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# On the non-emptiness of the Mas-Colell bargaining set☆

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**Abstract**

We introduce an extension of the Mas-Colell bargaining set and construct, by an elaboration on a voting paradox, a superadditive four-person non-transferable utility game whose extended bargaining set is empty. It is shown that this extension constitutes an upper hemicontinuous correspondence. We conclude that the Mas-Colell bargaining set of a non-levelled superadditive NTU game may be empty.


*JEL classification:* C71

*Keywords:* NTU games; Mas-Colell bargaining set

## 1. Introduction

Mas-Colell (1989) has introduced a bargaining set, which is defined also for finite games. In this paper we address the question of non-emptiness of the Mas-Colell bargaining set for

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superadditive NTU games. The problem is mentioned in Section 6 of Mas-Colell (1989) and in Holzman (2000). We construct a four-person majority voting game – majority voting games are automatically superadditive – with 10 alternatives whose Mas-Colell bargaining set is empty. In view of Vohra (1991) we include individual rationality in the definition of the bargaining set. However, the aforementioned result holds also in Mas-Colell's original model, i.e., without individual rationality.

Moreover, this voting game enables us to show the existence of a *non-levelled* superadditive NTU game whose bargaining set is empty, thereby solving an open problem raised by Vohra (1991). Indeed, we introduce an extension of the bargaining set, which is upper hemicontinuous and specifies the empty set when applied to our voting game.

The paper is organized as follows. Section 2 recalls the relevant definitions and introduces an extension of the Mas-Colell bargaining set, denoted by $\mathcal{MB}^*$. Section 3 presents the construction of the four-person voting game and the proof of emptiness of $\mathcal{MB}^*$ when applied to this game. In Section 4 we first prove that $\mathcal{MB}^*$ is an upper hemicontinuous correspondence. Moreover, we show that in any neighborhood of a superadditive NTU game there exists a non-levelled superadditive NTU game. Finally, we conclude that there exists a non-levelled superadditive four-person game whose (extended) bargaining set is empty.

## 2. Preliminaries

Let $N = \{1, \ldots, n\}$, $n \in \mathbb{N}$, be a set of players. For $S \subseteq N$ we denote by $\mathbb{R}^S$ the set of all real functions on $S$. So $\mathbb{R}^S$ is an $|S|$-dimensional Euclidean space. (Here and in the sequel, if $D$ is a finite set, then $|D|$ denotes the cardinality of $D$.) If $x \in \mathbb{R}^S$ and $T \subseteq S$, then $x^T$ denotes the restriction of $x$ to $T$. If $x, y \in \mathbb{R}^S$, then we write $x \geq y$ if $x^i \geq y^i$ for all $i \in S$. Moreover, we write $x > y$ if $x \geq y$ and $x \neq y$ and we write $x \gg y$ if $x^i > y^i$ for all $i \in S$. Denote $\mathbb{R}^S_+ = \{x \in \mathbb{R}^S \mid x \geq 0\}$. A set $C \subseteq \mathbb{R}^S$ is comprehensive if $x \in C$, $y \in \mathbb{R}^S$, and $y \leq x$ imply that $y \in C$. We are now ready to recall the definition of an NTU game.

**Definition 2.1.** An *NTU coalitional game* (a *game*) is a pair $(N, V)$ where $N$ is a set of players and $V$ is a function which associates with every $S \subseteq N$, $S \neq \emptyset$, a set $V(S) \subseteq \mathbb{R}^S$, $V(S) \neq \emptyset$, such that

(1) $V(S)$ is closed and comprehensive;
(2) $V(S) \cap (x + \mathbb{R}^S_+)$ is bounded for every $x \in \mathbb{R}^S$.

As we are working in the model of Vohra (1991), we shall restrict our attention to *weakly superadditive* games.

**Definition 2.2.** An NTU game $(N, V)$ is *weakly superadditive* if for every $i \in N$ and every $S \subseteq N \setminus \{i\}$ satisfying $S \neq \emptyset$, $V(S) \times V(\{i\}) \subseteq V(S \cup \{i\})$.

