On Bargaining Sets and Voting Games*

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Abstract

Using α-effectiveness we define the NTU games corresponding to simple majority voting, plurality voting, and approval voting. The Aumann-Davis-Maschler bargaining set of a simple majority voting game is nonempty if there are at most three alternatives and it may be empty for four or more alternatives, whereas the Mas-Colell bargaining set may be empty only for more than five alternatives. However, if the number of players tends to infinity, then the bargaining sets of simple majority voting games are likely to be nonempty. The emptiness of an upper hemicontinuous extension of the Mas-Colell bargaining set for a simple majority voting game with four persons is used to conclude that the Mas-Colell bargaining set of a non-levelled superadditive NTU game may be empty.

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1 Introduction

The Voting Paradox prevents us from applying the majority voting rule to choice problems with more than two alternatives. The standard way to avoid the paradox is to assume that the preferences of the voters are

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restricted so that the method of decision by majority yields no cycles (see Gaertner (2001) for a recent comprehensive survey). In this paper we follow a different path. It is well-known that the Voting Paradox is equivalent to the emptiness of the core of the corresponding cooperative majority voting game. We have chosen to investigate two bargaining sets which include the core: The Aumann-Davis-Maschler bargaining set and the Mas-Colell bargaining set. While it is well-known that the Aumann-Davis-Maschler bargaining set may be empty for superadditive NTU games, the problem of the non-emptiness of the Mas-Colell bargaining set (for the same class of games) was open when we started our investigation. Indeed, Example 4.5 provides for the first time a superadditive NTU game with an empty Mas-Colell bargaining set. Although the foregoing two bargaining sets may be empty, they perform much better than the core; for example, both are nonempty for the Voting Paradox and satisfy interesting asymptotic results.

We shall now review our results. At the end of the review we shall present our main conclusions.

In Section 2 we derive the exact form of the cooperative NTU games which correspond to simple majority voting, plurality voting, and approval voting (see Brams and Fishburn (1983)). We also recall the definitions of the Aumann-Davis-Maschler and Mas-Colell bargaining sets of cooperative NTU games. Throughout our study we focus, almost exclusively, on the foregoing two bargaining sets of simple majority voting games.

Section 3 deals with the Aumann-Davis-Maschler bargaining set. We report that it is nonempty for three alternatives. We show by means of an example that it may be empty (for a simple majority voting game), when there are four or more alternatives. Nevertheless, in a simple probabilistic model, if the number of alternatives is fixed, then the probability that the Aumann-Davis-Maschler bargaining set is nonempty tends to one as the number of voters tends to infinity.

Our main existence theorem is presented in Section 4: The Mas-Colell bargaining set of a simple majority voting game is nonempty for five (or less) alternatives. For six alternatives Example 4.5 shows that the Mas-Colell bargaining set (of a simple majority voting game) may be empty. Finally, we report in Section 4 the following result: If R^N is a profile of preferences of the n members of the set N of voters and if $k \geq n+2$, then the Mas-Colell bargaining set of any simple majority voting game that is derived from the k-th replication of R^N is nonempty.

In Section 5 we introduce an extension of the Mas-Colell bargaining set which is upper hemicontinuous. The emptiness of this extension for a four-person simple majority voting game with ten alternatives can be used to show the existence of a four-person non-levelled superadditive NTU game with an empty Mas-Colell bargaining set. This result solves an open problem of Vohra (1991).

Now we present our conclusions. Let (N, V) be a simple majority voting game and let x be an individually rational payoff vector. Then x is in a bargaining set if: (i) x is (weakly) Pareto optimal; and (ii) for every objection (in the sense of the bargaining set) there is a counter objection. Our study proves that the tension between (i) and (ii) is so strong that for six or more alternatives all bargaining sets may be empty. This is our first conclusion. Our second conclusion is more vague: If the number of players tends to infinity and the number of alternatives is held fixed, then the bargaining sets of (simple majority) voting games are likely to be non-empty.

Proofs of the results are contained in Peleg and Sudhölter (2004, 2005).

