GAMES AND ECONOMIC BEHAVIOR **18,** 277–285 (1997)
ARTICLE NO. GA970523

An Axiomatization of Nash Equilibria in Economic Situations*

Bezalel Peleg[†]

Center for Rationality and Interactive Decision Theory, The Hebrew University of Jerusalem, Feldman Building, Givat-Ram, 91904 Jerusalem, Israel

and

Peter Sudhölter[‡]

Institute of Mathematical Economics, University of Bielefeld, Postfach 100131, D-33501 Bielefeld, Germany

Received November 15, 1995

We consider the class of all abstract economies with convex and compact strategy sets, continuous and quasi-concave payoff functions, and continuous and convex-valued feasibility correspondences. We prove that the Nash correspondence is the unique solution on the foregoing class of abstract economies that satisfies nonemptiness, rationality in one-person economies, and consistency. *Journal of Economic Literature* Classification Numbers: C72, D50. © 1997 Academic Press

1. INTRODUCTION

Abstract economies were introduced by Debreu (1952) in order to help in the proof of existence of Walras equilibrium for competitive economies (Arrow and Debreu (1954)). They have since been used extensively in general equilibrium theory (see, e.g., Shafer and Sonnenschein (1975), Ichiishi (1983), and Border (1985)). Abstract economies are essentially strategic games in which the feasible set of strategies of a player may depend on the strategies chosen by the other players. Thus, an abstract

* We are indebted to M. Majumdar, D. Pallaschke, and J. Rosenmüller for several helpful conversations. We are also very grateful to two anonymous referees for insightful remarks.
[†] The work of the first author was supported by the WARSHOW endowment at Cornell University. Fax: (972)-2-6513681.
[‡] Fax: ++49-521-106-2997. E-mail: PSudhoelter@Wiwi.Uni-Bielefeld.De.

0899-8256/97 $25.00

economy is a strategic game combined with a finite sequence of feasibility correspondences, one for each player. The main solution concept for abstract economies is the social equilibrium (see Debreu (1952)), which is a generalization of the Nash equilibrium. We provide an axiomatization of the Nash correspondence of abstract economies.

We now briefly describe our result. Consider the class of all abstract economies with the following properties: (i) The strategy sets are convex and compact. (ii) The payoff functions are continuous and quasi-concave. (iii) The feasibility correspondences are continuous and have nonempty convex values. Then, the Nash correspondence is the unique solution on the foregoing class of abstract economies that satisfies nonemptiness, rationality in one-person games, and consistency.

The Nash correspondence has been axiomatized recently for various classes of games (see Peleg and Tijs (1996), Peleg *et al*. (1996), and Norde *et al*. (1996)). In particular, Norde *et al*. (1996) contains a complete characterization for the class of mixed extensions of finite games and for the class of games with continuous concave payoff functions. This paper is devoted to the characterization of the Nash correspondence of generalized games with *quasi*-concave payoff functions. Although we use some results of Peleg *et al*. (1996) and Norde *et al*. (1996), our result does not directly follow from the foregoing papers. The main difficulty is that the class of quasi-concave functions, unlike the class of concave functions, is not closed under addition (i.e., the sum of two quasi-concave functions may not be quasi-concave).

2. THE MODEL

Let Ω be an infinite set (the set of "potential players"). An *abstract economy* is a list

$$E = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N}, (F_i)_{i \in N} \rangle,$$

where $N = N(E) \subset \Omega$ is a finite nonempty set (the set of *players*); A_i is the (nonempty) set of *strategies* of player $i \in N$; u_i: $A(E) \to \mathbb{R}$ is the *payoff function* for $i \in N$ (here $A(E) = \times_{i \in N(E)} A_i$ and $\mathbb{R}$ is the real line); and F_i, $i \in N$, is a correspondence from $A_{-i} = \times_{j \neq i} A_j$ to A_i. F_i is the *feasibility correspondence* of $i \in N$. Thus, if $x \in A(E)$ then $F_i(x_{-i}) \subset A_i$ is the set of *feasible strategies* of i (when the rest of the players choose $x_{-i} = (x_j)_{j \in N \setminus \{i\}}$). $\hat{x} \in A(E)$ is a *Nash equilibrium* (NE), or a *social equilibrium* (SE), if (i) $\hat{x}_i \in F_i(\hat{x}_{-i})$ for all $i \in N$ and (ii) $u_i(\hat{x}) \geq u_i(x_i, \hat{x}_{-i})$ for all $x_i \in F_i(\hat{x}_{-i})$ and $i \in N$.

