

The positive core of a cooperative game

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Abstract The positive core is a nonempty extension of the core of transferable utility games. If the core is nonempty, then it coincides with the core. It shares many properties with the core. Six well-known axioms that are employed in some axiomatizations of the core, the prenucleolus, or the positive prekernel, and one new intuitive axiom, characterize the positive core for any infinite universe of players. This new axiom requires that the solution of a game, whenever it is nonempty, contains an element that is invariant under any symmetry of the game.

Keywords TU game · Solution concept · Core · Kernel · Nucleolus · Positive core

JEL Classification C71

Dedicated to our teacher and friend Michael Maschler.

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1 Introduction

The positive core is a set-valued solution of cooperative transferable utility games. Its definition strongly relates to the definition of the prenucleolus. A preimputation belongs to the prenucleolus of a game, if it lexicographically minimizes the excesses of the coalitions. The definition of the positive core differs from the definition of the prenucleolus only inasmuch as the excesses are replaced by their *positive* parts. Hence, the prenucleolus is a subsolution of the positive core. On the other hand, if the core of a game is nonempty, then the positive parts of all excesses at elements of the core are zero, hence the core coincides with the positive core, whenever the core is nonempty.

The concept of the positive core is not new. Indeed, [Sudhölter \(1993\)](#) used it to show that anonymity is logically independent of the remaining axioms in Sobolev's (1975) characterization of the prenucleolus. In 1993, in a letter to the second author, Peleg indicated that a student of Maschler, namely the first author, was aware of the importance of the positive core. Indeed, Maschler motivated the investigation of non-symmetric prekernels, that is, of solutions that satisfy the axioms of Peleg's (1986) axiomatization of the prekernel with the exception that the equal treatment property is replaced by single-valuedness for two-person games. In this context Maschler invented the expression "positive core" that was first mentioned in the first author's PhD thesis (see [Orshan 1994](#)). Though this solution was already regarded as an interesting core extension at that time, it served as an auxiliary solution only (see also [Orshan and Sudhölter 2003](#)). The present paper aims to show that the positive core is interesting in its own right.

Note that if an excess of a coalition S at a proposal x (see Sect. 2 for the formal definition) is positive, then it may be interpreted as the dissatisfaction of S when faced with x . If the excess is not positive, then the coalition is satisfied, that is, its dissatisfaction is 0. Hence, the positive core of a game consists of all preimputations that lexicographically minimize the dissatisfactions of the coalitions.

Though the core is regarded as one of the most intuitive solutions for cooperative games, there is (at least) one serious drawback: It specifies the empty set for many remarkable games. From a normative point of view, the elements of a solution of a game are interpreted as proposals of how to solve the game. If the solution applied to some game is empty, then this game cannot be solved. In this sense the core, when regarded a normative solution, should be applied to classes of balanced games (games with a nonempty core) only. As soon as non-balanced games have to be considered the core is a less suitable solution. It is then natural to replace the core by a solution that (a) contains the core as a subsolution and that (b) is nonempty when applied to any game under consideration. There are well-known nonempty core extensions. Indeed, the literature mentions various prebargaining sets (see, e.g., [Aumann and Maschler 1964](#); [Granot and Maschler 1997](#); or [Sudhölter and Potters 2001](#)) that contain the core. However, even if the core is nonempty, they can be much larger than the core. They may contain counter intuitive proposals that are ruled out by the core. Recently [Sudhölter and Peleg \(2000\)](#) introduced and axiomatized a smaller nonempty core extension, the positive prekernel (see Sect. 2 for the precise definition). Though this solution is a subsolution of the prebargaining sets, the core of a balanced game may be a proper subset of the positive prekernel of this game. It is the aim of this paper to show that the

positive core is characterized (see Remark 4.3 and Corollary 5.1) by intuitive simple axioms. Only one property (see Definition 3.2) is employed that is not used in some axiomatizations of the prekernel, the positive prekernel, the prenucleolus, or the core.

The paper is organized as follows: In Sect. 2 the notation and some definitions are presented. Moreover, some characterizations of solutions known from literature and relevant in the context of the positive core, are recalled.

The positive core satisfies nonemptiness, anonymity, covariance under strategic equivalence, and it is Pareto optimal and reasonable. The core satisfies three variants of reduced game properties: The reduced game property (RGP), the converse reduced game property (CRGP), and the reconfirmation property (RCP). The positive prekernel satisfies two of them, namely RGP and CRGP. In Sect. 3 it is shown that the positive core satisfies RGP and RCP. Moreover it shares a further property with the core, the (positive) prekernel, and the nucleolus. Indeed, these solutions allow for non-discrimination (ND). This axiom requires from the solution of a game that it, whenever nonempty, contains a preimputation that does not discriminate (i.e., that is invariant under all symmetries of the game).

The main result, Theorem 4.1, is formulated and proved in Sect. 4. It states that the prenucleolus, the positive core, and its relative interior, are the unique solutions that are nonempty-valued and satisfy reasonableness, covariance, RGP, RCP, and ND, provided that the universe of players is infinite. It turns out that ND may be replaced by anonymity and convex-valuedness. Moreover, the aforementioned results follow from Theorem 4.2 in which ND is replaced by anonymity and the condition that the solution contains the prenucleolus as a subsolution. Section 4.1 presents examples showing that each axiom in Theorem 4.1 and in Corollary 4.4 is logically independent of the remaining axioms. Section 4.2 is devoted to the proof of Theorem 4.2. We investigate the behavior and shape of solutions that are nonempty-valued and reasonable and satisfy covariance, RGP, and RCP. First it turns out (see Theorem 4.5) that the restriction of such a solution to one- and two-person games is a subsolution of the positive core. If it also satisfies anonymity then, restricted to two-person games, it coincides either with the prenucleolus, or the relative interior of the positive core, or the positive core. A technical part of the required proof occurs in the Appendix. In order to generalize this result to arbitrary games, it is shown (see Lemma 4.9) that the positive parts of the excesses at elements of the solution of a game have to coincide. For this and the preceding result the assumption of an infinite universe of potential players is crucial. Lemma 4.9 is used to prove that Theorem 4.5 can be generalized, if the solution is assumed to contain the prenucleolus. We conclude with some remarks in Sect. 5.

2 Notation and definitions

Let U be a set (the universe of players) containing, for simplicity, $1, \dots, k$ whenever $|U| \geq k$. A (cooperative transferable utility) *game* is a pair (N, v) such that $\emptyset \neq N \subseteq U$ is finite (N is the player set) and $v : 2^N \rightarrow \mathbb{R}$, $v(\emptyset) = 0$ (v is the coalition function). Here $2^N = \{S \subseteq N\}$ is the set of *coalitions*. Let (N, v) be a game. Then

$$X(N, v) = \{x \in \mathbb{R}^N \mid x(N) \leq v(N)\} \text{ and } \mathcal{I}^*(N, v) = \{x \in \mathbb{R}^N \mid x(N) = v(N)\}$$

denote the set of *feasible payoffs* and the set of *Pareto optimal feasible payoffs (preimputations)* of (N, v) , where we use $x(S) = \sum_{i \in S} x_i$ ($x(\emptyset) = 0$) for every $S \in 2^N$ and every $x \in \mathbb{R}^N$ as a convention. Additionally, x_S denotes the restriction of x to S , i.e., $x_S = (x_i)_{i \in S}$, and for disjoint coalitions S, T let $(x_S, x_T) = x_{S \cup T}$. The following abbreviations are used to recall the definitions of several well-known solutions that are relevant in the sequel. Let $S \subseteq N$, let $k, \ell \in N$ be distinct, and let $x \in \mathbb{R}^N$. Then $e(S, x, v) = v(S) - x(S)$ denotes the *excess* of S , $\mu(x, v) = \max_{S \subseteq N} e(S, x, v)$ denotes the *maximal excess* and

$$s_{k\ell}(x, v) = \max\{e(S, x, v) \mid S \subseteq N, \ell \notin S \ni k\}$$

denotes the *maximal surplus* of k against ℓ at x with respect to (w.r.t.) (N, v) .

The *core* of (N, v) is the set $\mathcal{C}(N, v) = \{x \in \mathcal{I}^*(N, v) \mid e(S, x, v) \leq 0 \ \forall S \subseteq N\}$ and the *prekernel* of (N, v) is the set

$$\mathcal{K}^*(N, v) = \{x \in \mathcal{I}^*(N, v) \mid s_{k\ell}(x, v) = s_{\ell k}(x, v) \ \forall k \in N \text{ and } \ell \in N \setminus \{k\}\}.$$

The *positive prekernel* of (N, v) is the set

$$\mathcal{K}_+^*(N, v) = \{x \in \mathcal{I}^*(N, v) \mid (s_{k\ell}(x, v))_+ = (s_{\ell k}(x, v))_+ \ \forall k \in N \text{ and } \ell \in N \setminus \{k\}\},$$

where $t_+ = \max\{t, 0\}$ denotes the positive part of a real t . The *prenucleolus* of (N, v) , denoted $\nu(N, v)$, is the set of preimputations that lexicographically minimize the non-increasingly ordered vector of excesses of the coalitions. The set $\nu(N, v)$ is a singleton $\{\nu(N, v)\}$.

In general, a *solution* σ is a mapping that associates with every game (N, v) a set $\sigma(N, v) \subseteq X(N, v)$.

Gillies (1959) introduced the core, for the prekernel and prenucleolus we refer to Maschler et al. (1972) and to Schmeidler (1969). The positive prekernel was introduced by Sudhölter and Peleg (2000). Now we are able to define the positive core.

