights can be exchanged. Both normalized representations are extreme points of $X_v$ and the first is in $\overline{X}_v$ but $\overline{X}_v$ contains the midpoint of these representations as extreme point, too.

(3) Applying each algorithm to the famous 12-person weighted majority zero-sum game introduced by Isbell (1959), which has two minimal representations such that the affected players 1 and 9 are of different type, we obtain one of the normalized minimal representations, i.e. both are extreme points of $X_v$ and $\overline{X}_v$.

(4) Both Algorithms I and II (a) can be modified in such a way that the shift minimal and shift maximal coalitions (i.e. the complements of the shift minimal coalitions) are identified with the types of these coalitions or profiles:

$$S \mapsto a(S) := (a_1(S), \ldots, a_{t(v)}(S)) ,$$

where

$$a_j(S) = |S \cap T_j| \qquad (1 \leq j \leq t(v)) \ ,$$

$T_j$ is defined according to Definition 1.5. Using the notation of Definition 4.4 $\tilde{I}(v)$ and $\bar{I}(v)$ must be substituted by the $(t(v) + k - 1) \times t(v)$ matrices

$$\left[\begin{array}{ccccc} & & a(S) & & \\ 1 & -1 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & -1 & 0 \\ 0 & 0 & \dots & 1 & -1 \end{array}\right]_{S \in I^s(v)} \quad \text{and}$$

$$\left[\begin{array}{ccccc} & & a(T) & & \\ -1 & 1 & \dots & 0 & 0 \\ 0 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & -1 & 1 \end{array}\right]_{T \in E_{k,n} - I^s_{(v)}} \quad \text{respectively} \ .$$

Note that it is very easy to compute the partition sets $T_j$ (see Sudhölter (1989), Section 4) and therefore this procedure will generically diminish the initial tableau and the Simplex steps. The disadvantage of the necessary computation of th $T_j$ will thus be compensated especially if the number of players is large. These new algorithms yield an extreme point of the nonvoid convex subset

$$\{x \in X_v | x_i = x_j \text{ if } i \sim j\} \qquad \text{of } \overline{X}_v \ ,$$

which is a singleton in the 9-person case.

## Appendix

Some figures and tables are sketched as illustrations of the presented algorithms. Fig. 4 sketches the lattice of directed $n$-person games ($n = 4, 5, 6$), considered as filters of $(\mathscr{P}(\Omega_n), \preceq)$ which are ordered by inclusion (see Section 2). The results of Table 1 have been developed with the help of a computer as follows:

The numbers of directed games ($n = 1, \dots, 8$) are obtained using the corresponding generating algorithm of Section 3. The number of edges in the corre-