Abstract economies were first studied in Debreu (1952) in connection with the Arrow–Debreu existence theorem of Walras equilibrium. They also later continued to play an important role in general equilibrium theory (see, e.g., Shafer and Sonnenschein (1975) and Border (1985, Chapters 19 and 20)).

We shall consider the class ξ_q (the subscript q indicates that we deal with quasi-concave utility functions) of abstract economies that have the following properties:

$$E = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N}, (F_i)_{i \in N} \rangle \in \xi_q$$

if

$$A_i \text{ is a nonempty, compact, and convex subset of some (finite-dimensional) Euclidean space } E_i \text{ for every } i \in N. \tag{2.1}$$

$$u_i \text{ is continuous on the graph of } F_i, \text{ gr } F_i, \text{ for every } i \in N. \ (\text{We recall that gr } F_i = \{x \in A(E) \mid x_i \in F_i(x_{-i})\}.) \tag{2.2}$$

$$\text{For every } i \in N \text{ and } x_{-i} \in A_{-i}, u_i(\cdot, x_{-i}) \text{ is a quasi-concave function on } F_i(x_{-i}). \tag{2.3}$$

$$\text{For every } i \in N, F_i \text{ is continuous (i.e., both upper hemi-continuous and lower hemi-continuous) on } A_{-i}. \tag{2.4}$$

$$\text{For every } i \in N \text{ and } x_{-i} \in A_{-i}, F_i(x_{-i}) \text{ is nonempty, closed, and convex.} \tag{2.5}$$

Without loss of generality we may assume

$$u_i(x) < u_i(y) \text{ if } x \in A(E) \backslash \text{gr } F_i \text{ and } y \in \text{gr } F_i \text{ for all } i \in N. \tag{2.6}$$

Indeed, the values of u_i on $A(E) \backslash \text{gr } F_i$ are completely arbitrary.

A *solution* on ξ_q is a function φ that assigns to each $E \in \xi_q$ a subset $\varphi(E)$ of $A(E)$. For example, the function SE, which assigns to each $E \in \xi_q$ its set of social equilibria SE(E), is a solution. A solution φ on ξ_q satisfies *nonemptiness* (NEM) if $\varphi(E) \neq \emptyset$ for every $E \in \xi_q$. (We remark that, by Theorem 4.3.1 in Ichiishi (1983), SE$(\cdot)$ satisfies NEM.) φ satisfies *one-person rationality* (OPR) if for every one-person abstract economy $E = \langle \{i\}, A_i, u_i, F_i \rangle$ in ξ_q,

$$\varphi(E) \subset \{x_i \in F_i \mid u_i(x_i) \geq u_i(y_i) \text{ for all } y_i \in F_i\}.$$

(F_i is constant for one-person abstract economies.) Obviously SE$(\cdot)$ satisfies OPR.

Let $E \in \xi_q$, $x \in A(E)$, and $S \subset N(E)$, $S \neq \emptyset$. The *reduced* abstract economy of E *with respect to* S *and* x, $E^{S,x}$ is given by

$$E^{S,x} = \left\langle S, (A_i)_{i \in S}, \left(u_i^{S,x}\right)_{i \in S}, \left(F_i^{S,x}\right)_{i \in S} \right\rangle,$$

where $u_i^{S,x}(a_S) = u_i(a_S, x_{N \setminus S})$ and $F_i^{S,x}(a_{S \setminus \{i\}}) = F_i(a_{S \setminus \{i\}}, x_{N \setminus S})$ for every $a_S \in A_S = \times_{j \in S} A_j$ and $i \in S$. (Here the notation $x_T = (x_j)_{j \in T}$ is used whenever $x \in A(E)$ and $T \subset N$.) Under the foregoing assumptions $E^{S,x} \in \xi_q$, that is, ξ_q is closed in the sense of Peleg and Tijs (1996). A solution φ on ξ_q is *consistent* (CONS) if for every $E \in \xi_q$, $S \subset N(E)$, $S \neq \emptyset$ and $x \in \varphi(E)$, $x_S \in \varphi(E^{S,x})$. The interpretation of the consistency property for abstract economies is actually the same as the interpretation for strategic games (see Peleg and Tijs (1996)). Indeed, φ is consistent if for every $E \in \xi_q$ the following condition is satisfied: If x is an "equilibrium," that is, $x \in \varphi(E)$, $S \subset N(E)$, $S \neq \emptyset$, and all the members of $N \setminus S$ announce their strategies x_i, $i \in N \setminus S$, and leave the economy E, then the members of S do not have to revise their strategies x_j, $j \in S$. As the reader may easily verify, SE$(\cdot)$ satisfies CONS on ξ_q.