2 Preliminaries

Let $N = \{1, \ldots, n\}$, $n \geq 3$, be a set of voters, also called players, and let $A = \{a_1, \ldots, a_m\}$, $m \geq 3$, be a set of m alternatives. For $S \subseteq N$ we denote by $\mathbb{R}^S$ the set of all real functions on S. So $\mathbb{R}^S$ is the $|S|$-dimensional Euclidean space. (Here and in the sequel, if D is a finite set, then $|D|$ denotes the cardinality of D.) If $x, y \in \mathbb{R}^S$, then we write $x \geq y$ if $x^i \geq y^i$ for all $i \in S$. Moreover, we write $x > y$ if $x \geq y$ and $x \neq y$ and we write

$x \gg y$ if $x^i > y^i$ for all $i \in S$. Denote $\mathbb{R}^S_+ = \{x \in \mathbb{R}^S \mid x \geq 0\}$. A set $C \subseteq \mathbb{R}^S$ is comprehensive if $x \in C$, $y \in \mathbb{R}^S$, and $y \leq x$ imply that $y \in C$. An *NTU game* with the player set N is a pair (N, V) where V is a function which associates with every *coalition* S (that is, $S \subseteq N$ and $S \neq \emptyset$) a set $V(S) \subseteq \mathbb{R}^S$, $V(S) \neq \emptyset$, such that $V(S)$ is closed and comprehensive and $V(S) \cap (x + \mathbb{R}^S_+)$ is bounded for every $x \in \mathbb{R}^S$.

We shall focus on choice by simple majority voting, by plurality voting, and by approval voting. The corresponding three strategic game forms leading to three kinds of NTU voting games may be described as follows. The first game form consists of the voters selecting an element of A. If a strict majority of voters agrees on $\alpha \in A$, then the outcome is α; otherwise no alternative is selected. The second game form is a multi-valued game form which differs from the first game form only inasmuch as the set of all alternatives that are announced by a maximal number of voters is selected. In the third game form each voter has to announce a nonempty subset – a ballot – of alternatives. The outcome is the set of alternatives that are members of a maximal number of ballots.

We shall now assume that each $i \in N$ has a linear preference R^i on A. Thus, for every $i \in N$, R^i is a complete, transitive, and antisymmetric binary relation on A. Moreover, let u^i, $i \in N$, be a utility function that represents R^i. With the exception of Section 5 we shall always assume that

$$\min_{\alpha \in A} u^i(\alpha) = 0 \text{ for all } i \in N. \tag{2.1}$$

As we are going to break ties by even-chance lotteries, we shall further assume that the utilities are *weakly cardinal*, that is, they satisfy the expected utility hypothesis for even-chance lotteries (see Fishburn (1972)). For each of the three strategic game forms any utility profile $u^N = (u^i)_{i \in N}$ that satisfies the foregoing assumptions determines its corresponding strategic game. These considerations motivate us to define the cooperative NTU *voting games* that are associated (via α-effectiveness) with our strategic games. Indeed, let u^N be a utility profile that satisfies (2.1). The NTU game (N, V_{u^N}) associated with choice by simple majority voting and called *simple majority voting game* (see Aumann (1967)) is defined by

$$V_{u^N}(S) = \{x \in \mathbb{R}^S \mid x \leq 0\} \text{ if } S \subseteq N, 1 \leq |S| \leq \frac{n}{2}; \tag{2.2}$$

$$V_{u^N}(S) = \{x \in \mathbb{R}^S \mid \exists \alpha \in A \text{ such that } x \leq u^S(\alpha)\} \text{ if } S \subseteq N, |S| > \frac{n}{2}. \tag{2.3}$$