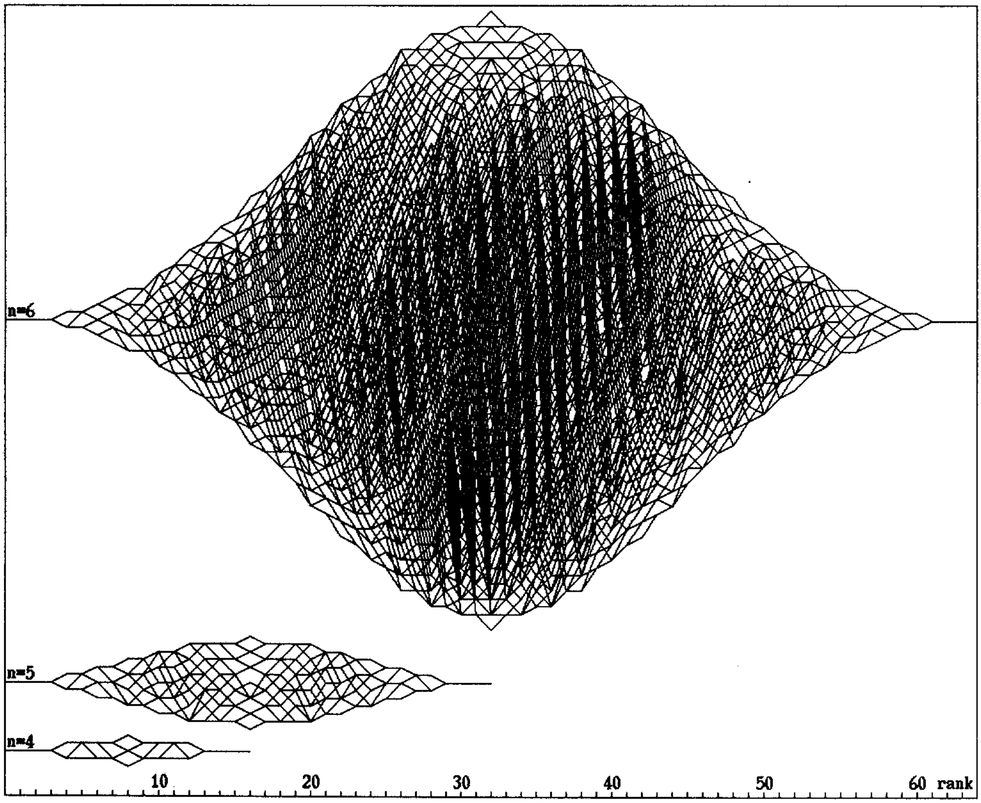


**Fig. 4**

**Table 1**

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| number of directed games | 3 | 5 | 10 | 27 | 119 | 1173 | 44315 | 16175190 | ? |
| number of edges in directed lattice | 2 | 4 | 10 | 36 | 224 | 3264 | 190162 | 110433364 | ? |
| number of weighted majority games | 3 | 5 | 10 | 27 | 119 | 1113 | 29375 | 2730166 | ? |
| number of directed zero-sum games | 1 | 1 | 2 | 3 | 7 | 21 | 135 | 2470 | 319124 |
| number of games in $Z_n^r$ | 1 | 1 | 2 | 3 | 7 | 21 | 135 | 2470 | 175428 |
| number of homogeneous games | 1 | 3 | 8 | 23 | 76 | 293 | 1307 | 6642 | 37882 |

sponding lattice are the numbers of occuring shift minimal coalitions, since two directed games are joined by an edge, iff the larger one arises from the smaller one by dropping one shift minimal coalition in the corresponding filter. Analogously, the numbers of directed $n$-person zero-sum games are computed using the corresponding algorithm of Section 3 for $n = 1, \ldots, 9$. Testing these games on representability (see e.g. Algorithm II of Section 4) yields the numbers of directed $n$-person weighted majority zero-sum games (see the sixth