3. A CHARACTERIZATION OF THE NASH CORRESPONDENCE ON ξ_q

Our main result is the following theorem.

THEOREM 3.1. *If a solution* φ *on* ξ_q *satisfies* NEM, OPR, *and* CONS, *then* $\varphi(E) = \text{SE}(E)$ *for every* $E \in \xi_q$.

The following simple result is needed for the proof of Theorem 3.1.

LEMMA 3.2. *Let* $E \in \xi_q$ *and* $x \in A(E)$. *If* $x_i \in \text{SE}(E^{\{i\},x})$ *for every* $i \in N(E)$, *then* $x \in \text{SE}(E)$.

Proof. Let $i \in N(E)$. Because $x_i \in \text{SE}(E^{\{i\},x})$ it follows that

$$x_i \in F_i^{\{i\},x} \quad \text{and} \quad u_i^{\{i\},x}(x_i) \geq u_i^{\{i\},x}(y_i) \quad \text{for all } y_i \in F_i^{\{i\},x}. \tag{3.1}$$

By the definition of reduced games, (3.1) is equivalent to

$$x_i \in F_i(x_{-i}) \quad \text{and} \quad u_i(x) \geq u_i(y_i, x_{-i})$$

for every $y_i \in F_i(x_{-i})$. Thus, $x \in \text{SE}(E)$. Q.E.D.

Now we shall prove that SE$(\cdot)$ is the maximum solution on ξ_q which satisfies OPR and CONS.

LEMMA 3.3. *If a solution* φ *on* ξ_q *satisfies OPR and CONS, then* $\varphi(E) \subset \text{SE}(E)$ *for every* $E \in \xi_q$.

Proof. Let $E \in \xi_q$ and $x \in \varphi(E)$. If $|N(E)| = 1$ then $\varphi(E) \subset \mathrm{SE}(E)$ by OPR. (If S is a finite set, then $|S|$ is the cardinality of S.) Assume now $|N(E)| \geq 2$. By CONS, $x_i \in \varphi(E^{\{i\}, x})$ for all $i \in N(E)$. Hence, by the first part of the proof, $x_i \in \mathrm{SE}(E^{\{i\}, x})$ for all $i \in N(E)$. Therefore, by Lemma 3.2, $x \in \mathrm{SE}(E)$. Q.E.D

In order to prove the converse inclusion we shall show that $\mathrm{SE}(\cdot)$ has the *ancestors property* (AP) on ξ_q (see Peleg *et al*. (1996) and Proposition 3 of Norde *et al*. (1996)).

LEMMA 3.4 (AP). *If $E \in \xi_q$ and $\hat{x} \in \mathrm{SE}(E)$, then there exists $H \in \xi_q$ such that the following conditions hold*:

$$N(H) \supset N(E); \tag{3.2}$$

$$H \textit{ has exactly one social equilibrium } y; \tag{3.3}$$

$$y_{N(E)} = \hat{x}; \tag{3.4}$$

$$H^{N(E), y} = E. \tag{3.5}$$

We postpone the proof of Lemma 3.4 and shall now prove Theorem 3.1.

Proof of Theorem 3.1. Let $E \in \xi_q$ and $\hat{x} \in \mathrm{SE}(E)$. By Lemma 3.4 there exists $H \in \xi_q$ such that (3.2)–(3.5) are satisfied. By NEM of φ and Lemma 3.3, $\varphi(H) = \{y\}$. By CONS of φ, (3.4), and (3.5) we obtain

$$\hat{x} = y_{N(E)} \in \varphi(H^{N(E), y}) = \varphi(E).$$

Thus, $\varphi(E) \supset \mathrm{SE}(E)$. Because of Lemma 3.3, $\varphi(E) = \mathrm{SE}(E)$. Q.E.D.