Definition 2.1 The *positive core* of a game (N, v) is the set $\mathcal{C}_+(N, v)$ defined by

$$\mathcal{C}_+(N, v) = \{x \in \mathcal{I}^*(N, v) \mid (e(S, x, v))_+ = (e(S, \nu(N, v), v))_+ \ \forall S \subseteq N\}.$$

The prenucleolus is the preimputation which minimizes the highest excess, then minimizes the number of coalitions attaining highest excess, then minimizes the second highest excess, and so on. Thus, the prenucleolus is yielded by iteratively solving a finite sequence of linear programs. The positive core can be computed by solving the same sequence of linear programs as long as the excesses are positive.

A member of the positive core lexicographically minimizes the non-increasingly ordered vector of *positive* excesses, whereas the prenucleolus lexicographically minimizes the non-increasingly ordered vector of *all* excesses. A member of the positive prekernel balances the maximal surplus of distinct players whenever it is *positive*, whereas a member of the prekernel balances the maximal surplus of *all* pairs of distinct players. In this sense the positive core of a game arises from the prenucleolus of the game in a similar way as the positive prekernel arises from the prekernel.

Clearly, $\mathcal{C}_+(N, v) = \mathcal{C}(N, v)$, if $\mathcal{C}(N, v) \neq \emptyset$, and $v(N, v) \in \mathcal{C}_+(N, v)$.

Now we recall the definition of the reduced game. For any $z \in X(N, v)$ and any $\emptyset \neq S \subseteq N$ the *reduced game* w.r.t. S and z , $(S, v^{S,z})$, is defined by

$$v^{S,z}(T) = \begin{cases} 0, & \text{if } T = \emptyset, \\ v(N) - z(N \setminus S), & \text{if } T = S, \\ \max_{Q \subseteq N \setminus S} (v(T \cup Q) - z(Q)), & \text{if } \emptyset \neq T \subsetneq S. \end{cases}$$

Some intuitive and well-known properties of a solution σ are as follows. The solution σ

- (1) is *covariant under strategic equivalence* (COV), if for any game (N, v) , any $z \in \mathbb{R}^N$, and any $\beta > 0$, $\sigma(N, \beta v + z) = \beta \sigma(N, v) + z$ (the games (N, v) and $(N, \beta v + z)$ are *strategically equivalent*, where, by convention, $z \in \mathbb{R}^N$ is identified with the *additive* coalitional function, again denoted by z , on the player set N defined by $z(S) = \sum_{i \in S} z_i$ for all $S \in 2^N$);
- (2) is *nonempty* (NE), if $\sigma(N, v) \neq \emptyset$ for every game (N, v) ;
- (3) is *Pareto optimal* (PO), if $\sigma(N, v) \subseteq \mathcal{I}^*(N, v)$ for every game (N, v) ;
- (4) is *single-valued* (SIVA), if $|\sigma(N, v)| = 1$ for every game (N, v) ;
- (5) is *anonymous* (AN), if the following condition is satisfied for all games (N, v) and (M, w) : If $\pi : N \rightarrow M$ is a bijection such that $\pi v = w$ (where $\pi v(S) = v(\pi^{-1}(S)) \forall S \subseteq M$), then $\sigma(M, w) = \pi(\sigma(N, v))$ (where, for any $x \in \mathbb{R}^N$, $\pi x \in \mathbb{R}^M$ is defined by $\pi x_j = x_{\pi^{-1}(j)} \forall j \in M$)—in this case the games (N, v) and (M, w) are *isomorphic*;
- (6) satisfies the *equal treatment property* (ETP), if for every game (N, v) , for every $x \in \sigma(N, v)$, $x_k = x_\ell$ for all substitutes $k, \ell \in N$ (where k and ℓ are *substitutes*, if $v(S \cup \{k\}) = v(S \cup \{\ell\}) \forall S \subseteq N \setminus \{k, \ell\}$);
- (7) satisfies the *zero inessential two-person game property* (ZIG), if for every game (N, v) with $|N| = 2$ and $v = 0$, $\sigma(N, v) \neq \emptyset$;
- (8) is *reasonable* (REAS), if, for every game (N, v) , for every $x \in \sigma(N, v)$, and for every $i \in N$,

$$\min_{S \subseteq N \setminus \{i\}} (v(S \cup \{i\}) - v(S)) \leq x_i \leq \max_{S \subseteq N \setminus \{i\}} (v(S \cup \{i\}) - v(S));$$

- (9) satisfies *weak unanimity for two-person games* (WUTPG), if, for every game (N, v) with $|N| = 2$, $\{x \in \mathcal{I}^*(N, v) \mid x_i \geq v(\{i\}) \forall i \in N\} \subseteq \sigma(N, v)$;
- (10) satisfies the *reduced game property* (RGP), if for every game (N, v) , $\emptyset \neq S \subseteq N$, and every $x \in \sigma(N, v)$, $x_S \in \sigma(S, v^{S,x})$;
- (11) satisfies the *converse reduced game property* (CRGP), if for every game (N, v) with $|N| \geq 2$ the following condition is satisfied for every $x \in \mathcal{I}^*(N, v)$: If, for every $S \subseteq N$ with $|S| = 2$, $x_S \in \sigma(S, v^{S,x})$, then $x \in \sigma(N, v)$;
- (12) satisfies the *reconfirmation property* (RCP), if for every game (N, v) , $\emptyset \neq S \subseteq N$, for every $x \in \sigma(N, v)$ and $y \in \sigma(S, v^{S,x})$, $(y, x_{N \setminus S}) \in \sigma(N, v)$.

With the help of these axioms the prekernel, the prenucleolus, the positive prekernel, and the core can be characterized. We recall some of the axiomatizations.

Theorem 2.2 (Peleg 1986) *The unique solution that satisfies NE, PO, COV, ETP, RGP, and CRGP is the prekernel.*

Theorem 2.3 (Sobolev 1975) *The unique solution that satisfies SIVA, COV, AN, and RGP is the prenucleolus, provided $|U| = \infty$.*

Theorem 2.4 (Sudhölter and Peleg 2000) *The unique solution that satisfies NE, AN, REAS, RGP, CRGP, and WUTPG is the positive prekernel, provided $|U| \geq 3$.*

Theorem 2.5 (Hwang and Sudhölter 2001) *The unique solution that satisfies ZIG, COV, AN, RGP, RCP, CRGP, and REAS is the core, provided $|U| \geq 5$.*

Remark 2.6 (1) In Theorem 2.2, CRGP is a substitute for maximality, i.e., each solution that satisfies the remaining axioms is contained in the prekernel (see Peleg 1986). Hence the prekernel is the *maximum* solution that satisfies the remaining five axioms.

- (2) Orshan (1993) shows that AN can be replaced by ETP in Theorem 2.3.
 (3) Note that the prenucleolus is also axiomatized by NE, COV, ETP, and RCP (see Orshan and Sudhölter 2003).

Remark 2.7 The positive core of a game is the set of all preimputations that lexicographically minimize the non-increasingly ordered vector of the positive parts of the excesses. A proof of this alternative “definition” of the positive core is straightforward.

The following simple results are useful. Let Γ^2 be the set of games with at most two players.

Remark 2.8 Let (N, v) be a game, $\emptyset \neq S \subseteq N$, $k, \ell \in S$, $k \neq \ell$, and let $x \in \mathbb{R}^N$. Then

$$s_{k\ell}(x, v) = s_{k\ell}(x_S, v^{S,x}). \quad (2.1)$$

Remark 2.9 (cf. Lemma 4.3 of Hwang and Sudhölter 2001) If σ satisfies REAS and RGP, then σ also satisfies PO.

Remark 2.10 (Lemma 7.1 of Sudhölter and Peleg 2000) Let σ^1, σ^2 be solutions. If σ^1 satisfies PO and RGP, if σ^2 satisfies CRGP, and if $\sigma^1(N, v) \subseteq \sigma^2(N, v)$ for every game $(N, v) \in \Gamma^2$, then σ^1 is a subsolution of σ^2 .

3 Properties and examples

This section serves to show that the positive core has many properties in common with the core, the (positive) prekernel, and the nucleolus. Moreover, an algebraic characterization by balanced collections of coalitions is provided. We shall also use the *relative interior*, denoted $\text{rint } \mathcal{C}_+$ of the positive core. Let (N, v) be a game. Then

$$\text{rint } \mathcal{C}_+(N, v) = \{x \in \mathcal{C}_+(N, v) \mid \forall S \subseteq N : e(S, v(N, v), v) < 0 \Rightarrow e(S, x, v) < 0\}.$$

The positive core satisfies NE and, hence, ZIG, because it contains the prenucleolus. Clearly, it satisfies COV, AN, and PO. It is a subsolution of the positive prekernel which satisfies REAS (see [Sudhölter and Peleg 2000](#)), thus it satisfies REAS as well. The relative interior of the positive core satisfies the same axioms. For two-person games with a non-empty core, C_+ coincides with the core and for two-person games with an empty core it coincides with the prenucleolus. Therefore it satisfies, different from $\text{rint } C_+$, WUTPG.

Remark 3.1 It is well-known that the prekernel of any three-person game coincides with its prenucleolus. Hence, by Theorem 4.1 of [Sudhölter and Peleg \(2000\)](#), the positive core of any three-person game coincides with its positive prekernel.

A further property of the positive core is interesting. Let (N, v) be a game. A *symmetry* of (N, v) is a permutation π of N such that $\pi v = v$. Let $\text{SYM}(N, v)$ denote the group of symmetries of (N, v) .

Definition 3.2 A solution σ allows for non-discrimination (ND), if the following property is valid for every game (N, v) : If $\sigma(N, v) \neq \emptyset$, then there exists $x \in \sigma(N, v)$ such that $\pi x = x$ for every $\pi \in \text{SYM}(N, v)$.