In particular we shall be interested in superadditive games. A game $(N, V)$ is *superadditive* if for every pair of disjoint coalitions $S, T$ (a coalition is a nonempty subset of $N$), $V(S) \times V(T) \subseteq V(S \cup T)$.

We shall restrict our attention to *zero-normalized* games, that is, to games $(N, V)$ that satisfy $V(\{i\}) = -\mathbb{R}_+^{\{i\}} (= \{x \in \mathbb{R}^i \mid x \leq 0\})$ for all $i \in N$.

Let $(N, V)$ be a zero-normalized weakly superadditive game and $x \in \mathbb{R}^N$. We say that $x$ is

- *individually rational* if $x \geq 0$;
- *Pareto optimal* (with respect to $V(N)$) if $x \in V(N)$ and if $y \in V(N)$ and $y \geq x$ imply $x = y$;
- *weakly Pareto optimal* (with respect to $V(N)$) if $x \in V(N)$ and if for every $y \in V(N)$ there exists $i \in N$ such that $x^i \geq y^i$;
- a *preimputation* if $x \in V(N)$ and $x$ is weakly Pareto optimal;
- an *imputation* if $x$ is an individually rational preimputation.

Note that the set of imputations of a weakly superadditive game is nonempty. Mas-Colell (1989) has introduced the following bargaining set. Let $(N, V)$ be an NTU game and let $x$ be an imputation. A pair $(P, y)$ is an *objection* at $x$ if $\emptyset \neq P \subseteq N$, $y$ is Pareto optimal with respect to $V(P)$, and $y > x^P$. The pair $(Q, z)$ is a *counter objection* to the objection $(P, y)$ if $Q \subseteq N$, $Q \neq \emptyset, P$, if $z \in V(Q)$, and if $z > (y^{P\cap Q}, x^{Q\setminus P})$. An objection is *justified* if it cannot be countered.

**Definition 2.3.** Let $(N, V)$ be an NTU game. The *Mas-Colell bargaining set* of $(N, V)$, $\mathcal{MB}(N, V)$, is the set of all imputations $x$ such that there are no justified objections at $x$.

Let $(N, V)$ be an NTU game. It should be noted that the Mas-Colell *pre*bargaining set of $(N, V)$ is the set of all *pre*imputations of $(N, V)$ that do not have justified objections. So, $\mathcal{MB}(N, V)$ is the intersection of the Mas-Colell prebargaining set of $(N, V)$ and the set of imputations of $(N, V)$.

In view of Vohra (1991) we restrict our attention to the members of $\mathcal{MB}(N, V)$ rather than to the members of the Mas-Colell prebargaining set of $(N, V)$. Hence we may restrict our attention to the individually rational subsets of the sets $V(S)$. Indeed, let $(N, V)$ be a zero-normalized weakly superadditive NTU game. For $\emptyset \neq S \subseteq N$ denote $V^+(S) = V(S) \cap \mathbb{R}_+^S$. Then $V^+$ is nonempty-valued, compact-valued, and (restricted) comprehensive, that is, for every coalition $S$, if $x \in V^+(S)$ and $y \in \mathbb{R}_+^S$, $y \leq x$, then $y \in V^+(S)$. Hence, we shall call $(N, V^+)$ an NTU game as well.

**Remark 2.4.** If $(N, V)$ is a weakly superadditive zero-normalized NTU game, then $\mathcal{MB}(N, V) = \mathcal{MB}(N, V^+)$.

**Proof.** The sets of imputations of $(N, V)$ and of $(N, V^+)$ coincide. Let $x$ be an imputation. Then the sets of objections at $x$ with respect to $(N, V)$ and with respect to $(N, V^+)$ coincide. Finally, let $(P, y)$ be an objection at $x$. The observation that the sets of counter objections to

$(P, y)$ with respect to $(N, V)$ and with respect to $(N, V^+)$ also coincide proves the foregoing remark. □

Let $(N, V)$ be a weakly superadditive zero-normalized NTU game. We say that $(N, V)$ is *non-levelled* if

$$\begin{array}{l}\text{for each coalition } S \text{ every weakly Pareto optimal element} \\ \text{with respect to } V^+(S) \text{ is Pareto optimal with respect to } V^+(S).\end{array} \tag{2.1}$$

In this case we shall also say that $V^+$ is non-levelled. (Note that in Vohra (1991) the foregoing property is called *strong comprehensiveness*.)