Proof of Lemma 3.4. Let

$$E = \langle N(E), (A_i)_{i \in N(E)}, (u_i)_{i \in N(E)}, (F_i)_{i \in N(E)} \rangle$$

be a member of ξ_q and $\hat{x} \in \mathrm{SE}(E)$. We define

$$H = \langle N(H), (A_i^*)_{i \in N(H)}, (u_i^*)_{i \in N(H)}, (F_i^*)_{i \in N(H)} \rangle$$

in the following way:

$$N(H) = N \cup N^+ \qquad \text{where } N \cap N^+ = \emptyset, |N| = |N^+|, \text{ and } N = N(E). \tag{3.6}$$

The existence of $N(H) \subset \Omega$ is guaranteed by the fact that Ω is infinite.

We denote by t a bijection of N onto N^+ and we choose

$$A_i^* = A_{t(i)}^* = A_i \qquad \text{for all } i \in N. \tag{3.7}$$

In order to define u_i^* and F_i^*, $i \in N(H)$, we need the following concepts and notations. A continuous function $\pi_i: E_i \to A_i$, $i \in N(E)$ (see (2.1)), is a *retraction* if $\pi_i(x_i) = x_i$ for all $x_i \in A_i$. Because A_i is a convex and compact subset of E_i there exits a retraction π_i of E_i on A_i. Thus we may define

$$F_i^*(x_{N\setminus\{i\}}, x_{N^+}) = F_i\Big(\big(\pi_j\big(x_j + \hat{x}_j - x_{t(j)}\big)\big)_{j \in N\setminus\{i\}}\Big) \tag{3.8}$$

for every $i \in N$ and $x \in A(H)$ and

$$F_{t(i)}^*(x_{-t(i)}) = A_i \tag{3.9}$$

for every $i \in N$ and every $x_{-t(i)} \in A_{-t(i)}^*$, where

$$A_{-t(i)}^* = \bigtimes_{j \in N} A_j \times \bigtimes_{j \neq i} A_{t(j)}^*.$$

As the reader may verify, F_i^* satisfies (2.4) and (2.5) for all $i \in N(H)$.

In order to define u_i^*, $i \in N(E)$, we first introduce auxiliary functions

$$v_i^*(x) = u_i\Big(x_i, \big(\pi_j\big(x_j + \hat{x}_j - x_{t(j)}\big)\big)_{j \in N\setminus\{i\}}\Big) \tag{3.10}$$

for all $x \in A(H)$ and $i \in N$. $v_i^*(\cdot, x_{-i})$ is quasi-concave on

$$F_i^*(x_{-i}) = F_i\Big(\big(\pi_j\big(x_j + \hat{x}_j - x_{t(j)}\big)\big)_{j \in N\setminus\{i\}}\Big)$$

for every $x_{-i} \in A_{-i}^*$, and v_i^* continuous on $\operatorname{gr} F_i^*$. Using (3.10) we further define

$$\beta_i: \bigtimes_{j \neq i} A_j \times \bigtimes_{h \in N^+} A_h^* \to \mathbb{R}$$

by

$$\beta_i(x_{N\setminus\{i\}}, x_{N^+}) = \beta_i(x_{-i}) = \max\big\{v_i^*(x_i, x_{-i}) \,|\, x_i \in F_i^*(x_{-i})\big\}$$

for $i \in N$. By the Maximum Theorem (see Section 2.3 of Ichiishi (1983)), $\beta_i(\cdot)$ is continuous. Furthermore

$$\beta_i\Big((x_j)_{j \in N\setminus\{i\}}, (\hat{x}_h)_{h \in N^+}\Big) = \max\Big\{u_i(x_i, x_{N\setminus\{i\}}) \,|\, x_i \in F_i(x_{N\setminus\{i\}})\Big\}. \tag{3.11}$$

(Here $\hat{x} \in A(H)$ is defined by $\hat{x}_{t(i)} = \hat{x}_i$ for $i \in N$.) Now we introduce our second family of auxiliary functions. For every $i \in N$ let $w_i^*: A(H) \to \mathbb{R}$ be given by

$$w_i^*(x) = w_i^*(x_i, x_{-i}) = \beta_i(x_{-i}) - \|x_{t(i)} - \hat{x}_i\| \, \|x_i - \hat{x}_i\|. \tag{3.12}$$

(Here $\|\cdot\|$ denotes the Euclidean norm on E_i.)