The coalition function of the *plurality voting game*, that is, the NTU game associated with choice by plurality voting, is denoted by $V^{pl}_{u^N}$ and it may differ from V_{u^N} only for coalitions $S \subseteq N$ such that $|S| = n/2$ and for the grand coalition N. Indeed, we define

$$V^{pl}_{u^N}(S) = \left\{ x \in \mathbb{R}^S \,\middle|\, \exists\, \alpha \in A \text{ such that } x \leq \frac{1}{2}u^S(\alpha) \right\} \text{ for all } S \subseteq N, |S| = \frac{n}{2}, \tag{2.4}$$

and

$$V^{pl}_{u^N}(N) = \left\{ x \in \mathbb{R}^N \,\middle|\, \begin{array}{c} \exists\, B \subseteq A \text{ such that } 1 \leq |B| \leq n, \\ \left(\left[\frac{n}{|B|}\right] - 1\right)|A| + |B| \geq n, \text{ and } x \leq \frac{\sum_{\beta \in B} u^N(\beta)}{|B|} \end{array} \right\}, \tag{2.5}$$

where $[r]$ denotes the largest integer less than or equal to r. Indeed, if $|S| = n/2$ and all members of S select the same alternative α, then a player $i \in S$ cannot be prevented from the utility $u^i(\alpha)/2$ even if all members of $N \setminus S$ select i's worst alternative (see (2.1)). Moreover, if B is the set of alternatives that are announced by a maximal number t of voters, then $0 \leq n - t|B| \leq (t-1)(|A| - |B|)$ and, hence, $t \leq [n/|B|]$ and

$$n - |B| \leq ([n/|B|] - 1)|A|. \tag{2.6}$$

If $B \subseteq A$ satisfies (2.6), then there exists a profile of strategies that results in the outcome B.

Now, if approval voting is employed, if $S \subseteq N$ satisfies $|S| = n/2$, and if each member j of S selects a ballot B^j, then the strategies of the players in $N \setminus S$ may induce the following sets of outcomes: (1) Any subset of $\bigcup_{j \in S} B^j$ and (2) any superset of $\bigcap_{j \in S} B^j$. Hence, if $i \in S$, then $N \setminus S$ may prevent i from receiving more than the utility

$$\min \left\{ \min_{\beta \in \bigcup_{j \in S} B^j} u^i(\beta), \min_{C \supseteq \bigcap_{j \in S} B^j} \sum_{\gamma \in C} \frac{u^i(\gamma)}{|C|} \right\} \leq$$

$$\leq \min\left\{\min_{\beta\in B^j} u^i(\beta), \min_{C\supseteq B^j} \sum_{\gamma\in C} \frac{u^i(\gamma)}{|C|}\right\} \; \forall j \in S.$$

Also, if all members of the grand coalition select $B\subseteq A$, then the resulting utility profile is $\sum_{\beta\in B} u^N(\beta)/|B|$. Hence, the NTU game associated with choice by approval voting, $(N, V^{ap}_{u^N})$, called *approval voting game*, differs from (N, V_{u^N}) only inasmuch as for any $S \subseteq N, |S| = \frac{n}{2}$,

$$V^{ap}_{u^N}(S) = \left\{x \in \mathbb{R}^S \middle| \exists\, \emptyset \neq B \subsetneq A \text{ such that } x^i \leq \min\left\{\min_{\beta\in B} u^i(\beta), \min_{\emptyset\neq C\subseteq A\setminus B} \frac{\sum_{\beta\in B\cup C} u^i(\beta)}{|B|+|C|}\right\} \forall\, i \in S\right\}, \tag{2.7}$$

and

$$V^{ap}_{u^N}(N) = \left\{x \in \mathbb{R}^N \middle| \exists\, \emptyset \neq B \subseteq A \text{ such that } x \leq \frac{\sum_{\beta\in B} u^N(\beta)}{|B|}\right\}. \tag{2.8}$$

Hence, for each coalition S, $V_{u^N}(S)$ (or $V^{pl}_{u^N}(S), V^{ap}_{u^N}(S)$, respectively) consists of all vectors $x \in \mathbb{R}^S$ that S can get, regardless of the strategies chosen by the members of $N\setminus S$, with respect to choice by simple majority voting (or plurality voting, approval voting, respectively). Note that the selection of no alternative in the context of choice by simple majority voting is assumed to result in the utility 0 for each voter.