ND has the following simple interpretation: If the players of a game (N, v) are able to agree upon a proposal to solve the game (that is, if $\sigma(N, v) \neq \emptyset$), then ND requires that they should as well be able to agree upon a proposal, which is invariant under all symmetries of the game and which, hence, does not discriminate.

There is a “natural” strong version of ND. Indeed, a solution σ satisfies *strong non-discrimination* (SND), if every $x \in \sigma(N, v)$ is invariant under symmetries. Note that any solution σ that satisfies SIVA and AN also satisfies SND. Hence the prenucleolus satisfies SND and the (positive) prekernel, the core, the (relative interior of the) positive core, and many other well-known solutions satisfy ND. Note that there are also set-valued solutions that satisfy SND (e.g., the maximal satisfaction solution introduced by [Sudhölter and Peleg 1998](#)).

The following approach results in a characterization of the positive core by balanced collections of coalitions and allows to deduce RGP and RCP.

For every game (N, v) , for every $x \in \mathbb{R}^N$ and every $\alpha \in \mathbb{R}$ denote $\mathcal{D}(\alpha, x, v) = \{S \subseteq N \mid e(S, x, v) \geq \alpha\}$. Let $S \subseteq N$. The *characteristic vector* χ^S of S is the member of \mathbb{R}^N which is given by

$$\chi_i^S = \begin{cases} 1, & \text{if } i \in S, \\ 0, & \text{if } i \in N \setminus S. \end{cases}$$

A collection $\mathcal{D} \subseteq 2^N$ of coalitions is *balanced* (over N), if there are coefficients $\gamma_S > 0$, $S \in \mathcal{D}$, such that $\sum_{S \in \mathcal{D}} \gamma_S \chi^S = \chi^N$. The collection $(\delta_S)_{S \in \mathcal{D}}$ is called a system of *balancing weights*.

We first recall a result of [Kohlberg \(1971\)](#) characterizing the prenucleolus.

Theorem 3.3 *Let (N, v) be a game and $x \in \mathcal{I}^*(N, v)$. Then the following assertions are equivalent: (a) $x = v(N, v)$. (b) For every $\alpha \in \mathbb{R}$ and every $y \in \mathbb{R}^N$ satisfying*

$y(N) = 0$ and $y(S) \geq 0$ for all $S \in \mathcal{D}(\alpha, x, v)$, $y(S) = 0$ for all $S \in \mathcal{D}(\alpha, x, v)$. (c) For every $\alpha \in \mathbb{R}$, $\mathcal{D}(\alpha, x, v)$ is balanced or empty.

The equivalence of (a) and (b) may be proved directly and the equivalence of (b) and (c) is a direct consequence of the duality theorem of linear programming. In a completely analogous way the following result may be proved.

Theorem 3.4 *Let (N, v) be a game and $x \in \mathcal{I}^*(N, v)$. Then the following assertions are equivalent: (a) $x \in \mathcal{C}_+(N, v)$. (b) For every positive $\alpha \in \mathbb{R}$ and every $y \in \mathbb{R}^N$ satisfying $y(N) = 0$ and $y(S) \geq 0$ for all $S \in \mathcal{D}(\alpha, x, v)$, $y(S) = 0$ for all $S \in \mathcal{D}(\alpha, x, v)$. (c) For every positive $\alpha \in \mathbb{R}$, $\mathcal{D}(\alpha, x, v)$ is balanced or empty.*

Theorem 3.4 implies (see Theorem 6.3.14 in Peleg and Sudhölter 2003) that the positive core satisfies RGP and RCP. Moreover, the relative interior of the positive core may be characterized by modifying Theorem 3.4 only inasmuch as \mathcal{C}_+ has to be replaced by $\text{rint } \mathcal{C}_+$ and positive has to be replaced twice by nonnegative. This modification implies that the relative interior of the positive core satisfies RGP and RCP as well. Hence, the following theorem is valid.

Theorem 3.5 *The positive core and its relative interior satisfy RGP and RCP.*

We shall now present an example which shows that the positive core does not satisfy CRGP. Sudhölter and Peleg (2002) use a similar example.

Example 3.6 Let $N = \{1, 2, 3, 4\}$ and $v(S)$, $S \subseteq N$, be defined by

$$v(S) = \begin{cases} 0, & \text{if } S \in \{\emptyset, N\}, \\ 2, & \text{if } S \in \{S^i \mid i = 1, 2, 3, 4\}, \\ -2, & \text{otherwise,} \end{cases}$$

where $S^1 = \{1, 2\}$, $S^2 = \{2, 3\}$, $S^3 = \{3, 4\}$, and $S^4 = \{1, 4\}$. Note that $\text{SYM}(N, v)$ contains the subgroup generated by the cyclic permutation, which maps 1 to 2, 2 to 3, 3 to 4, and 4 to 1; thus the game is *transitive*. (A game is called *transitive*, if its symmetry group is transitive.) As v satisfies ND and PO, $v(N, v) = 0$. Thus $e(S, v(N, v), v) = 2$ for $S \in \{S^i \mid i = 1, \dots, 4\}$ and $e(T, v(N, v), v) \leq 0$ for all other coalitions. Therefore,

$$\mathcal{C}_+(N, v) = \text{convh}\{(1, -1, 1, -1), (-1, 1, -1, 1)\},$$

where “convh” means “convex hull”. Let $x \in \text{convh}\{(2, -2, 2, -2), (-2, 2, -2, 2)\}$, $k \in N$, and $\ell \in N \setminus \{k\}$. Then $s_{k\ell}(x, v) = 2$ and $s_{k\ell}(x, v)$ is attained by some S^i , $i = 1, \dots, 4$. Thus

$$\text{convh}\{(2, -2, 2, -2), (-2, 2, -2, 2)\} \subseteq \mathcal{K}^*(N, v).$$

The current example together with Theorem 2.4 shows that the positive core does not satisfy CRGP when $|U| \geq 4$.

Remark 3.7 The positive core shares another property with the core: The positive core is *convex-valued* (CON). Indeed, Definition 2.1 directly implies that the positive core of a game is a compact convex polyhedral set.

In the next section we shall need the following easy result.

Remark 3.8 (see, e.g., Remark 2.7 of *Sudhölter 1997*) Let $\emptyset \neq N$ be a finite set and let \mathcal{D} be a balanced collection over N . If $T \subseteq N$ satisfies $\chi^T \in \langle \{\chi^S \mid S \in \mathcal{D}\} \rangle$, then $\mathcal{D} \cup \{T\}$ is balanced. Here $\langle \dots \rangle$ denotes the linear span. (We say that \mathcal{D} spans $\mathcal{D} \cup \{T\}$.)

4 A characterization of the positive core

This section is organized as follows. The present part is devoted to a discussion of our main result, i.e., Theorem 4.1. We show that this result is implied by Theorem 4.2 and that it may be used to characterize the positive core by simple and intuitive axioms (see Corollary 5.1). In Sect. 4.2 we prove Theorem 4.2 and in Sect. 4.1 the logical independence of the employed axioms is discussed. Throughout we shall assume that σ is a solution.

Theorem 4.1 *Assume that $|U| = \infty$. The solution σ satisfies NE, REAS, COV, RGP, RCP, and ND, if and only if σ coincides*

- (1) *with the prenucleolus or*
- (2) *with the relative interior of the positive core or*
- (3) *with the positive core.*

We now show that NE and ND are implying AN and that the prenucleolus is a subsolution, i.e., that Theorem 4.1 is implied by the following result.

Theorem 4.2 *Assume that $|U| = \infty$. The solution σ satisfies REAS, COV, AN, RGP, RCP, and \mathbf{v} is a subsolution of σ , if and only if $\sigma \in \{\mathbf{v}, \text{rint } C_+, C_+\}$.*

Proof of Theorem 4.1 The three mentioned solutions satisfy the required properties. Let σ satisfy the axioms. By Theorems 4.2 it suffices to prove (a) that \mathbf{v} is a subsolution of σ and (b) that σ satisfies AN. Let (N, v) be a game. By COV we may assume without loss of generality that $v(N, v) = 0 \in \mathbb{R}^N$.

We first prove (a). By our infinity assumption on $|U|$ and according to *Sobolev (1975)* there exists a game (\tilde{N}, w) satisfying the following properties:

- (1) $v(\tilde{N}, w) = 0 \in \mathbb{R}^{\tilde{N}}$.
- (2) $N \subseteq \tilde{N}$ and $w^{N,0} = v$.
- (3) (\tilde{N}, w) is transitive.

By NE and ND and transitivity of (\tilde{N}, w) there exists $z \in \sigma(\tilde{N}, w)$ such that $z_i = z_j$ for all $i, j \in \tilde{N}$. By Remark 2.9, σ satisfies PO, thus $z = v(\tilde{N}, w)$. By RGP of \mathbf{v} , $z_N = v(N, v)$. By RGP of σ , $z_N \in \sigma(N, v)$.