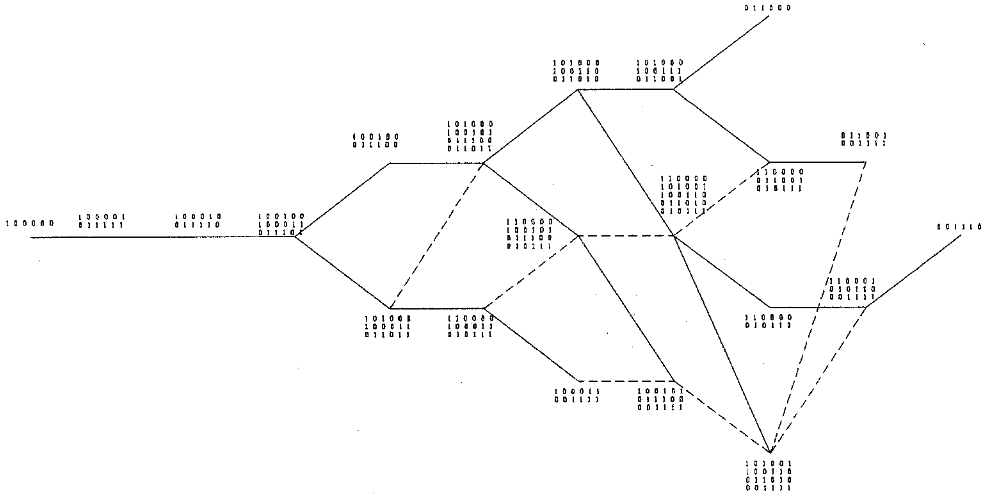

**Fig. 5**

**Table 2**

| $\frac{1}{2-p};\frac{m}{2-p}$ | | | | | | | | | | a minimal representation | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 30 ; | 17 | 9 | 8 | 6.5 | 6.5 | 5 | 3 | 2 | 2 | 30 ; | 17 | 9 | 8 | 7 | 6 | 5 | 3 | 2 | 2 |
| 25 ; | 13 | 7 | 6 | 6 | 4 | 4 | 4 | 2.5 | 2.5 | 25 ; | 13 | 7 | 6 | 6 | 4 | 4 | 4 | 3 | 2 |
| 27 ; | 14 | 9 | 6.5 | 6.5 | 5 | 5 | 3 | 2 | 2 | 27 ; | 14 | 9 | 7 | 6 | 5 | 5 | 3 | 2 | 2 |
| 33 ; | 17 | 12 | 8 | 8 | 6.5 | 6.5 | 3 | 2 | 2 | 33 ; | 17 | 12 | 8 | 8 | 7 | 6 | 3 | 2 | 2 |
| 28 ; | 13 | 9 | 7 | 7 | 6 | 4 | 4 | 2.5 | 2.5 | 28 ; | 13 | 9 | 7 | 7 | 6 | 4 | 4 | 3 | 2 |
| 24 ; | 11 | 9 | 6 | 6 | 4 | 4 | 4 | 1.5 | 1.5 | 24 ; | 11 | 9 | 6 | 6 | 4 | 4 | 4 | 2 | 1 |
| 28 ; | 13 | 11 | 8 | 6 | 6 | 4 | 4 | 1.5 | 1.5 | 28 ; | 13 | 11 | 8 | 6 | 6 | 4 | 4 | 2 | 1 |
| 28 ; | 13 | 11 | 7 | 7 | 5 | 5 | 4 | 1.5 | 1.5 | 28 ; | 13 | 11 | 7 | 7 | 5 | 5 | 4 | 2 | 1 |
| 32 ; | 15 | 13 | 9 | 7 | 7 | 5 | 4 | 1.5 | 1.5 | 32 ; | 15 | 13 | 9 | 7 | 7 | 5 | 4 | 2 | 1 |
| 31 ; | 14.5 | 12.5 | 9.5 | 7.5 | 6 | 4 | 4 | 1.5 | 1.5 | 32 ; | 15 | 13 | 10 | 8 | 6 | 4 | 4 | 2 | 1 |
| 35 ; | 16.5 | 14.5 | 10.5 | 8.5 | 7 | 5 | 4 | 1.5 | 1.5 | 36 ; | 17 | 15 | 11 | 9 | 7 | 5 | 4 | 2 | 1 |
| 34 ; | 16 | 14 | 11 | 9 | 6 | 4 | 4 | 1.5 | 1.5 | 34 ; | 16 | 14 | 11 | 9 | 6 | 4 | 4 | 2 | 1 |
| 38 ; | 18 | 16 | 12 | 10 | 7 | 5 | 4 | 1.5 | 1.5 | 38 ; | 18 | 16 | 12 | 10 | 7 | 5 | 4 | 2 | 1 |
| 33 ; | 13 | 11 | 10 | 8 | 6 | 6 | 4.5 | 4.5 | 2 | 33 ; | 13 | 11 | 10 | 8 | 6 | 6 | 5 | 4 | 2 |

row). The numbers of directed $n$-person weighted majority games (see the fourth row) are obtained by considering the types of the corresponding zero-sum extensions due to Corollary 1.8.

In order to illustrate the extraordinary growth of the numbers of games of the just mentioned classes we additionally show the numbers of homogeneous games, which are easily computed using the recursive formulae of Sudhölter (1989). For further results concerning homogeneity see Ostmann (1987a) and Rosenmüller (1987).

Fig. 5 sketches the Hasse diagram of the directed 6-person zero-sum games. The tree, consisting of all vertices, i.e. the corresponding shift minimal matrices, and the "straight line" edges, is generated by the original algorithm presented after Proposition 3.4. The additional edges result from the corresponding modified algorithm.