Notation 2.1 In the sequel let $L = L(A)$ denote the set of linear preferences on A. If $R^N \in L^N$, then denote

$$\mathcal{U}^{R^N} = \{(u^i)_{i\in N} \mid u^i \text{ is a representation of } R^i \text{ satisfying (2.1) } \forall i \in N\}.$$

Remark 2.2 Let $R^N \in L^N$. Then the associated simple majority voting games are derived from each other by ordinal transformations. The associated plurality voting games and the associated approval voting games may not be derived from each other by an ordinal transformation, because weakly cardinal utilities may not be covariant under monotone transformations.

Let (N, V) be an NTU game. The pair (N, V) is *zero-normalized* if $V(\{i\}) = -\mathbb{R}^{\{i\}}_{+}(= \{x \in \mathbb{R}^i \mid x \leq 0\})$ for all $i \in N$. Also, (N, V) is *superadditive* if for every pair of disjoint coalitions S, T, $V(S) \times V(T) \subseteq V(S \cup T)$. It should be remarked that the three foregoing NTU games are zero-normalized and superadditive.

Now we shall recall the definitions of two bargaining sets introduced by Davis and Maschler (1967) and by Mas-Colell (1989). Let (N, V) be a zero-normalized NTU game and $x \in \mathbb{R}^N$. We say that x is

- *individually rational* if $x \geq 0$;
- *Pareto optimal* (in $V(N)$) if $x \in V(N)$ and if $y \in V(N)$ and $y \geq x$ imply $x = y$;
- *weakly Pareto optimal* (in $V(N)$) if $x \in V(N)$ and if for every $y \in V(N)$ there exists $i \in N$ such that $x^i \geq y^i$;
- a *preimputation* if x is weakly Pareto optimal in $V(N)$;
- an *imputation* if x is an individually rational preimputation.

A pair (P, y) is an *objection* at x if $\emptyset \neq P \subseteq N$, y is Pareto optimal in $V(P)$, and $y > x^P$. An objection (P, y) is *strong* if $y \gg x^P$. The pair (Q, z) is a *weak counter objection* to the objection (P, y) if $Q \subseteq N$, $Q \neq \emptyset, P$, if $z \in V(Q)$, and if $z \geq (y^{P \cap Q}, x^{Q \setminus P})$. A weak counter objection (Q, z) is a *counter objection* to the objection (P, y) if $z > (y^{P \cap Q}, x^{Q \setminus P})$. A strong objection (P, y) is *justified in the sense of the bargaining set* if there exist players $k \in P$ and $\ell \in N \setminus P$ such that there does not exist any weak counter objection (Q, z) to (P, y) satisfying $\ell \in Q$ and $k \notin Q$. The *bargaining set* of (N, V), $\mathcal{M}(N, V)$, is the set of all imputations x that do not have strong justified objections at x in the sense of the bargaining set (see Davis and Maschler (1967)). An objection (P, y) is *justified in the sense of the Mas-Colell bargaining set* if there does not exist any counter objection to (P, y). The *Mas-Colell bargaining set* of (N, V), $\mathcal{MB}(N, V)$, is the set of all imputations x that do not have a justified objection at x in the sense of the Mas-Colell bargaining set (see Mas-Colell (1989)).

Notation 2.3 If $R^N \in L^N$ and $\alpha, \beta \in A$, $\alpha \neq \beta$, then α *dominates* β (abbreviated $\alpha \succ_{R^N} \beta$) if $|\{i \in N \mid \alpha \; R^i \; \beta\}| > \frac{n}{2}$. For $R \in L$ and for $k \in \{1, \ldots, m\}$, let $t_k(R)$ denote the k-th alternative in the order R. Also, for $B \subseteq A$ let $R_{|B}$ denote the restriction of R to B.