It remains to show (b). Let $\tilde{N} \subseteq U$ be such that there is a bijection $\pi : N \rightarrow \tilde{N}$. It has to be proved that $\pi(\sigma(N, v)) \subseteq \sigma(\tilde{N}, \pi v)$. By the infinity assumption on $|U|$ there

is a further set of players of the same cardinality in U disjoint from both, N and \bar{N} , thus we may assume that N and \bar{N} are disjoint. Let $x \in \sigma(N, v)$. Then $\pi(x) \in \sigma(\bar{N}, \pi v)$ remains to be shown. We are going to define a “replicated” game $(N \cup \bar{N}, u)$ such that the reduced games w.r.t. N and \bar{N} and the prenucleolus coincide with the games (N, v) and $(\bar{N}, \pi v)$ respectively. Put $\alpha = \min_{R, S \subseteq N} \min\{e(S, v(N, v), v), e(S, x, v) + x(R)\}$ and define, for all $S \subseteq N, T \subseteq \bar{N}$,

$$u(S \cup T) = \begin{cases} v(S), & \text{if } T = \pi(S), \\ \alpha, & \text{otherwise.} \end{cases}$$

Let $y = 0 \in \mathbb{R}^{N \cup \bar{N}}$ and observe that the equation

$$\mathcal{D}(\beta, y, u) = \begin{cases} \{S \cup \pi(S) \mid S \in \mathcal{D}(\beta, v(N, v), v)\}, & \text{if } \beta > \alpha, \\ 2^{N \cup \bar{N}}, & \text{otherwise} \end{cases}$$

is valid, thus $y = v(N \cup \bar{N}, u)$ by Theorem 3.3. By (a), $y \in \sigma(N \cup \bar{N}, u)$. Observe that $u^{N,y} = v$ by construction, thus $z := (x, y_{\bar{N}}) \in \sigma(N \cup \bar{N}, u)$ by RCP. Again the construction of u yields $u^{\bar{N},z} = \pi v - \pi(x)$, thus $\pi x \in \sigma(\bar{N}, \pi v)$ by COV and RGP. □

Remark 4.3 In Theorem 4.2 the property that v is a subsolution of σ may be replaced by NE and CON (see Remark 3.7). We may prove this statement by first proceeding as in the proof of Theorem 4.1 till and including the construction of (\tilde{N}, w) . Then, by NE, there exists $y \in \sigma(\tilde{N}, w)$, and, by Remark 2.9, $y(\tilde{N}) = 0$. By AN, $\pi(y) \in \sigma(\tilde{N}, w)$ for every $\pi \in \text{SYM}(\tilde{N}, w)$. By CON,

$$z = \frac{1}{|\text{SYM}(\tilde{N}, w)|} \sum_{\pi \in \text{SYM}(\tilde{N}, w)} \pi(y) \in \sigma(\tilde{N}, w).$$

By transitivity, $z = 0 = v(\tilde{N}, w)$. By RGP of v , $z_N = v(N, v)$. By RGP of σ , $v(N, v) \in \sigma(N, v)$ so that the proof is complete.

We say that σ is *closed-valued* (CLOS) if $\sigma(N, v)$ is closed for any game (N, v) . With the help of CLOS the positive core may be characterized as follows.

Corollary 4.4 *Assume that $|U| = \infty$ and $\sigma \neq v$. Then σ satisfies NE, REAS, COV, RGP, RCP, CLOS, and ND, if and only if $\sigma = C_+$.*

4.1 On the independence of the axioms

Six examples are presented which show that each of the axioms (1) NE or “ v is a subsolution of σ ”, (2) REAS, (3) COV, (4) RGP, (5) RCP, and (6) ND or AN is logically independent of the remaining axioms in Theorems 4.1 and 4.2. Hence, Corollary 4.4 may be regarded as an axiomatization of the positive core.

Let U be a set, let $t \in \mathbb{R}, t \leq 0$, and let (N, v) be a game. Let $\sigma^i, i = 1, \dots, 5$, be defined by

$$\begin{aligned} \sigma^1(N, v) &= \mathcal{C}(N, v), \\ \sigma^2(N, v) &= \mathcal{I}^*(N, v), \\ \sigma^3(N, v) &= \{x \in \mathcal{I}^*(N, v) \mid \max\{e(S, x, v), t\} = \max\{e(S, v(N, v), v), t\} \forall S \subseteq N\}, \\ \sigma^4(N, v) &= \{x \in \mathcal{I}^*(N, v) \mid \mu(x, v) = \mu(v(N, v), v)\}, \text{ and} \\ \sigma^5(N, v) &= \mathcal{K}_+^*(N, v). \end{aligned}$$

Let \leq be a total order of U . For every finite set $N \subseteq U$ let \leq_N be the restriction of \leq to N and let \leq_{lex}^N be the induced lexicographical order on \mathbb{R}^N . Then σ^6 is defined by

$$\sigma^6(N, v) = \{x \in \mathcal{C}_+(N, v) \mid x \leq_{\text{lex}}^N y \forall y \in \mathcal{C}_+(N, v)\}.$$

It is straightforward to check that $\sigma^i, i = 1, \dots, 6$, satisfies all properties except the i th one. If $|U| \geq 4$ and $t \neq 0$, then none of the solutions coincides with \mathcal{C}_+ or $\text{rint } \mathcal{C}_+$.

4.2 The proof of Theorem 4.2

We recall that Γ^2 is the set of games with at most two players. We shall first derive some results for solutions when restricted to Γ^2 and assume throughout that $|U| \geq 3$.

Theorem 4.5 *The solution σ satisfies NE, REAS, COV, AN, RGP, and RCP, only if it coincides on Γ^2 with v or $\text{rint } \mathcal{C}_+$ or with \mathcal{C}_+ .*

We first show that a solution σ satisfies $\sigma(N, v) \subseteq \mathcal{C}_+(N, v)$ for all $(N, v) \in \Gamma^2$, if σ satisfies the axioms of Theorem 4.5 (where AN is not needed). The technical proof of the remaining part of Theorem 4.5 is contained in Sect. 6.

Lemma 4.6 *If σ satisfies NE, REAS, COV, RGP, and RCP, then $\sigma(N, v) \subseteq \mathcal{C}_+(N, v)$ for all $(N, v) \in \Gamma^2$.*

Proof By Remark 2.9, σ satisfies PO and the assertion is true for any 1-person game. Let $(N, v), |N| = 2$, be a game.

If $\mathcal{C}(N, v) \neq \emptyset$, then, by COV, we may assume that there exists $c \geq 0$ such that $v(\{\ell\}) = -c \leq 0$ for all $\ell \in N$ and $v(N) = 0$. If $x \in \sigma(N, v)$, then, by REAS, $-c \leq x_\ell \leq c$ so that $x \in \mathcal{C}_+(N, v)$ by PO.

If $\mathcal{C}(N, v) = \emptyset$, then, by COV we may assume that $v(\{i\}) = 1$ for $i \in N$ and $v(N) = 0$. Let $x \in \sigma(N, v)$. It remains to show that $x = 0$. Let $N = \{i, j\}$. By PO and REAS, $-1 \leq x_i \leq 1$ and $x_j = -x_i$. As $|U| \geq 3$, there exists $k \in U \setminus N$. Define $\tilde{N} = N \cup \{k\}$ define (\tilde{N}, w) by $w(S) = v(S \cap N)$ for all $S \subseteq \tilde{N}$. By NE there exists $z \in \sigma(\tilde{N}, w)$ and, by REAS, $z_k = 0$. Hence, $(N, w^{N,z}) = (N, v)$. Let $y \in \mathbb{R}^{\tilde{N}}$ be defined by $y_N = x$ and $y_k = 0$. By RCP, $y \in \sigma(\tilde{N}, w)$. Let $\pi : N \rightarrow \{i, k\}$ be the bijection that satisfies $\pi(i) = i$ and let $\tilde{x} \in \mathbb{R}^{\{i,k\}}$ be given by $\tilde{x}_i = 0$ and $\tilde{x}_k = x_i$. Then $\pi v = w^{\{i,k\},y} - \tilde{x}$. Hence, by RGP and COV, $\pi x \in \sigma(\{i, k\}, \pi v)$. By interchanging N with $\{j, k\}$ and k with i , we receive that, with $\tilde{\pi} : N \rightarrow \{i, k\}$ defined by

$\tilde{\pi}(i) = k$ and $\tilde{\pi}(j) = i$, $\tilde{\pi}x \in \sigma(\{i, k\}, \pi v)$. Let $\tilde{z} \in \mathbb{R}^{\tilde{N}}$ be defined by $\tilde{z}_i = \tilde{z}_j = x_j$ and $\tilde{z}_k = 2x_i$. By RCP and COV, $\tilde{z} \in \sigma(\tilde{N}, w)$ so that, by REAS, $x_i = 0$. \square

We shall now turn to the general case.

Lemma 4.7 *If σ satisfies NE, REAS, COV, RGP, and RCP, then it is a subsolution of the positive prekernel.*

Proof By Lemma 4.6, $\sigma(N, v) \subseteq C_+(N, v) \subseteq \mathcal{K}_+^*(N, v)$ for all $(N, v) \in \Gamma^2$. By Theorem 2.4, \mathcal{K}_+^* satisfies CRGP. Therefore Remark 2.10 applied to $\sigma^1 = \sigma$ and $\sigma^2 = \mathcal{K}_+^*$ completes the proof. \square

If σ is a subsolution of the positive prekernel and if σ satisfies COV, RGP, and RCP, then it is shown that a coalition which has a positive excess at some member of the solution when applied to the game, has the same excess at any member of the solution of the game. The following simple lemma is useful.

Lemma 4.8 *Let (N, v) be a game, let $x \in \mathcal{K}_+^*(N, v)$, and let $k \in N$. If $\mu(x, v) > 0$ then there exist $S^k, S^{-k} \subseteq N$ such that*

$$e(S^k, x, v) = e(S^{-k}, x, v) = \mu(x, v), \quad k \in S^k, \text{ and } k \notin S^{-k}.$$

Proof It suffices to show that there exists $\ell \in N \setminus \{k\}$ such that

$$s_{k\ell}(x, v) = s_{\ell k}(x, v) = \mu(x, v). \tag{4.1}$$

Let $S \subseteq N$ be a coalition attaining $\mu(x, v)$. As $\mu(x, v) > 0$, $\emptyset \neq S \neq N$. If $k \in S$, then (4.1) holds for every $\ell \in N \setminus S$. If $k \notin S$, then (4.1) holds for every $\ell \in S$. \square

The following Lemma is proved in Sect. 6.