It is obvious that w_i^* satisfies (2.2) and (2.3). Hence

$$u_i^*(x) = \min\{v_i^*(x), w_i^*(x)\}$$

is an admissible payoff function for $i \in N(E)$. Finally, we define

$$u_{t(i)}^*(x) = -\|x_i - x_{t(i)}\| \qquad \text{for all } i \in N(E). \tag{3.13}$$

This completes the definition of H. Clearly, $H \in \xi_q$. Now we claim

$$\text{If } y \in \mathrm{SE}(H) \qquad \text{then } y_i = y_{t(i)} = \hat{x}_i, \qquad \text{for all } i \in N(E). \tag{3.14}$$

Indeed, if $y \in \mathrm{SE}(H)$ then $y_i = y_{t(i)}$ for all $i \in N(E)$ by (3.13). Hence $v_i^*(y) = u_i(y_i, \hat{x}_{N\setminus\{i\}})$ and $w_i^*(y) = u_i(\hat{x}_N) - \|y_i - \hat{x}_i\|^2$. Thus, for $i \in N(E)$, $y_i = \hat{x}_i$, because y_i is a best response to y_{-i}. Therefore, our claim is proved. So far we have proved (3.2)–(3.4). In order to prove (3.5) we observe that for all $i \in N(E)$:

$$\begin{aligned}
\min&\Big\{v_i^*\Big(\big(x_j\big)_{j\in N}, \big(\hat{x}_h\big)_{h\in N^+}\Big), w_i^*\Big(\big(x_j\big)_{j\in N}, \big(\hat{x}_h\big)_{h\in N^+}\Big)\Big\} \\
&= \min\Big\{u_i(x_N), \beta_i\Big(\big(x_j\big)_{j\in N\setminus\{i\}}, \big(\hat{x}_h\big)_{h\in N^+}\Big)\Big\} \\
&= \min\Big\{u_i(x_N), \max\big\{u_i(z_i, x_{N\setminus\{i\}}) \mid z_i \in F_i(x_{N\setminus\{i\}})\big\}\Big\} \\
&= u_i(x_N)(\text{see } (3.11) \text{ and } (2.6)).
\end{aligned}$$

Also,

$$F_i^*\Big(\big(x_j\big)_{j\in N\setminus\{i\}}, \big(\big(\hat{x}_h\big)_{h\in N^+}\Big) = F_i\Big(\big(x_j\big)_{j\in N\setminus\{i\}}\Big)$$

for each $i \in N(E)$. Q.E.D.

4. CONCLUDING REMARKS

We considered the set of abstract economies with convex and compact strategy sets, continuous and quasi-concave payoff functions, and continuous and convex-valued feasibility correspondences. The Nash correspondence is completely characterized on the foregoing class of economies by the following three axioms: nonemptiness, rationality for one-person games, and consistency.

Let $E = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}, (F_i)_{i\in N}\rangle$ be an abstract economy. E is a *game* if for each $i \in N$ and $x_{-i} \in A_{-i}$, $F_i(x_{-i}) = A_i$. Thus, E is a game if there are no feasibility constraints. If E is a game, then we shall also write

$E = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$. Denote by Γ_q the set of all games in ξ_q. By modifying the proof of Theorem 3.1 we can show the following result.

THEOREM 4.1. *If a solution* φ *on* Γ_q *satisfies* NEM, OPR, *and* CONS, *then* $\varphi(G) = \text{NE}(G)$ *for every game* $G \in \Gamma_q$ (*for* $G \in \Gamma_q$ *we denote* $\text{NE}(G) = \text{SE}(G)$).

The axiomatization of the Nash correspondence on Γ_q is not covered by the results of Norde *et al.* (1996). Also, our results may be generalized to infinite-dimensional strategy spaces. Indeed, one may replace (2.1) by a weaker assumption. First, we recall that a normed linear space $\underline{\overline{X}}$ (with norm $\|\cdot\|$), is *strictly convex* if for every pair of linearly independent vectors $x, y \in \underline{\overline{X}}$ it holds that $\|x + y\| < \|x\| + \|y\|$. The following result is useful.