In Section 3 we shall construct an example of a superadditive game whose Mas-Colell bargaining set is empty. However, this NTU game is not non-levelled. In order to show that the Mas-Colell bargaining set may be empty even for a non-levelled superadditive game, it is useful to define the following extension of $\mathcal{MB}$. Let $(N, V)$ be an NTU game and let $x$ be an imputation. An objection $(P, y)$ at $x$ is a *strong* objection if $y \gg x^P$. A pair $(Q, z)$ is a *weak counter objection* to the objection $(P, y)$ if $\emptyset, P \neq Q \subseteq N$, $z \in V(Q)$, and $z \geq (y^{P\cap Q}, x^{Q\setminus P})$. A strong objection is *strongly justified* if it has no weak counter objection.

**Definition 2.5.** Let $(N, V)$ be an NTU game. The *extended bargaining set* of $(N, V)$, $\mathcal{MB}^*(N, V)$, is the set of all imputations $x$ such that are no strongly justified strong objections at $x$.

Let $(N, V)$ be an NTU game. Note that $\mathcal{MB}(N, V)$ and $\mathcal{MB}^*(N, V)$ remain unchanged if we do not require in the definition of an objection $(P, y)$ that $y$ is Pareto optimal with respect to $V(P)$. However, the present definition of an objection $(P, y)$ automatically excludes any counter objection that uses the same coalition $P$. In the definition of weak counter objection the objecting coalition has to be excluded explicitly, because otherwise any objection has a weak counter objection. Hence, the requirement of Pareto optimality in the definition of objections guarantees that counter objections are weak counter objections. In particular, the following result is an immediate consequence of the foregoing definitions.

**Remark 2.6.** Let $(N, V)$ be an NTU game. Then

$$\mathcal{MB}(N, V) \subseteq \mathcal{MB}^*(N, V). \tag{2.2}$$

Further, if $(N, V)$ is weakly superadditive and zero-normalized, then

$$\mathcal{MB}^*(N, V) = \mathcal{MB}^*(N, V^+). \tag{2.3}$$

**Proof.** The inclusion (2.2) is implied by the facts that (a) any strong objection at an imputation $x$ is an objection at $x$ and that (b) any counter objection to an objection at $x$ is a weak counter objection to that objection as well. The proof of the second assertion is similar to the proof of Remark 2.4. Only objections have to be replaced by strong objections and counter objections have to be replaced by weak counter objections. □

Table 1
Preference profile

| $R^1$ | $R^2$ | $R^3$ | $R^4$ |
|---|---|---|---|
| $a_1$ | $a_4$ | $a_3$ | $a_2$ |
| $a_2$ | $a_1$ | $a_4$ | $a_3$ |
| $a_2^*$ | $a_1^*$ | $a_4^*$ | $a_3^*$ |
| $a_1^*$ | $c$ | $a_3^*$ | $a_2^*$ |
| $c$ | $a_4^*$ | $c$ | $b$ |
| $b$ | $b$ | $b$ | $a_4^*$ |
| $a_3^*$ | $a_2^*$ | $a_1^*$ | $a_4$ |
| $a_3$ | $a_2$ | $a_1$ | $c$ |
| $a_4^*$ | $a_3^*$ | $a_2^*$ | $a_1^*$ |
| $a_4$ | $a_3$ | $a_2$ | $a_1$ |

## 3. The example

Let $N = \{1, 2, 3, 4\}$ be the set of players and let

$$A = \{a_1, a_2, a_3, a_4, a_1^*, a_2^*, a_3^*, a_4^*, b, c\}$$

be a set of 10 alternatives. In the corresponding strategic game the players simultaneously announce an alternative. If there is a majority (of three or more players) for an alternative, then that alternative is chosen. Otherwise, everybody gets 0. Let the linear preferences on $A$ of the players, $R^i$, $i = 1, \ldots, 4$, be specified by Table 1. Thus, for every $i \in N$, $R^i$, is a complete, transitive, and antisymmetric binary relation on $A$. These preferences will be used to define our NTU game.