Lemma 4.9 *Let $|U| = \infty$, let σ be a subsolution of the positive prekernel, let (N, v) be a game, let $x, y \in \sigma(N, v)$, and let $\bar{S} \subseteq N$. If σ satisfies NE, COV, RGP, and RCP, then*

$$e(\bar{S}, x, v) > 0 \Rightarrow e(\bar{S}, y, v) = e(\bar{S}, x, v).$$

Corollary 4.10 *If $|U| = \infty$, if the prenucleolus is a subsolution of σ , and if σ satisfies REAS, COV, RGP, and RCP, then σ is a subsolution of the positive core.*

Proof By Lemma 4.7 the solution σ is a subsolution of the positive prekernel, hence Lemma 4.9 can be applied. \square

Now we are ready to finish the desired proof.

Proof of Theorem 4.2 The three solutions satisfy the desired properties. Let, now, σ be a solution that satisfies the axioms and contains the prenucleolus. By Theorem 4.5 the following three cases (1)–(3) may occur.

(1) The solution σ coincides with the prenucleolus on the set of all 2-person games. By (2.1), σ must be a subsolution of the prekernel, thus σ satisfies ETP. By Theorem 4.2 of Orshan and Sudhölter (2003), σ is the prenucleolus.

(2) The solution σ coincides with the positive core on the set of all 2-person games. By Corollary 4.10, σ is a subsolution of the positive core. Let (N, v) be a game. It remains to show $\mathcal{C}_+(N, v) \subseteq \sigma(N, v)$. Without loss of generality we may assume that

$$N = \{1, \dots, n\}, N^* := \{1^*, \dots, n^*\} = \{n + 1, \dots, 2n\} \subseteq U.$$

By COV we may assume that $v(N, v) = 0 \in \mathbb{R}^N$. If $\mathcal{C}_+(N, v) = \mathbf{v}(N, v)$, then NE completes the proof. Hence, let $x \in \mathcal{C}_+(N, v) \setminus \mathbf{v}(N, v)$ and put $\tilde{N} = N \cup N^*$. It is no loss of generality to assume that

$$x_1 \geq \dots \geq x_n. \tag{4.2}$$

We are going to define a game (\tilde{N}, w) which allows to show that $\sigma(\tilde{N}, w)$ contains a member y such that

- (a) the reduced game $(N, w^{N,y})$ coincides with (N, v) and
- (b) the restriction y_N coincides with x .

Indeed the proof is finished by RGP as soon as (\tilde{N}, w) and y are constructed.

Let $k \in \mathbb{N}$ and $e_i \in \mathbb{R}$, $i = 1, \dots, k$, be defined by

$$\{e_i \mid i = 1, \dots, k\} = \{v(S) \mid v(S) < 0\}, e_1 > \dots > e_k,$$

i.e., e_1, \dots, e_k are the different negative excesses of (N, v) at the prenucleolus in the strictly decreasing order. Moreover, define $\mathcal{S}^i, \mathcal{T}^i, \beta_i$ for all $i = 1, \dots, k$ by the requirements

$$\mathcal{S}^i = \{S \subseteq N \mid v(S) = e_i\}, \tag{4.3}$$

$$\mathcal{T}^i = \left\{ T \subseteq \bigcup_{j=1}^i \bigcup_{S \in \mathcal{S}^j} S \right\}, \tag{4.4}$$

$$\beta_i = \max_{S \in \mathcal{T}^i} x(S). \tag{4.5}$$

Note that \mathcal{S}^i is the i th negative level set. By construction

$$e_k < \dots < e_1 < 0 = x(\emptyset) \leq \beta_1 \leq \dots \leq \beta_k.$$

Put $T^0 = \{\emptyset\}$ and $\beta_{k+1} = \max_{S \subseteq N} v(S) + \max_{T \subseteq N} x(T)$. We are now ready to define w . For $S, T \subseteq N$, let $w(S \cup T^*)$ be given by the following equation:

$$w(S \cup T^*) = \begin{cases} v(S), & \text{if } v(S) \geq 0 \text{ and } T = S, \\ e_i - \beta_i, & \text{if } \left(\begin{array}{l} S \in \mathcal{S}^i, T \in \mathcal{T}^i \text{ or} \\ S = \emptyset, T \in \mathcal{T}^i \setminus \mathcal{T}^{i-1} \end{array} \right) \text{ for some } 1 \leq i \leq k, \\ e_k - \beta_{k+1}, & \text{otherwise.} \end{cases}$$

We are going to show that

$$v(\tilde{N}, w) = 0 \in \mathbb{R}^{\tilde{N}} \tag{4.6}$$

and

$$y^\ell := (\underbrace{x_1, \dots, x_\ell}_\ell, \underbrace{0, \dots, 0}_{n-\ell}, \underbrace{-x_1, \dots, -x_\ell}_\ell, \underbrace{0, \dots, 0}_{n-\ell}) \in \sigma(\tilde{N}, w) \quad \forall \ell = 1, \dots, n. \tag{4.7}$$

- (a) In order to show (4.6), it suffices to show in view of Theorem 3.3 that, for any $R \subseteq \tilde{N}$,

$$\mathcal{D}(w(R), 0, w) = \{Q \subseteq \tilde{N} \mid w(Q) \geq w(R)\}$$

is balanced (over \tilde{N}). If $w(R) \geq 0$, i.e., $R = S \cup S^*$ for some S with $v(S) \geq 0$, then the required balancedness follows immediately from the fact that 0 is the prenucleolus of (N, v) and the observation that

$$\{Q \subseteq \tilde{N} \mid w(Q) \geq w(R)\} = \{T \cup T^* \mid T \subseteq N, v(T) \geq v(S)\}.$$

Thus it remains to show that

$$\widehat{\mathcal{D}}^i := \{Q \subseteq \tilde{N} \mid w(Q) \geq e_i - \beta_i\}$$

is balanced for any $i = 1, \dots, k$. By Remark 3.8 it suffices to show that $\widehat{\mathcal{D}}^i$ contains a balanced subset \mathcal{D}^i that spans $\widehat{\mathcal{D}}^i$. In order to define \mathcal{D}^i note that

$$S \cup \emptyset \in \widehat{\mathcal{D}}^i \quad \forall S \in \bigcup_{j=1}^i \mathcal{S}^j \text{ and } \emptyset \cup \{t^*\} \in \widehat{\mathcal{D}}^i \quad \forall \{t\} \in \mathcal{T}^i$$

for $i = 1, \dots, k$. Hence we put

$$\mathcal{D}^i := \{S \cup S^* \mid v(S) \geq 0\} \cup \bigcup_{j=1}^i \mathcal{S}^j \cup \{\{t^*\} \mid \{t\} \in \mathcal{T}^i\} \quad \forall i = 1, \dots, k.$$

By Theorem 3.3, $\{S \subseteq N \mid v(S) \geq 0\} \cup \bigcup_{j=1}^i S^j$ is balanced over N , hence $\{S^* \subseteq N^* \mid v(S) \geq 0\} \cup \{S^* \subseteq N^* \mid S \in \bigcup_{j=1}^i S^j\}$ is balanced over N^* . The observation that $\{\{t^*\} \mid \{t\} \in \mathcal{T}^i\} = \{\{t^*\} \mid t \in \bigcup_{j=1}^i \bigcup_{S \in \mathcal{S}^j} S\}$ shows that \mathcal{D}^i is balanced. Also, it is straightforward to show that \mathcal{D}^i spans $\widehat{\mathcal{D}}^i$ for every $i = 1, \dots, k$.

- (b) Assertion (4.7) is shown recursively. Denote the prenucleolus $v(\widetilde{N}, w) = 0$ by y^0 . By the definition of y^ℓ we conclude for every $T \subseteq N$ that $y^\ell(T^*) = -x(T \cap \{1, \dots, \ell\}) \geq -\beta_{k+1}$. Also, by (4.5), $y^\ell(T^*) \geq -\beta_i$ for every $T \in \mathcal{T}^i$ and every $i = 1, \dots, k$. Also, by (4.2), we have $y^\ell(S) \geq \min\{x(S), 0\}$ for all $S \subseteq N$. We conclude for $S, T \subseteq N$ that

$$e(S \cup T^*, y^\ell, w) \leq \max\{e(S, x, v), e_i\}, \text{ if } S \in \mathcal{S}^i \text{ for some } 1 \leq i \leq k. \tag{4.8}$$

Also,

$$e(S \cup T^*, y^\ell, w) < 0, \text{ if } w(S \cup T^*) = e_k - \beta_{k+1} \tag{4.9}$$

We conclude that

$$\left(e(S \cup T^*, y^\ell, w) \right)_+ = (w(S \cup T^*))_+.$$

Let $w^{\ell-1} := w^{\{\ell, \ell^*\}, y^{\ell-1}} \forall \ell = 1, \dots, n$. denote the coalition function of the ‘‘bilateral’’ reduced game w.r.t. $\{\ell, \ell^*\}$ and $y^{\ell-1}$. By (4.8) and (4.9),

$$(x^\ell, -x^\ell) \in \mathcal{C}(\{\ell, \ell^*\}, w^{\ell-1}) \forall \ell = 1, \dots, n. \tag{4.10}$$

The facts that the solution σ coincides with the positive core on 2-person games and that it contains the prenucleolus show that RCP, when applied recursively to $\ell = 1, \dots, n$, yields the following ‘‘chain’’ of arguments:

$$y^{\ell-1} \in \sigma(\widetilde{N}, w) \Rightarrow y^\ell \in \sigma(\widetilde{N}, w)$$

Now the proof of this case can be finished. Put $y = y^n$ and observe that $w^{N,y} = v$ by the definition of w . Thus RGP implies $y_N \in \sigma(N, v)$. Clearly $y_N = x$ is satisfied.