Table 2 shows the 14 games mentioned in the Concluding Remarks (2) of Section 4. The left hand side representation is computed using Algorithm I and coincides, up to normalization, with the nucleolus. Rows 10 and 11 are the "pure" exceptions.

## References


Aumann RJ, Peleg B, Rabinowitz P (1965) A method for computing the kernel of $n$-person games. Mathematics of Computation 19:531–551

Brickmann L (1989) Mathematical introduction to linear programming and game theory. Springer-Verlag New York

Dubey P, Shapley LS (1978) Mathematical properties of the banzhaf power index. The Rand Corporation 99–131

Einy E (1985) The desirability relation of simple games. Math Social Sc 10:155–168

Einy E, Lehrer E (1989) Regular simple games. Int Journal of Game Theory 18:195–208

Engel K, Gronau HD (1985) Sperner theory in partially ordered sets. Teubner Verlag Leipzig

Erdös P (1965) Extremal problems in number theory. Theory of numbers. Whiteman AL (Ed) Amer Math Soc Proc Symp Pure Math 8:181–189

Euler L (1750) De partitione numerorum. Commentationes Arithmeticae

Isbell JR (1959) On the enumeraton of majority games. Math Tables Aids Comput 13:21–28

Kopelowitz A (1967) Computation of the kernel of simple games and the nucleolus of $n$-person games. Research program in game theory and math economics. Dept of Math The Hebrew University of Jerusalem RM 31

Maschler M, Peleg B (1966) A characterization, existence proof and dimension bounds for the kernel of a game. Pacific J Math 18:289–328

Maschler M, Peleg B, Shapley LS (1979) Geometric properties of the kernel, nucleolus and related solution concepts. Math of Operations Research 4:303–338

von Neumann J, Morgenstern O (1944) Theory of games and economic behavior. Princeton University Press New Jersey

Ostmann A (1987) Life-length of a process with elements of decreasing importance. Working Paper 156 Inst of Math Ec University of Bielefeld

Ostmann A (1987a) On the minimal representation of homogeneous games. Int Journal of Game Theory 16:69–81

Ostmann A (1989) Simple games: On order and symmetrie. Working Paper 169 Inst of Math Ec University of Bielefeld

Peleg B (1968) On weights of constant-sum majority games. SIAM J of Appl Math 16:527–532

Peleg B, Rosenmüller J (1992) The least-core, nucleolus, and kernel of homogeneous weighted majority games. Games and Economic Behaviour 4:588–605

Peleg B, Rosenmüller J, Sudhölter P (1994) The kernel of homogeneous games with steps. Essays in game theory in honor of Michael Maschler, Jerusalem. Nimrod M (ed) 163–192

Proctor RA (1982a) Representations of sl (2, $\mathbb{C}$) on posets and the sperner property. SIAM J Algebraic Discrete Methods 3:275–280

Proctor RA (1982b) Solution of two difficult combinatorial problems with linear algebra. The American Mathematical Monthly 89:721–734

Rosenmüller J (1981) The theory of games and markets. North Holland Publ Comp

Rosenmüller (1987) Homogeneous games: Recursive structure and computation. Math of Operations Research 12:309–330

Rosenmüller J, Sudhölter P (1994) The nucleolus of homogeneous games with steps. Discrete Applied Mathematics 50:53–76

Schmeidler D (1966) The nucleolus of a characteristic function game. Research program in game theory and math economics. The Hebrew University of Jerusalem RM 23

Shapley LS (1962) Simple games: An outline of the descriptive theory. Behavioral Sci 7:59–66

Stanley RP (1980) Weyl groups, the hard lefschetz theorem, and the sperner property. SIAM J Algebraic Discrete Methods 1:168–184

Sudhölter P (1989) Homogeneous games as anti step functions. Int Journal of Game Theory 18: 433–469

Wolsey LA (1976) The nucleolus and kernel of simple games or special valid inequalities for 0–1 linear integer programs. Int Journal of Game Theory 5:227–238

Received: April 1993
Revised version received: October 1994