Remark 2.4 Let $u^N \in \mathcal{U}^{R^N}$, let $B \subsetneq A$, let $i \in N$, and let

$$(t_1({R^i}_{|A\setminus B}), \ldots, t_{m-|B|}({R^i}_{|A\setminus B})) = (\alpha_1, \ldots, \alpha_{m-|B|})$$

be the vector of alternatives in $A \setminus B$ ordered by R^i. For $j = 1, \ldots, m - |B|$, define

$$z_j = \frac{1}{m-j+1}\left(\sum_{\beta \in B} u^i(\beta) + \sum_{k=j}^{m-|B|} u^i(\alpha_k)\right).$$

It can be deduced that the sequence $(z_j)_{j=1}^{m-|B|}$ is unimodal, i.e., there exists $t \in \{1, \ldots, m - |B|\}$ such that $z_k > z_{k+1}$ for $k \leq t - 1$, $z_k < z_{k+1}$ for $k > t$, and $z_t \leq z_{t+1}$ if $t < m - |B|$. We conclude that

$$\min_{\emptyset \neq C \subseteq A \setminus B} \sum_{\beta \in B \cup C} \frac{u^i(\beta)}{|B| + |C|} = \min_{j=1,\ldots,m-|B|} z_j = z_t.$$

This remark enables us to easily compute (2.7), taking (2.1) into account, that is,

$$t_m(R^i) \in B \;\Rightarrow\; \min_{\beta \in B} u^i(\beta) = 0 \leq z_t, \tag{2.9}$$

$$t_m(R^i) \notin B \;\Rightarrow\; u^i(\alpha_{m-|B|}) = u^i(t_m(R^i)) = 0. \tag{2.10}$$

We shall say that an alternative $\alpha \in A$ is a *weak Condorcet winner* (with respect to R^N) if $\beta \not\succ_{R^N} \alpha$ for all $\beta \in A$.

3 The Aumann-Davis-Maschler Bargaining Set

Throughout this section and Section 4 let $R^N \in L(A)^N$, $u^N \in \mathcal{U}^{R^N}$ (see Notation 2.1), $V = V_{u^N}$ (see (2.2) and (2.3)) and let $\succ \; = \; \succ_{R^N}$ (see Notation 2.3).

Theorem 3.1 *If $|A| = 3$, then $\mathcal{M}(N, V_{u^N}) \neq \emptyset$.*

In order to partially characterize the bargaining set, for $\alpha, \beta \in A$, $\alpha \neq \beta$, let

$$D_{\alpha\beta}(R^N) = D_{\alpha\beta} = \{i \in N \mid \alpha \; R^i \; \beta\}.$$

Theorem 3.2 *Let $A = \{a, b, c\}$. Assume that $a \succ b$, $b \succ c$, $c \succ a$, and that*

$$|D_{\alpha\beta}| > \frac{n}{2} + 1 \text{ for all } (\alpha, \beta) \in \{(a,b),(b,c),(c,a)\}.$$

If $x \in \mathbb{R}^N$ satisfies

$$0 \leq x \leq u^N(\alpha) \text{ for some } \alpha \in A \tag{3.1}$$

and

$$x^i \leq u^i(t_2(R^i)) \text{ for all } i \in N, \tag{3.2}$$

then $x \in \mathcal{M}(N, V)$.

Remark 3.3 In fact $|D_{ca}| > \frac{n}{2} + 1$ is not used when $x \leq u^N(a)$. Thus, the following stronger result may be deduced.

Corollary 3.4 *Let $A = \{a, b, c\}$. Assume that $x \in \mathbb{R}^N$ satisfies $0 \leq x \leq (u^i(t_2(R^i)))_{i \in N}$ and assume that $a \succ b$, $b \succ c$, and $c \succ a$. Then $x \in \mathcal{M}(N, V)$ in each of the following three cases:*

$$\left(x \leq u^N(a) \text{ and } |D_{ab}|, |D_{bc}| > \tfrac{n}{2} + 1\right), \quad \text{or}$$
$$\left(x \leq u^N(b) \text{ and } |D_{bc}|, |D_{ca}| > \tfrac{n}{2} + 1\right), \quad \text{or}$$
$$\left(x \leq u^N(c) \text{ and } |D_{ca}|, |D_{ab}| > \tfrac{n}{2} + 1\right).$$

By means of an example we shall show that $\mathcal{M}(N, V_{u^N})$ may be empty for any $u^N \in \mathcal{U}^{R^N}$, provided $|A| \geq 4$.