- (3) The solution σ coincides with the relative interior of the positive core on the set of all 2-person games.

By Corollary 4.10, σ is a subsolution of the positive core. By copying the relevant part of the proof of the second case, it can be shown that the relative interior of the positive core is a subsolution of σ . Indeed, a careful inspection of the definition of the game (\widetilde{N}, w) , especially of (4.8) and (4.9), shows that (4.10) may be replaced by

$$(x^\ell, -x^\ell) \in \text{int } \mathcal{C}(\{\ell, \ell^*\}, w^{\ell-1}) \forall \ell = 1, \dots, n,$$

whenever $x \in \text{rint } \mathcal{C}_+(N, v)$ is assumed. (“int \mathcal{C} ” means the “interior of \mathcal{C} relative to \mathcal{T}^{**} ”, i.e., for every game (M, u) , $\text{int } \mathcal{C}(M, u) = \{x \in \mathcal{C}(M, u) \mid e(S, x, u) < 0 \forall \emptyset \neq S \subsetneq M\}$.)

Hence it remains to show that $\sigma(N, v) \subseteq \text{rint } \mathcal{C}_+(N, v)$ for any game (N, v) satisfying $\mathcal{C}_+(N, v) \setminus \text{rint } \mathcal{C}_+(N, v) \neq \emptyset$. Take $x \in \mathcal{C}_+(N, v) \setminus \text{rint } \mathcal{C}_+(N, v)$ and let $\bar{S} \subseteq N$ be a coalition satisfying $0 = e(\bar{S}, x, v) > e(\bar{S}, v, v)$, where $v = v(N, v)$ denotes the prenucleolus. Without loss of generality we may assume $N = \{1, \dots, n\}$, $1 \in \bar{S}$ and $* = n + 1 \in U$. Moreover, let $(N \cup \{*\}, w)$ defined by

$$w(S) = \begin{cases} v(S \setminus \{*\}), & \text{if } S \subseteq N \setminus \{1\} \text{ or } 1, * \in S, \\ v(\bar{S}), & \text{if } S = \bar{S}, \\ e(\bar{S}, v, v), & \text{if } S = \{*\}, \\ v(S \setminus \{*\}) - \alpha, & \text{otherwise,} \end{cases}$$

where $\alpha := \mu(x, v) - \min_{S \subseteq N} e(S, v, v) - e(\bar{S}, v, v)$.

Claim $z := v(N \cup \{*\}, w) = (v, 0)$

For $\beta > e(\bar{S}, v, v)$,

$$\mathcal{D}(\beta, z, w) = \{S \in \mathcal{D}(\beta, v, v) \mid 1 \notin S\} \cup \{S \cup \{*\} \mid 1 \in S \in \mathcal{D}(\beta, v, v)\}.$$

For $-\alpha < \beta \leq e(\bar{S}, v, v)$,

$$\mathcal{D}(\beta, z, w) = \{S \in \mathcal{D}(\beta, v, v) \mid 1 \notin S\} \cup \{\bar{S}, \{*\}\} \cup \{S \cup \{*\} \mid 1 \in S \in \mathcal{D}(\beta, v, v)\}.$$

For $\beta \leq -\alpha$, $\mathcal{D}(\beta, z, w) = 2^{N \cup \{*\}}$. Hence our claim follows from Theorem 3.3.

The reduced game $(N, w^{N,z})$ coincides with (N, v) . Assuming, on the contrary, that $x \in \sigma(N, v)$, yields $\tilde{z} := (x, 0) \in \sigma(N \cup \{*\}, w)$ by RCP. Applying RGP yields

$$(x_1, 0) \in \sigma(\{1, *\}, w^{\{1, *\}, \tilde{z}}).$$

Moreover, this reduced game can easily be computed as

$$w^{\{1, *\}, \tilde{z}}(\{1\}) = e(\bar{S}, x, v) + x_1, w^{\{1, *\}, \tilde{z}}(\{*\}) = e(\bar{S}, v, v) < 0, w^{\{1, *\}, \tilde{z}}(\{1, *\}) = x_1.$$

Therefore the interior of the core of this reduced game is nonempty. The fact that $(x_1, 0)$ is not a member of this interior shows the desired contradiction. \square

5 Remarks

It is possible to characterize the positive core as the *maximum* solution that satisfies several axioms. Indeed, in view of Theorem 4.2, Remark 4.3, and Theorem 4.1 the positive core is the maximum solution satisfying NE, REAS, COV, AN, CON, RGP, and RCP, or, alternatively, satisfying NE, REAS, COV, ND, RGP, and RCP, provided that $|U| = \infty$. Moreover, we may replace the requirement of being maximum by weak unanimity for two-person games.

Corollary 5.1 *If $|U| = \infty$, then there is a unique solution that satisfies WUTPG, NE, REAS, COV, RGP, RCP, and ND, and it is the positive core.*

Peleg and Sudhölter (2003, Remark 6.3.3) show that Theorem 2.3 is no longer valid if $4 \leq |U| < \infty$. Similarly, the infinity assumption on $|U|$ is crucial in Theorems 4.2 and 4.1, in Remark 4.3, in Corollary 4.4, and in the aforementioned characterizations of the positive core. Indeed, if $|U| < \infty$, then define for any game (N, v) ,

$$\sigma(N, v) = \begin{cases} \mathcal{C}_+(N, v), & \text{if } N \subsetneq U, \\ \{x \in \mathcal{I}^*(N, v) \mid x_S \in \mathcal{C}_+(S, v^{S,x}) \forall \emptyset \neq S \subsetneq N\}, & \text{if } N = U. \end{cases}$$

In order to show that σ satisfies RCP, let (N, v) be a game, let $\emptyset \neq S \subsetneq N$, let $x \in \sigma(N, v)$, let $y \in \sigma(S, v^{S,x})$, and define $z = (y, x_{N \setminus S})$. If $N \subsetneq U$, then $z \in \sigma(N, v)$ by the definition of σ , because the positive core satisfies RCP. Hence, we may assume that $N = U$. Let $T \subseteq N$. We claim that $(e(T, x, v))_+ = (e(T, z, v))_+$. If $S \subseteq T$ or $S \cap T = \emptyset$, then our claim is obvious. Hence, we assume that $S \setminus T \neq \emptyset \neq S \cap T$ and observe that $e := (v^{S,x}(T \cap S) - x(T \cap S))_+ = (v^{S,x}(T \cap S) - y(T \cap S))_+$ so that

$$e = \max_{Q \subseteq N \setminus S} (e((T \cap S) \cup Q, x, v))_+ = \max_{Q \subseteq N \setminus S} (e((T \cap S) \cup Q, z, v))_+.$$

If $e > 0$, then $x(S \cap T) = z(S \cap T)$ and our claim follows. If $e \leq 0$, then $e(T, x, v), e(T, z, v) \leq 0$ so that our claim is always valid. Now RCP follows from Theorem 3.4, because $\mathcal{D}(\alpha, x_T, v^{T,x}) = \mathcal{D}(\alpha, z_T, v^{T,z})$ for any $\alpha > 0$ and all $\emptyset \neq T \subsetneq N$.

It is straightforward to show that σ satisfies the remaining properties that are employed in the mentioned results.

For $4 \leq |U| < \infty$, by Exercises 6.3.2 and 6.3.3 of the aforementioned book, there exist a game (U, v) and $x \in \mathcal{I}^*(U, v)$ such that $x \notin \mathcal{C}_+(U, v)$ and x_S is the prenucleolus of $(S, v^{S,x})$ for any $\emptyset \neq S \subsetneq U$. We conclude that \mathcal{C}_+ is a proper subsolution of σ when $|U| \geq 4$.

We conjecture that CON is *not* logically independent of the remaining axioms in Remark 4.3. Also, we do not know whether the aforementioned characterizations and Corollary 5.1 are axiomatizations, because we do not have examples which show that AN, CON, or ND are independent of the remaining axioms.

6 Appendix

Proof of Theorem 4.5 Let (N, v) be a 2-person game and assume that $N = \{1, 2\}$. If the core of (N, v) is empty or single valued, then Lemma 4.6 finishes the proof. Hence, by COV, we may assume that (N, v) is given by $v(\{1\}) = v(\{2\}) = v(\emptyset) = 0$ and $v(N) = 1$. Assume that $\sigma(N, v)$ does not coincide with the prenucleolus. It remains to show that $\sigma(N, v) \supseteq \text{rint } \mathcal{C}_+(N, v)$. By AN and PO there is $x^0 \in \sigma(N, v)$ satisfying

$$0 \leq x_1^0 < x_2^0 = 1 - x_1^0 \leq 1 \tag{6.1}$$

It suffices to show that

- (1) there is a convergent sequence $(\bar{x}^k)_{k \in \mathbb{N}}$ satisfying $\bar{x}^k \in \sigma(N, v) \forall k \in \mathbb{N}$ with limit $(0, 1)$ and
- (2) $\sigma(N, v)$ is convex.