If $\alpha, \beta \in A$, $\alpha \neq \beta$, then $\alpha$ *dominates* $\beta$, written $\alpha \succ \beta$, if

$$|\{i \in N \mid \alpha R^i \beta\}| \geq 3.$$

The entire domination relation $\succ$ is depicted in Table 2.

For each $i \in N$ let $u^i : A \to \mathbb{R}$ be a utility function that represents $R^i$, that is, $u^i(\alpha) \geq u^i(\beta)$ if and only if $\alpha R^i \beta$, for all $\alpha, \beta \in A$. Furthermore we assume that

$$\min_{\alpha \in A} u^i(\alpha) > 0 \text{ for all } i \in N. \tag{3.1}$$

We are now able to define our NTU game $(N, V)$. For each $S \subseteq N$, $S \neq \emptyset$, let

$$V(S) = \{x \in \mathbb{R}^S \mid x \leq 0\}, \text{ if } |S| = 1, 2, \tag{3.2}$$

Table 2
Domination relation

| | | | |
|---|---|---|---|
| $a_1 \succ a_2$ | $a_2 \succ a_3$ | $a_3 \succ a_4$ | $a_4 \succ a_1$ |
| $a_1 \succ a_2^*$ | $a_2 \succ a_3^*$ | $a_3 \succ a_4^*$ | $a_4 \succ a_1^*$ |
| | $a_4 \succ c$ | $c \succ b$ | |

Table 3
Constructions of strong objections

| | | | |
|---|---|---|---|
| $u^{S_1}(a_1) \gg u^{S_1}(a_2)$ | $u^{S_2}(a_2) \gg u^{S_2}(a_3)$ | $u^{S_3}(a_3) \gg u^{S_3}(a_4)$ | $u^{S_4}(a_4) \gg u^{S_4}(a_1)$ |
| $u^{S_1}(a_1) \gg u^{S_1}(a_2^*)$ | $u^{S_2}(a_2) \gg u^{S_2}(a_3^*)$ | $u^{S_3}(a_3) \gg u^{S_3}(a_4^*)$ | $u^{S_4}(a_4) \gg u^{S_4}(a_1^*)$ |
| | $u^{S_4}(a_4) \gg u^{S_4}(c)$ | $u^{S_1}(c) \gg u^{S_1}(b)$ | |

$$V(S) = \left\{ x \in \mathbb{R}^S \;\middle|\; \begin{array}{l}\text{there exists}\, \alpha \in A \\ \text{such that}\, x \leq u^S(\alpha)\end{array} \right\}, \text{ if } |S| \geq 3, \tag{3.3}$$

where $u^S(\alpha) = (u^i(\alpha))_{i \in S}$. As the reader may easily verify, $(N, V)$ is a zero-normalized and superadditive NTU game. Moreover, every imputation $x$ of $(N, V)$ satisfies $x \geq 0$ and

$$x^i \geq u^i(b) \text{ for some } i \in N, \tag{3.4}$$

among other inequalities. Indeed (3.4) is satisfied, because $x$ is weakly Pareto optimal and $u^N(b) \in V(N)$.

We shall now prove the main result of this section.

**Theorem 3.1.** $\mathcal{MB}^*(N, V) = \emptyset$.

**Proof.** Assume, on the contrary, that there exists an imputation $x$ in the set $\mathcal{MB}^*(N, V)$. Let $S_1 = \{1, 2, 3\}$, $S_2 = \{1, 2, 4\}$, $S_3 = \{1, 3, 4\}$, and $S_4 = \{2, 3, 4\}$. Frequently used strong objections that use some of the foregoing coalitions may be constructed with the help of Table 3 which is deduced from Table 1 (see also Table 2). By (3.3), $x \leq u^N(\alpha)$ for some $\alpha \in A$. As $A$ has 10 elements, we proceed by distinguishing the arising 10 possibilities. First we shall consider the following case:

$$x \leq u^N(a_1). \tag{3.5}$$

As $a_4 \succ a_1$, $(S_4, u^{S_4}(a_4))$ is a strong objection at $x$ (see Table 3). Note that $a_4$ is the first, that is, the most preferred, alternative of player two, the second alternative of player three, and note that $a_3$ is the first alternative of player three. So, if $\alpha \in A \setminus \{a_3, a_4\}$, then

$$|S_4 \cap \{i \in N \mid a_4 R^i \alpha\}| \geq 2.$$

Thus the foregoing objection can be weakly countered only by $(S_3, y)$ for some $y \leq u^{S_3}(a_3)$, or by $(T, z)$ for some $|T| \geq 3$ such that $1 \in T$ and some $z \leq u^T(a_4)$, or by $(\{1\}, 0)$ (if $x^1 = 0$). Hence $x^1 \leq u^1(a_3)$. From Table 1 we conclude that $(S_3, u^{S_3}(a_3^*))$ is a strong objection at $x$. Let $\alpha \in A$. If

$$|S_3 \cap \{i \in N \mid a_3^* R^i \alpha\}| < 2,$$

then $\alpha \in \{a_2, a_3^*, a_3\}$. Thus, if $(P, y)$ is a weak counter objection to $(S_3, u^{S_3}(a_3^*))$, then $2 \in P$ and $y^2 \leq u^2(a_2)$. As $x \in \mathcal{MB}(N, V)$ is assumed, there exists a weak counter objection $(P, y)$ to the foregoing strong objection. We conclude that $x^2 \leq y^2 \leq u^2(a_2)$. Thus, $x \ll u^N(b)$ and the desired contradiction has been obtained (see (3.4)).

The following three cases may be treated similarly to (3.5):

$$x \le u^N(\alpha) \text{ for some } \alpha \in \{a_2, a_3, a_4\}. \tag{3.6}$$

Indeed, if $i \in \{2, 3, 4\}$ and $x \le u^N(a_i)$, then Table 3 shows that

$$(S_{i-1}, u^{S_{i-1}}(a_{i-1}))$$

is a strong objection at $x$. A careful inspection of the tables allows to specify one further strong objection, namely $(S_{i-2}, u^{S_{i-2}}(a^*_{i-2}))$ if $i \ne 2$ and $(S_4, u^{S_4}(a^*_4))$ if $i = 2$, and again the existence of a weak counter objections implies that $x \ll u^N(b)$.

The next case is the following case

$$x \le u^N(a^*_1). \tag{3.7}$$

As $(S_4, u^{S_4}(a_4))$ is a strong objection at $x$ (see Table 3), a careful inspection of Table 1 shows that we may proceed as in (3.5).

The following three cases may be treated similarly to (3.7):

$$x \le u^N(\alpha) \text{ for some } \alpha \in \{a^*_2, a^*_3, a^*_4\}. \tag{3.8}$$

Now we shall consider the 9th possibility:

$$x \le u^N(b). \tag{3.9}$$

In this case Table 3 shows that $(S_1, u^{S_1}(c))$ is a strong objection at $x$. If $(P, y)$ is a weak counter objection to the foregoing strong objection, then an inspection of Table 1 shows that $(P, y)$ satisfies at least one of the following properties:

$$y \le u^P(c) \text{ and } 4 \in P;$$
$$y \le u^P(a_1) \text{ and } P = S_2;$$
$$y \le u^P(a^*_1) \text{ and } P = S_2;$$
$$y \le u^P(a_4) \text{ and } P = S_4.$$

Therefore $x^4 \le u^4(a_4)$. We conclude that $(S_4, u^{S_4}(a^*_4))$ is a strong objection at $x$. Then

$$\{\alpha \in A \mid |S_4 \cap \{i \in N \mid a^*_4 R^i \alpha\}| < 2\} = \{a_3, a_4, a^*_4\}.$$