Example 3.5 Let $A = \{a, b, c, d\}$, let $n = 3$, let R^N be given by Table 3.1, let $u^N \in \mathcal{U}^{R^N}$, and let $V = V_{u^N}$. Then $\mathcal{M}(N, V) = \emptyset$. Example 3.5 shows that the tension between (weak) Pareto optimality and stability (à la Aumann and Maschler (1964)) may result in an empty bargaining set.

Table 3.1: Preference Profile of a 4-Alternative Voting Problem

R^1	R^2	R^3
a	c	b
b	a	c
d	d	d
c	b	a

Example 3.5 may be generalized to any number $m \geq 4$ of alternatives. Indeed, let $A = \{a, b, c, d_1, \ldots, d_k\}$, where $k = m - 3$, and define R^N by

$$\begin{aligned} R^1 &= (a, b, d_1, \ldots, d_k, c), \\ R^2 &= (c, a, d_1, \ldots, d_k, b), \\ R^3 &= (b, c, d_1, \ldots, d_k, a), \end{aligned}$$

and note that $\mathcal{M}(N, V_{u^N}) = \emptyset$ for any $u^N \in \mathcal{U}^{R^N}$. More interestingly, Example 3.5 can be generalized to yield an empty bargaining set for simple majority voting games on four alternatives with infinitely many numbers of voters.

Example 3.6 (Example 3.5 generalized) Let

$$R_1 = (a, b, d, c), \quad R_2 = (a, c, d, b), \quad R_3 = (b, a, d, c),$$
$$R_4 = (b, c, d, a), \quad R_5 = (c, a, d, b), \quad R_6 = (c, b, d, a),$$

and let $k \in \mathbb{N}$. Let $N = \{1, \ldots, 6k - 3\}$ and let $R^N \in L^N$ satisfy

$$|\{j \in N \mid R^j = R_i\}| = \begin{cases} k & \text{, if } i = 1, 4, 5, \\ k - 1 & \text{, if } i = 2, 3, 6. \end{cases}$$

Then $\mathcal{M}(N, V_{u^N}) = \emptyset$ for any $u^N \in \mathcal{U}^{R^N}$. Indeed, $k = 1$ coincides with Example 3.5.

Notwithstanding Example 3.5, there is a simple probabilistic model in which most preference profiles lead to a nonempty bargaining set $\mathcal{M}$ as the number of players becomes large. Let $|A| = m \geq 4$ and let $L(A) = L$. Assume that each $R \in L$ appears with positive probability $p_R > 0$ in the population of potential voters, where $\sum_{R \in L} p_R = 1$. Now let $(\mathcal{R}^i)_{i \in \mathbb{N}}$ be a sequence of independent and identically distributed random variables such that $Pr(\{\mathcal{R}^i = R\}) = p_R$ for all $i \in \mathbb{N}$, $R \in L$. Call $R^N \in L^N$ *good* if for all $\alpha \in A$ there exists $i \in N$ such that $\alpha = t_m(R^i)$. If R^N is good, then $\left(u^i(t_m(R^i))\right)_{i \in N} \in \mathcal{M}(N, V_{u^N})$ for any $u^N \in \mathcal{U}^{R^N}$. By the law of large numbers, $\lim_{n \to \infty} Pr\left(\{\mathcal{R}^N \text{ is good}\}\right) = 1$, where $\mathcal{R}^N = \left(\mathcal{R}^1, \ldots, \mathcal{R}^n\right)$. Hence, $\lim_{n \to \infty} Pr\left(\{\mathcal{M}\left(N, V\left(\mathcal{R}^N\right)\right) \neq \emptyset\}\right) = 1$, where $\left(N, V\left(\mathcal{R}^N\right)\right)$ is a random NTU game which is a simple majority voting game V_{u^N}, $u^N \in \mathcal{U}^{R^N}$, for any realization R^N of $\mathcal{R}^N$.

4 The Mas-Colell Bargaining Set

Remark 4.1 If there exists a weak Condorcet winner with respect to R^N, then $\mathcal{MB}(N, V_{u^N})$ contains the set of the utility profiles of all weak Condorcet winners.

