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The modiclus and core stability

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Abstract. The modiclus, a relative of the prenucleolus, assigns a singleton to any cooperative TU game. We show that the modiclus selects a member of the core for any exact orthogonal game and for any assignment game that has a stable core. Moreover, by means of an example we show that there is an exact TU game with a stable core that does not contain the modiclus.

Key words: Core, stable set, modiclus.

1. Introduction

The prenucleolus and the core are widely accepted solutions for cooperative transferable utility games. The prenucleolus selects a unique member of the core, whenever the core is nonempty. A further interesting solution, the modiclus, is a relative of the prenucleolus. The prenucleolus of a game is obtained by lexicographically minimizing the non-increasingly ordered vector of excesses of the coalitions within the set of Pareto optimal payoff vectors. Analogously, the modiclus is obtained by lexicographically minimizing the non-increasingly ordered vector of differences of excesses. When comparing the definitions of the prenucleolus and the nucleolus, the excesses, i.e., the "dissatisfactions", of the coalitions are replaced by the bi-excesses (differences of excesses) of the pairs of coalitions. The bi-excess between two coalitions S and T may be regarded as the mutual envy between coalitions S and T. For the precise definition see Section 2.

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The modiclus has many properties in common with the prenucleolus. For its nice behavior on the class of weighted majority games see Sudhölter (1996). Unlike the prenucleolus, for any general TU game, the modiclus may not select a core element, even if the core is nonempty. E.g., the modiclus does not select a core member of an asymmetric glove game. Instead it assigns the same amount to both the left-hand glove owners and the right-hand glove owners (see Sudhölter (2001)). Due to this kind of "equal treatment property" of groups of players the modiclus has an advantage over any selection of the core like the prenucleolus.

However, the core may certainly be regarded as a "reference solution" on convex games: If a solution to a convex game does not at least intersect the core of that game, then it may be difficult to justify it. The modiclus indeed selects an element of the core of any convex game (see Sudhölter (1997)). Hence it is natural to investigate a possible generalization of the aforementioned result. We emphasize two properties of a convex game: The core is its unique (von Neumann-Morgenstern) stable set. Moreover, it is exact. Hence we may raise the following question. To what extent does the foregoing result extend to exact games with a stable core?

We show that the modiclus is an element of the core for two significant classes of games that have a stable core or that are exact. However, it turns out that a further generalization of these results is problematic. Indeed, we show by means of an example that there exists an exact game whose core is stable, but it does not contain the modiclus.

A glove game is both an assignment game and an orthogonal game. Similarly to an assignment game, an orthogonal game allows for a canonical partition of the players into groups (see Section 4). Seen as a market game, agents of different groups initially own different types of commodities. An orthogonal game is exact if there are equal aggregate amounts of the types of commodities. For a generalization to a continuum of players and for further interpretations see Einy, Holzman, Monderer and Shitovitz (1996)

We shall show that the modiclus is an element of the core for any orthogonal game that is exact and for any assignment game whose core is stable.

We now briefly review the contents of the paper. Section 2 recalls definitions of some relevant solutions and of stability. In Section 3 we show that the modified least core (containing the modiclus) is a subset of the core of any assignment game whose core is stable. Section 4 is devoted to the discussion of orthogonal games. We show that any orthogonal game with a stable core is exact. Moreover, we deduce that the modified least core is contained in the core of any exact orthogonal game. Finally, Section 5 presents the aforementioned counterexample.

2. Notation and definitions

A (cooperative TU) *game* is a pair (N, v) such that $\emptyset \neq N$ is finite and $v : 2^N \to \mathbb{R}$, $v(\emptyset) = 0$. For any game (N, v) let $X(N, v) = \{x \in \mathbb{R}^N \,|x(N) = v(N)\}$ denote the set of *Pareto optimal allocations* (*preimputations*). We use $x(S) = \sum_{i \in S} x_i$ ($x(\emptyset) = 0$) for every $S \in 2^N$ and every $x \in \mathbb{R}^N$. Additionally, x_S denotes the restriction of x to S, i.e. $x_S = (x_i)_{i \in S}$. For $x \in \mathbb{R}^N$ and $S \subseteq N$ let $e(S, x, v) = v(S) - x(S)$ denote the *excess* of S at x with respect to (N, v). For