Hence $x^1 \le u^1(a_3)$. Thus, $(S_3, u^{S_3}(a^*_3))$ is a strong objection at $x$. The observation that

$$\{\alpha \in A \mid |S_3 \cap \{i \in N \mid a^*_3 R^i \alpha\}| < 2\} = \{a_2, a_3, a^*_3\},$$

shows that $x^2 \le u^2(a_2)$ and, thus, $(S_2, u^{S_2}(a^*_2))$ is a strong objection at $x$. We compute

$$\{\alpha \in A \mid |S_2 \cap \{i \in N \mid a^*_2 R^i \alpha\}| < 2\} = \{a_1, a_2, a^*_2\}.$$

Thus, if $(P, y)$ is a weak counter objection to $(S_2, u^{S_2}(a^*_2))$, then $3 \in P$ and $y^3 \le u^3(a_1)$. We conclude that $x^3 \le u^3(a_1)$. Therefore, again, $x \ll u^N(b)$.

Finally, we have to consider the following case

$$x \le u^N(c). \tag{3.10}$$

Then $(S_4, u^{S_4}(a_4))$ is a strong objection at $x$ (see Table 3). If $(P, y)$ is a weak counter objection to $(S_4, u^{S_4}(a_4))$, then (1) $P = S_3$ and $y \le u^P(a_3)$ or (2) $1 \in P$ and $y \le u^P(a_4)$. Hence, $x^1 \le u^1(a_3)$ and $(S_3, u^{S_3}(a_3^*))$ is a strong objection at $x$. We may now continue as in (3.9) and deduce that $x \ll u^N(b)$.

By (3.3), the domain defined by (3.5)–(3.10) is equal to V(N). Hence, we have derived a contradiction to the required weak Pareto optimality in all possible 10 cases. □

## 4. Non-levelled games

Let $N$ be a finite nonempty set and denote

$$\Gamma^+ = \{V^+ \mid (N, V) \text{ is a zero-normalized weakly superadditive NTU game}\}$$

(for the definition of $V^+$ see Section 2). Let $V_1^+, V_2^+ \in \Gamma^+$. The *distance* between $V_1^+$ and $V_2^+$ is

$$\delta(V_1^+, V_2^+) = \max_{\emptyset \ne S \subseteq N} d_S(V_1^+(S), V_2^+(S)),$$

where $d_S(\cdot, \cdot)$ is the Hausdorff distance between nonempty compact subsets of $\mathbb{R}^S$.

**Lemma 4.1.** *$\mathcal{MB}^*$ is an upper hemicontinuous correspondence on $\Gamma^+$.*

**Proof.** It is sufficient to prove that $\mathcal{MB}^*$ has a closed graph. Thus let $V^+, V_k^+ \in \Gamma^+, k \in \mathbb{N}$, such that $\lim_{k\to\infty} \delta(V^+, V_k^+) = 0$, and let $x_k \in \mathcal{MB}^*(V_k^+), k \in \mathbb{N}$, such that $\lim_{k\to\infty} x_k = x$. It remains to show that $x \in \mathcal{MB}^*(V^+)$. Note that $x$ is a weakly Pareto optimal element of $V^+(N)$. Assume, on the contrary, that $x \notin \mathcal{MB}^*(V^+)$. Then there exists a strongly justified strong objection $(P, y)$ at $x$. Thus, $y$ is a Pareto optimal element of $V^+(P)$ and for every $S \subseteq N$ such that $S \ne \emptyset, P$ and any $z \in V^+(S)$,

$$f_S(x, y, z, V^+) = \min\{\min_{i \in S\cap P}(z^i - y^i), \min_{i \in S\setminus P}(z^i - x^i)\} < 0.$$

The mapping $g_S$ defined by $g_S(x, y, V^+) = \max_{z\in V^+(S)} f_S(x, y, z, V^+)$ is a continuous function of $x$, $y$, and $V^+$. Choose, for $k \in \mathbb{N}$, a Pareto optimal member $y_k$ of $V_k^+$ such that $\lim_{k\to\infty} y_k = y$. By continuity of $g_S$ there exists a sufficiently large $k_0 \in \mathbb{N}$ such that for every $k > k_0$, $g_S(x_k, y_k, V_k^+) < 0$ for all $S \subseteq N$, $S \ne \emptyset, P$, and $y_k^i > x_k^i$ for all $i \in P$. Thus, $(P, y_k)$ is a strongly justified strong objection at $x_k$ for $k > k_0$. As $x_k \in \mathcal{MB}^*(V_k^+)$, the desired contradiction has been obtained. □