In the case of three alternatives we may deduce the following results.

Theorem 4.2 *If $|A| = 3$ and if there is no weak Condorcet winner with respect to R^N and if $x \in \mathbb{R}^N$ satisfies*

$$0 \leq x^i \leq u^i(t_2(R^i)) \text{ for all } i \in N; \tag{4.1}$$

$$\text{there exists } \alpha \in A \text{ such that } x \leq u^N(\alpha), \tag{4.2}$$

then $x \in \mathcal{MB}(N, V)$.

Corollary 4.3 *If $|A| = 3$ and there is no weak Condorcet winner with respect to R^N, then $\mathcal{M}(N, V) \subseteq \mathcal{MB}(N, V)$.*

Examples show that the inclusion in the foregoing corollary may be strict.

Theorem 4.4 *If $m \leq 5$, then $\mathcal{MB}\left(N, V_{u^N}\right) \neq \emptyset$ for all $u^N \in \mathcal{U}^{R^N}$.*

Table 4.1: Preference Profile leading to an empty $\mathcal{MB}$

R^1	R^2	R^3	R^4
a_1	a_4	a_3	a_2
a_2	a_1	a_4	a_3
c	c	c	b
b	b	b	a_4
a_3	a_2	a_1	c
a_4	a_3	a_2	a_1

We shall now present an example of a simple majority voting game on six alternatives whose Mas-Colell bargaining set is empty.

Example 4.5 Let $n = 4$, $A = \{a_1, \ldots, a_4, b, c\}$, let $R^N \in L^N$ be given by Table 4.1 and let $u^N \in \mathcal{U}^{R^N}$. It may be verified that $\mathcal{MB}(N, V_{u^N}) = \emptyset$.

Example 4.5 may be generalized to any number $m \geq 6$ of alternatives. Also, if $R_i = R^i$ for $i = 1, \ldots, 4$, if

$$R_5 = (a_2, a_1, c, b, a_3, a_4), R_6 = (a_4, a_3, c, b, a_1, a_2),$$

if $n = 4 + 2k$ for some $k \in \mathbb{N}$, if $\widetilde{R}^N \in L^N$ such that

$$|\{j \in N \mid \widetilde{R}^j = R_i\}| = \begin{cases} k & \text{, if } i = 5, 6, \\ 1 & \text{, if } i = 1, 2, 3, 4, \end{cases}$$

then $\mathcal{MB}(N, V_{u^N}) = \emptyset$ for all $u^N \in \mathcal{U}^{\widetilde{R}^N}$.

In what follows we shall show that a suitable choice of utilities in Example 4.5 shows that the Mas-Colell bargaining set of a plurality or of a approval voting game on six alternatives may be empty.

Example 4.6 (Example 4.5 continued) We now specify a utility representation $u^N \in \mathcal{U}^{R^N}$ by

$$u^i(t_j(R^i)) = 6^5 - 6^{j-1} \text{ for all } i \in N \text{ and } j = 1, \ldots, 6.$$

Let (N, V) the corresponding plurality or approval voting game, that is, $V \in \{V^{pl}_{u^N}, V^{ap}_{u^N}\}$.

Then $\mathcal{MB}(N, V) = \emptyset$.

Remark 4.7 It is possible to modify the utility profile u^N of the foregoing example in such a way that the Mas-Colell bargaining sets of the approval or the plurality voting game are nonempty. Indeed, if we just replace u^i, $i = 1, 2$, by $\widetilde{u}^i$ which differs from u^i only inasmuch as $\widetilde{u}^i(t_j(R^i)) = 12 - 2j$ for $j = 4, 5$, then

$$x = \left(\frac{\widetilde{u}^1(a_3) + \widetilde{u}^1(a_4)}{2}, \frac{u^2(a_1)}{2}, u^{\{3,4\}}(a_4)\right) = (1, 3885, 7770, 7560) \in \mathcal{MB}(N, V).$$

In order to replicate the simple majority voting game (N, V_{u^N}), let $k \in \mathbb{N}$ and denote

$$kN = \{(j, i) \mid i \in N, \; j = 1, \ldots, k\}.$$

Furthermore, let $R^{(j,i)} = R^i$ and $u^{(j,i)} = u^i$ for all $i \in N$ and $j = 1, \ldots, k$. Then $(kN, V_{u^{kN}})$ is the *k-fold replication* of (N, V_{u^N}).