Indeed, if $x \in \text{rint } \mathcal{C}(N, v)$, then choose $k \in \mathbb{N}$ such that $\bar{x}_1^k \leq \min\{x_1, x_2\}$ and notice that $(\bar{x}_2^k, \bar{x}_1^k) \in \sigma(N, v)$ is true by AN. Hence x is a convex combination of these two vectors.

ad 1 Let $\tilde{N} = \{1, 2, 3\}$ and let (\tilde{N}, w) be defined by

$$w(S) = \begin{cases} 0, & \text{if } S = \emptyset, \tilde{N}, \\ -1, & \text{otherwise,} \end{cases}$$

for all $S \subseteq \tilde{N}$. Moreover, let

$$X^{\leq} = \{x \in \mathcal{C}(\tilde{N}, w) \mid x_1 \leq x_2 \leq 0 \leq x_3\}, X^{\geq} = \{x \in \mathcal{C}(\tilde{N}, w) \mid x_1 \leq 0 \leq x_2 \leq x_3\},$$

and define $g^{\leq} : X^{\leq} \rightarrow \mathbb{R}^{\tilde{N}}$ by $g^{\leq}(x) = (-\alpha - x_3, \min\{\alpha, x_3\}, \max\{\alpha, x_3\})$, where

$$\alpha = \frac{(1 - x_1)(2 - x_3)}{2 + x_2} - 1 = x_1 \frac{x_3 - 1}{2 + x_2} \forall x \in X^{\leq},$$

and $g^{\geq} : X^{\geq} \rightarrow \mathbb{R}^{\tilde{N}}$ by $g^{\geq}(y) = (\min\{\beta, y_1\}, \max\{\beta, y_1\}, -\beta - y_1)$, where

$$\beta = \frac{(1 + y_1)(2 + y_1)}{2 - y_2} - 1 - y_1 = -y_3 \frac{1 + y_1}{2 - y_2} \forall y \in X^{\geq}.$$

Note that the denominators are positive, because x and y are assumed to be elements of the core. (In fact, $\mathcal{C}(\tilde{N}, w)$ is the convex hull of all permutations of $(-1, 0, 1)$.)

For any $x \in X^{\leq}$ and for any $y \in X^{\geq}$ we observe that $(\{1, 2\}, w^{\{1,2\},x})$ and $(\{2, 3\}, w^{\{2,3\},y})$, respectively, are isomorphic (by interchanging players 2 and 3 and by interchanging 1 and 2, respectively) to

$$\left(\{1, 3\}, \left(w^{\{1,3\},x} + (x_2 + 1, x_2 + 1) \right) \frac{2 - x_3}{2 + x_2} - (1, 1) \right)$$

and to

$$\left(\{1, 3\}, \left(w^{\{1,3\},y} + (1, 1) \right) \frac{2 + y_1}{2 - y_2} - (1 + y_1, 1 + y_1) \right),$$

respectively. Thus, by AN, RGP, RCP, and COV of the core, we conclude (see the proof of Lemma 4.6) that

$$g^{\leq}(x) \in \mathcal{C}(\tilde{N}, w) \forall x \in X^{\leq} \text{ and } g^{\geq}(y) \in \mathcal{C}(\tilde{N}, w) \forall y \in X^{\geq}.$$

The proof of the following properties is straightforward and skipped:

$$g^{\leq}(x) \in X^{\geq}, \quad g_1^{\leq}(x) \leq x_1, \quad \text{and} \quad g_3^{\leq}(x) \geq x_3 \quad \forall x \in X^{\leq} \tag{6.2}$$

$$\forall x \in X^{\leq} : g^{\leq}(x) = x \Leftrightarrow x = 0 \text{ or } x = (-1, 0, 1) \tag{6.3}$$

$$g^{\geq}(y) \in X^{\leq}, \quad g_1^{\geq}(y) \leq y_1, \quad \text{and} \quad g_3^{\geq}(y) \geq y_3 \quad \forall y \in X^{\geq} \tag{6.4}$$

$$\forall y \in X^{\geq} : g^{\geq}(y) = y \Leftrightarrow y = 0 \text{ or } y = (-1, 0, 1) \tag{6.5}$$

Therefore the composition $g = g^{\geq} \circ g^{\leq} : X^{\leq} \rightarrow X^{\leq}$ is well-defined and satisfies “monotonicity”, i.e., it satisfies $g_1(x) \leq x_1$ and $g_3(x) \geq x_3$ for all $x \in X^{\leq}$. Hence for any $x \in X^{\leq} \setminus \{0\}$ the sequence $(g^k(x))_{k \in \mathbb{N}}$ converges to $(-1, 0, 1)$.

In order to construct the sequence $(\bar{x}^k)_{k \in \mathbb{N}}$ with the desired properties we first show that X^{\leq} contains a nonzero member x^1 of $\sigma(\tilde{N}, w)$. For this purpose choose $z \in \sigma(\tilde{N}, w) \cap (X^{\leq} \cup X^{\geq})$ which is possible by NE and Lemma 4.7. If $z \neq 0$, then put

$$x^1 = \begin{cases} z, & \text{if } z \in X^{\leq}, \\ g^{\geq}(z), & \text{otherwise,} \end{cases}$$

and observe that x^1 belongs to $\sigma(\tilde{N}, w)$ by AN, RGP, RCP, and COV.

If $z = 0$, then observe that the reduced game $(N, w^{N,z})$ is strategically equivalent to (N, v) , thus by NE, RGP, RCP, COV, and the assumption that $\sigma(N, v) \neq \{v(N, v)\}$, there is some $y \in \sigma(\tilde{N}, w) \setminus \{0\}$. By Lemma 4.6 we may assume that y belongs to $X^{\leq} \cup X^{\geq}$, thus we define $x^1 = y$ in the first and $x^1 = g^{\geq}(y)$ in the latter case.

Now the proof of this part may be finished by defining the sequence $(\bar{x}^k)_{k \in \mathbb{N}}$ as follows. Put $x^k = g^{k-1}(x^1) \forall k \in \mathbb{N}$ and define

$$\bar{x}^k = \left(\frac{x_1^k + 1}{2 - x_3^k}, \frac{x_2^k + 1}{2 - x_3^k} \right) \forall k \in \mathbb{N}.$$

Then the sequence $(\bar{x}^k)_{k \in \mathbb{N}}$ converges to $(0, 1)$, because $(x^k)_{k \in \mathbb{N}}$ converges to $(-1, 0, 1)$. It remains to show that $\bar{x}^k \in \sigma(N, v)$ for any natural number k . This assertion is implied by COV, RGP, and the observation that $v = \left(w^{N,x^k} + (1, 1) \right) \frac{1}{2-x_3^k}$.

ad 2 In view of AN it suffices to show that $(\alpha, 1 - \alpha)$ belongs to $\sigma(N, v)$ for any $0 < \alpha \leq \frac{1}{2}$. By (1), which was just proved, there exists $(a, 1 - a) \in \sigma(N, v)$ satisfying $0 \leq a \leq \alpha$. Additionally, we shall assume that $\frac{1-a}{2-a} \geq a$ (which is true, e.g., if $a \leq 1/4$). For any β such that $\frac{1-a}{2-a} \leq \beta \leq 1$ define (\tilde{N}, w_β) by

$$w_\beta(S) = \begin{cases} 1, & \text{if } S = \tilde{N}, \\ \beta, & \text{if } S = N, \\ 0, & \text{otherwise,} \end{cases}$$

and choose $x^0 \in \sigma(\tilde{N}, w_\beta)$. Then $x^0 \in \mathcal{K}_+^*(\tilde{N}, w_\beta)$ by Lemma 4.7. It is well-known (see [Sudhölter and Peleg 2000](#)) that the positive prekernel of a three-person game

coincides with its positive core. Hence $x^0 \in \mathcal{C}(\tilde{N}, w_\beta)$. We proceed by recursively applying RGP, RCP, and COV four times. In (2) and (4) AN is employed in addition. For a sketch of the four steps see Fig. 1.

- (1) Let (N, w^0) be the reduced game of (\tilde{N}, w_β) w.r.t. N and x^0 . Then

$$w^0(\{1\}) = w^0(\{2\}) = 0, \quad w^0(N) = 1 - x_3^0,$$

thus $x^1 = (a(1 - x_3^0), (1 - a)(1 - x_3^0), x_3^0) \in \sigma(\tilde{N}, w_\beta)$. (If $\beta = 1$, then $x_3^0 = 0$ and $w^0 = v$.)

- (2) Let $(\{2, 3\}, w^1)$ be the reduced game of (\tilde{N}, w_β) w.r.t. $\{2, 3\}$ and x^1 . The inequalities $\beta \geq \frac{1-a}{2-a} \geq a \geq x_1^1$ imply $w^1(\{2\}) = \max\{0, \beta - x_1^1\} = \beta - x_1^1$. Also, $w^1(\{3\}) = 0$ and $w^1(\{2, 3\}) = 1 - x_1^1$. Thus

$$x^2 = \left(a(1 - x_3^0), (1 - \beta)a + \beta - x_1^1, (1 - \beta)(1 - a) \right) \in \sigma(\tilde{N}, w_\beta).$$

- (3) Let (N, w^2) be the reduced game of (\tilde{N}, w_β) w.r.t. N and x^2 . Then

$$w^2(\{1\}) = w^2(\{2\}) = 0, \quad w^2(N) = 1 - x_3^2,$$

thus $x^3 = (a(1 - x_3^2), (1 - a)(1 - x_3^2), x_3^2) \in \sigma(\tilde{N}, w_\beta)$. Note that x^3 does not depend on x^0 , because $x_3^2 = (1 - \beta)(1 - a)$.