Let $V^+ \in \Gamma^+$, let $\epsilon > 0$, let $K = \max_{\emptyset \neq S \subseteq N} \max_{x \in V^+(S)} \max_{i \in S} x^i$. For every $\emptyset \neq S \subseteq N$ define $h_S^\epsilon : \mathbb{R}_+^S \to \mathbb{R}$ by

$$h_S^\epsilon(x) = 1 + \frac{\epsilon}{2 + \sum_{i \in S} x^i + K|N \setminus S|}. \tag{4.1}$$

Using the foregoing equation define

$$V_\epsilon^+(S) = \{h_S^\epsilon(x)x \mid x \in V^+(S)\}. \tag{4.2}$$

We shall say that $V^+$ is *p-non-levelled* if, for each coalition $S$, any weakly Pareto optimal element $x \gg 0$ with respect to $V^+(S)$ is Pareto optimal. Hence a non-levelled game is p-non-levelled (see (2.1)).

**Lemma 4.2.** *Let $V^+ \in \Gamma^+$ be superadditive and let $\epsilon > 0$. Then $V_\epsilon^+$ is a superadditive p-non-levelled game such that $\delta(V_\epsilon^+, V^+) < \epsilon$.*

**Proof.** Let $S \subseteq N$, $S \neq \emptyset$. By Wooders (1983, Theorem 4),

$$d_S(V^+(S), V_\epsilon^+(S)) < \epsilon,$$

$V_\epsilon^+(S)$ is restricted comprehensive, and $V_\epsilon^+$ is p-non-levelled. In order to show that $V_\epsilon^+$ is superadditive, let $S, T \subseteq N$, $S, T \neq \emptyset$, and $S \cap T = \emptyset$. If $x_\epsilon \in V_\epsilon^+(S)$ and $y_\epsilon \in V_\epsilon^+(T)$, then let $x \in V^+(S)$ and $y \in V^+(T)$ be defined by $h_S^\epsilon(x)x = x_\epsilon$ and $h_T^\epsilon(y)y = y_\epsilon$. By superadditivity of $V^+$, $(x, y) \in V^+(S \cup T)$. Moreover,

$$h_{S \cup T}^\epsilon(x, y) \geq \max\{h_S^\epsilon(x), h_T^\epsilon(y)\}.$$

Thus, $(x_\epsilon, y_\epsilon) \leq h_{S \cup T}^\epsilon(x, y)(x, y)$. By restricted comprehensiveness, $V_\epsilon^+$ is superadditive. □

We are now ready to prove the main result of this paper.

**Theorem 4.3.** *There exists a superadditive and non-levelled four-person game $U^+$ such that $\mathcal{MB}(U^+) = \emptyset$.*

**Proof.** Let $V$ be the game of the example defined in Section 3. As $\mathcal{MB}^*$ is upper hemicontinuous and $\mathcal{MB}^*(V^+) = \emptyset$, there exists $\epsilon > 0$ such that $\mathcal{MB}^*(W^+) = \emptyset$ for any $W^+ \in \Gamma^+$ such that $\delta(V^+, W^+) < \epsilon$. By Lemma 4.2, $V_\epsilon^+ \in \Gamma^+$ is a superadditive p-non-levelled game and $\delta(V^+, V_\epsilon^+) < \epsilon$. By (3.1), $V_\epsilon^+$ is non-levelled. Remark 2.6 completes the proof. □

## References

Holzman, R., 2000. The comparability of the classical and the Mas-Colell bargaining set. International Journal of Game Theory 29, 543–553.

Mas-Colell, A., 1989. An equivalence theorem for a bargaining set. Journal of Mathematical Economics 18, 129–139.

Vohra, R., 1991. An existence theorem for a bargaining set. Journal of Mathematical Economics 20, 19–34.

Wooders, M., 1983. The epsilon core of a large replica game. Journal of Mathematical Economics 11, 277–300.