Remark 4.8 *If α is a weak Condorcet winner with respect to R^N, then $u^{kN}(\alpha) \in \mathcal{MB}(kN, V_{u^{kN}})$.*

Theorem 4.9 *If* $k \geq \left\{ \begin{array}{ll} n+2 & \text{, if } n \text{ is odd,} \\ \frac{n}{2}+2 & \text{, if } n \text{ is even,} \end{array} \right\}$ *then* $\mathcal{MB}(kN, V_{u^{kN}}) \neq \emptyset$.

It should be remarked that the foregoing theorem remains valid for any $u^{kN} \in \mathcal{U}^{R^{kN}}$.

5 A Non-Levelled Superadditive Game with an Empty $\mathcal{MB}$

In this section we show that there exists a **non-levelled**[1] superadditive NTU game whose Mas-Colell bargaining set is empty. Note that simple majority

[1] A zero-normalized NTU game (N, V) is *non-levelled* if for every coalition S every weakly Pareto optimal element of $V(S) \cap \mathbb{R}^S_+$ is Pareto optimal.

voting games are levelled. We shall now extend $\mathcal{MB}$ and modify the game of Example 4.5 suitably. Let (N, V) be a zero-normalized superadditive NTU game and let x be an imputation. A strong objection at x is *strongly justified* if it has no weak counter objection. The *extended bargaining set* $\mathcal{MB}^*(N, V)$ is the set of all imputations that do not have strongly justified strong objections. Clearly, $\mathcal{MB}(N, V) \subseteq \mathcal{MB}^*(N, V)$.

If (N, V) is the game of Example 4.5, then $(u^{\{1,2,3\}}(b), u^4(a_4)) \in \mathcal{MB}^*(N, V)$. However, the following example presents a game whose extended bargaining set is empty.

Example 5.1 Let $n = 4$, $A = \{a_1, \ldots, a_4, a_1^*, \ldots, a_4^*, b, c\}$, let $R^N \in L^N$ be given by Table 5.1, let u^N represent R^N such that $\min_{\alpha \in A} u^i(\alpha) > 0$ for all

Figure 5.1: A Preference Profile

R^1	R^2	R^3	R^4
a_1	a_4	a_3	a_2
a_2	a_1	a_4	a_3
a_2^*	a_1^*	a_4^*	a_3^*
a_1^*	c	a_3^*	a_2^*
c	a_4^*	c	b
b	b	b	a_4^*
a_3^*	a_2^*	a_1^*	a_4
a_3	a_2	a_1	c
a_4^*	a_3^*	a_2^*	a_1^*
a_4	a_3	a_2	a_1

$i \in N$, and let $V = V_{u^N}$. It may be verified that $\mathcal{MB}^*(N, V) = \emptyset$.

Let N be a finite nonempty set and let Γ denote the set of all superadditive zero-normalized NTU games (N, V).

The following lemma may be shown directly.

Lemma 5.2 *$\mathcal{MB}^*$ is an upper hemicontinuous correspondence*[2] *on Γ.*

With the help of Theorem 4 of Wooders (1983) it is possible to deduce the following desired result.

Theorem 5.3 *There exists a superadditive and non-levelled four-person game U such that $\mathcal{MB}(U) = \emptyset$.*

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[2] Here we identify any game $(N, V) \in \Gamma$ with V^+, defined by $V^+(S) = V(S) \cap \mathbb{R}^S_+$ for all $S \subseteq N$, $S \neq \emptyset$. The *distance* between two games V_1^+ and V_2^+ is the number $\delta(V_1^+, V_2^+) = \max_{\emptyset \neq S \subseteq N} d_S(V_1^+(S), V_2^+(S))$, where $d_S(\cdot, \cdot)$ is the Hausdorff distance between nonempty compact subsets of $\mathbb{R}^S$.

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