- (4) Let $(\{1, 3\}, w^3)$ be the reduced game of (\tilde{N}, w_β) w.r.t. $\{1, 3\}$ and x^3 . The inequality $\beta \geq x_2^3$ is true, because $\beta \geq \frac{1-a}{2-a}$. Hence, $w^3(\{1\}) = \max\{0, \beta - x_2^3\} = \beta - x_2^3$. Also, $w^3(\{3\}) = 0$ and $w^3(\{1, 3\}) = 1 - x_2^3$. Thus

$$x^4 = \left((1 - a)(1 - \beta) + \beta - x_2^3, x_2^3, a(1 - \beta) \right) \in \sigma(\tilde{N}, w_\beta).$$

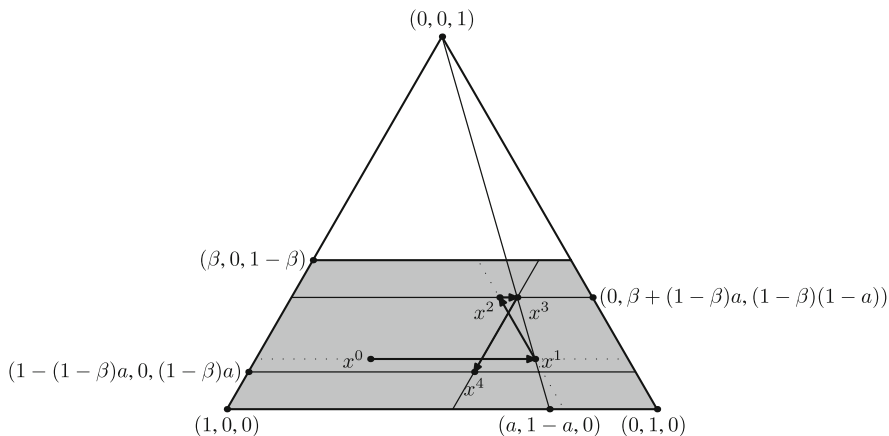


Fig. 1 Sketch of the steps (1)–(4) in ad 2

Now the proof may be finished by considering the reduced game (N, w^4) of (\tilde{N}, w_β) w.r.t. x^4 . Indeed, we have

$$w^4(\{1\}) = w^4(\{2\}) = 0, \quad w^4(N) = 1 - x_3^4,$$

thus

$$y^\beta = \frac{1}{1 - x_3^4}(x_1^4, x_2^4) \in \sigma(N, v).$$

A careful inspection of the construction immediately shows that

$$y_1^\beta = \begin{cases} a, & \text{if } \beta = 1, \\ \frac{1}{2}, & \text{if } \beta = (1 - a)/(2 - a). \end{cases}$$

The continuity of the mapping $\beta \mapsto y^\beta$ shows that there exist some $\tilde{\beta}, \frac{1-a}{2-a} \leq \tilde{\beta} \leq 1$, such that $y_1^{\tilde{\beta}} = \alpha$. □

Proof of Lemma 4.9 Assume without loss of generality that $1 \in \tilde{S}$ and that $N = \{1, \dots, n\}$. Choose $*$ $\in U \setminus N$, let us say $* = n + 1$, and put $\tilde{N} = N \cup \{*\}$.

Choose $\alpha > (n + 1) \max_{P, Q \subseteq N} (v(P) - v(Q))$ and define the game (\tilde{N}, w) by

$$w(S) = \begin{cases} v(S \setminus \{*\}), & \text{if } S \subseteq N \setminus \{1\} \text{ or } 1, * \in S, \\ v(\tilde{S}), & \text{if } S = \tilde{S}, \\ v(N), & \text{if } S = N, \\ 0, & \text{if } S = \{*\}, \\ v(S \setminus \{*\}) - \alpha, & \text{otherwise.} \end{cases}$$

By NE there exists $z \in \sigma(w)$. It is the aim to show the following assertions:

- (1) $(N, w^{N,z})$ is strategically equivalent to (N, v) .
- (2) $s_{*1}(z, w) > 0$ and $s_{*1}(z, w)$ is only attained by the coalition $\{*\}$.
- (3) $s_{1*}(z, w)$ is only attained by \tilde{S} .

Then we shall use RGP and RCP to “insert” x or y and immediately deduce the desired coincidence of the excesses. In order to prove (1) we shall show the following

Claim $z_* \leq 0$

Assume, on the contrary, $z_* > 0$.

Step 1 It is first proved that

$$\exists \bar{k} \in N \setminus \{1\} : z_{\bar{k}} < - \max_{P, Q \subseteq N} (v(P) - v(Q)). \tag{6.6}$$

Indeed, the assumption yields

$$s_{1*}(z, w) \geq e(N, z, w) = v(N) - z(N) = z_* > 0.$$

As $z \in \mathcal{K}_+^*(\tilde{N}, w)$, $s_{*1}(z, w) > 0$. Choose $T \subseteq N \setminus \{1\}$ such that $e(T \cup \{*\}, z, w) > 0$. As $T \neq \emptyset$,

$$0 < v(T) - \alpha - z(T) - z_* \leq v(T) - \alpha - z(T) \\ = (v(T) - v(\emptyset)) - \alpha - z(T) < -n \max_{P, Q \subseteq N} (v(P) - v(Q)) - z(T).$$

Hence there is $\bar{k} \in T \subseteq N \setminus \{1\}$ with $z_{\bar{k}} < -\max_{P, Q \subseteq N} (v(P) - v(Q))$.

Step 2 Let $S \subseteq N \cup \{*\}$ attain $\mu(z, w)$. By Step 1 there exists $i \in N \setminus \{1\}$ such that $z_i < -\max_{P, Q \subseteq N} (v(P) - v(Q))$. The following assertions are shown:

- (1) \bar{S} attains $\mu(z, w)$.
- (2) If $S \neq \bar{S}$, then $S \supseteq (N \cup \{*\}) \setminus \bar{S}$.
- (3) Player i does not belong to \bar{S} .

In order to show (1) the following claim is shown:

$$S \neq \bar{S} \Rightarrow i \in S. \tag{6.7}$$

note that $\{*\}$ does not attain $\mu(z, w)$ because $\mu(z, w)$ is positive. Let $T \subseteq (N \cup \{*\}) \setminus \{i\}$ such that $T \neq \bar{S}$ and $T \neq \{*\}$. If $\bar{S} \neq T \cup \{i\} \neq N$, then

$$e(T \cup \{i\}, z, w) - e(T, z, w) = v((T \cup \{i\}) \setminus \{*\}) - v(T \setminus \{*\}) - z_i > 0.$$

Also, if $T \cup \{i\} \in \{N, \bar{S}\}$, then

$$e(T \cup \{i\}, z, w) - e(T, z, w) = v(T \cup \{i\}) - v(T) - z_i + \alpha > 0.$$

Hence (6.7) is shown. By Lemma 4.8 there exists $S^{-i} \subseteq (N \cup \{*\}) \setminus \{i\}$ attaining $\mu(z, w)$. By (6.7), $S^{-i} = \bar{S}$, thus (1) is shown.

Now this step may be completed. In order to show (2) we assume, on the contrary, that there is some coalition $S \neq \bar{S}$ attaining maximal excess such that $S \not\supseteq (N \cup \{*\}) \setminus \bar{S}$. By (6.7), $i \in S$. Choose $j \in (N \cup \{*\}) \setminus (\bar{S} \cup S)$. Then $\mu(z, w) = e(S, z, w) = s_{ij}(z, w)$ so that $s_{ji}(z, w) = \mu(z, w)$ which is impossible by (6.7). Moreover, (3) is valid because of the existence of S^{-i} .

Step 3 By the preceding steps there is some coalition S attaining maximal excess such that $1 \notin S \ni *$. Moreover, $\{*\} \neq S$, hence

$$0 \geq e(S \cup \{1\}, z, w) - e(S, z, w) = v((S \cup \{1\}) \setminus \{*\}) - v(S \setminus \{*\}) + \alpha - z_1.$$

Thus $z_1 > n \max_{P, Q \subseteq N} (v(P) - v(Q))$. By (1) of the preceding step (the maximality of $e(\bar{S}, z, w)$) we obtain

$$0 < e(\bar{S}, z, w) = v(\bar{S}) - z_1 - z(\bar{S} \setminus \{1\}) \\ < -((n - 1) \max_{P, Q \subseteq N} (v(P) - v(Q))) - z(\bar{S} \setminus \{1\})$$

and, thus, there is $i \in \bar{S} \setminus \{1\}$ with $z_i < -\max_{P, Q \subseteq N} (v(P) - v(Q))$. The last observation contradicts (3) of the preceding step. Thus our claim is shown.

Now the proof may be finished. Let $u = w^{N, z}$ be the reduced game. By reasonableness of the positive prekernel, $z_* \geq -\alpha$. Hence, $z_* \leq 0$ implies that $u = v + (-z_*, \underbrace{0, \dots, 0}_{n-1})$. By RCP and COV

$$\begin{aligned} z^1 &= (-z_*, 0, \dots, 0, z_*) + (x, 0) \in \sigma(N \cup \{*\}, w) \quad \text{and} \\ z^2 &= (-z_*, 0, \dots, 0, z_*) + (y, 0) \in \sigma(N \cup \{*\}, w). \end{aligned}$$

By REAS of the positive prekernel, $z_i^t \geq -\max_{P, Q \subseteq N} (v(P) - v(Q))$ for all $i \in N \setminus \{1\}$ and $t = 1, 2$. Therefore $s_{*1}(z^t, w)$ must be attained by $\{*\}$, thus this surplus is non-negative and

$$s_{*1}(z^1, w) = -z_* = \left(s_{1*}(z^1, w)\right)_+ \geq (e(\bar{S}, x, v) + z_*)_+.$$

As $e(\bar{S}, x, v) > 0, z_* < 0$. Hence, $e(N, z^t, w) < 0$ and $s_{1*}(z^t, w)$ must be attained by \bar{S} for $t = 1, 2$.

Therefore $e(\bar{S}, z^1, w) = e(\bar{S}, x, v) + z_* = e(\{*\}, z^1, w) = -z_*$ and, similarly, $e(\bar{S}, z^2, w) = -z_*$. Thus,

$$z_* = -\frac{e(\bar{S}, x, v)}{2} \quad \text{and} \quad z_* = -\frac{e(\bar{S}, y, v)}{2}.$$

Hence we conclude that $e(\bar{S}, x, v) = e(\bar{S}, y, v)$ is satisfied. □

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