LEMMA 4.2. *If* A *is a nonempty, convex, and compact subset of a convex compact subset* B *of a normed space* $\underline{\overline{X}}$, *then there is a retraction of* B *on* A.

Proof. It is well known that a compact set is separable, and hence the *affine hull* of B (the set

$$\text{aff } B = \left\{ \sum_{i=1}^{r} \gamma_i b_i \,\middle|\, r \in \mathbb{N},\ \sum_{i=1}^{r} \gamma_i = 1,\ \gamma_i \in \mathbb{R},\ b_i \in B \text{ for all } i \text{ with } 1 \le i \le r \right\}\Bigg)$$

is separable. Indeed, if $\{b_i \mid i \in \mathbb{N}\}$ is a countable dense subset of B, then

$$\left\{ \sum_{i=1}^{r} \gamma_i b_i \,\middle|\, r \in \mathbb{N},\ \sum_{i=1}^{r} \gamma_i = 1,\ \gamma_i \in \mathbb{Q} \right\}$$

is a countable dense subset of aff B.

If there is a retraction of $B - \{b_0\}$ on $A - \{b_0\}$ for some $b_0 \in B$, then there is a retraction of B on A, because translations are continuous. Therefore we assume w.l.o.g. that 0 is a member of B, hence aff B is a linear subspace of $\underline{\overline{X}}$. Moreover, it can be assumed that aff B is a Banach space. Otherwise take a "smallest" completion of aff B. This new space inherits separability and convex or compact subsets of the original space retain these properties (see p. 130 of Köthe (1966)). Thus aff B has a strictly convex isomorphic norm (see p. 160 of Day (1973)). The proof of existence of a retraction of a strictly convex normed space on a nonvoid convex compact subset is the same as the proof when the normed space is Euclidean. Q.E.D.

Now, using the generalizations of existence of social equilibria and the Maximum Theorem for infinite-dimensional spaces (see Section 2.3 and Theorem 4.7.2 of Ichiishi (1983)), we obtain

COROLLARY 4.3. *Theorem* 3.1 *remains true if* (2.1) *is replaced by*:

$$\text{For every } i \in N,\ A_i \text{ is a nonempty, compact, and convex subset of a normed linear space.} \tag{4.1}$$

Indeed, in view of (3.10) it is sufficient to verify the existence of a retraction of $A_i + A_i - A_i$ to A_i for $i \in N(E)$. This existence is guaranteed by Lemma 4.2 applied to $B = A_i + A_i - A_i$ and $A = A_i$.

Finally, it should be remarked that obvious examples show the logical independence of NEM, OPR, and CONS in the results (Theorems 3.1 and 4.1 and Corollary 4.3).

REFERENCES

Arrow, K. J., and Debreu, G. (1954). "Existence of an Equilibrium for a Competitive Economy," *Econometrica* **22**, 265–290.

Border, K. C. (1985). *Fixed Point Theorems with Applications to Economics and Game Theory*. Cambridge, UK: Cambridge Univ. Press.

Day, M. M. (1973). *Normed Linear Spaces*. Berlin: Springer-Verlag.

Debreu, G. (1952). "A Social Equilibrium Existence Theorem," *Proc. Nat. Acad. Sci.* **38**, 886–893.

Ichiishi, T. (1983). *Game Theory for Economic Analysis*, New York: Academic Press.

Köthe, G. (1966). *Topologische lineare Räume*. Berlin: Springer-Verlag.

Norde, H., Potters, J., Reijnierse, H., and Vermeulen, D. (1996). "Equilibrium Selection and Consistency," *Games Econ. Behav.* **12**, 219–225.

Peleg, B., Potters, J., and Tijs, S. (1996). "Minimality of Consistent Solutions for Strategic Games, in Particular for Potential Games," *Econ. Theory* **7**, 81–93.

Peleg, B., and Tijs, S. (1996). "The Consistency Principle for Games in Strategic Form," *Int. J. Game Theory* **25**, 13–34.

Shafer, W. J., and Sonnenschein, H. (1975). "Equilibrium in Abstract Economies without Ordered Preferences," *J. Math. Econ.* **2**, 345–348.