$X \subseteq \mathbb{R}^N$ let $\mathcal{N}((N,v);X)$ denote the *nucleolus* of (N,v) with respect to X, i.e. the set of members of X that lexicographically minimize the non-increasingly ordered vector of excesses of the coalitions (see Schmeidler (1969)). It is well known that the nucleolus with respect to $X(N,v)$ is a singleton, called the *prenucleolus* of (N,v) and denoted by $v(N,v)$. In order to define the modiclus of (N,v), let, for every pair $(S,T) \in 2^N \times 2^N$, the *bi-excesses* of (S,T) at x, $e^b(S,T,x,v)$, be given by

$$e^b(S,T,x,v) = e(S,x,v) - e(T,x,v).$$

The *modiclus* of (N,v) is the set of members of $X(N,v)$ that lexicographically minimize the non-increasingly ordered vector of bi-excesses of the pairs of coalitions. The modiclus of (N,v) is a singleton denoted by $\psi(N,v)$ (see Sudhölter (1996)).

The *dual game* (N,v^*) of (N,v) is defined by $v^*(S) = v(N) - v(N \setminus S)$ for all $S \subseteq N$. The *preventive power* of a coalition $S \subseteq N$ may be measured by $v^*(S) = v(N) - v(N \setminus S)$. For any $x \in X(N,v)$ and any pair (S,T) of coalitions,

$$e(N \setminus T,x,v^*) = v(N) - v(T) - x(N) + x(T) = -e(T,x,v).$$

Hence $e^b(S,T,x,v) = e(S,x,v) + e(N \setminus T,x,v^*)$. Thus, the modiclus lexicographically minimizes the vector of sums of excesses with respect to the game and its dual. We conclude that $\psi(N,v) = \psi(N,v^*)$. The following notation is useful. For $x \in \mathbb{R}^N$ let

$$\mu(x,v) = \max\{e(S,x,v) \mid S \subseteq N\}. \tag{2.1}$$

The *modified least core* of (N,v), $\mathcal{MLC}(N,v)$, is defined by

$$\mathcal{MLC}(N,v) = \{x \in X(N,v) \mid \mu(x,v) + \mu(x,v^*) \leq \mu(y,v) + \mu(y,v^*) \text{ for all } y \in X(N,v)\}. \tag{2.2}$$

Hence, $\psi(N,v) \in \mathcal{MLC}(N,v)$ by definition.

The modiclus has several interesting and desirable properties. E.g., it assigns a representation to a weighted majority game (see Sudhölter (1996)). In order to mention further properties, we first recall that the *core* of (N,v), $\mathcal{C}(N,v)$, is defined by

$$\mathcal{C}(N,v) = \{x \in X(N,v) \mid x(S) \geq v(S) \text{ forall } S \subseteq N\}.$$

So, if (N,v) is a weighted majority game whose core is nonempty, that is, if (N,v) has a veto player, then every core element distributes $v(N) = 1$ solely to the veto players. In particular, if $i \in N$ is not a veto player, then any core element assigns 0 to i. However, if a representation of (N,v) assigns 0 to i, then i is a null player, that is, $v(S \cup \{i\}) = v(S)$ for all $S \subseteq N$. Thus, the modiclus may not select a core element of a weighted majority game that has a veto player. However, the modiclus lies in the core of any convex game (see Sudhölter (1997)).

We recall that a game is *balanced* if its core is nonempty (see Bondareva (1963) and Shapley (1967)). A game (N,v) is *totally balanced* if all subgames (S,v), $\emptyset \neq S \subseteq N$, are balanced. A game (N,v) is *exact* if for any $S \subseteq N$ there exists $x \in \mathcal{C}(N,v)$ with $v(S) = x(S)$. Exact games were introduced by Shapley (1971) (see also Schmeidler (1972)).

Finally, we recall the definition of core stability. Let $x, y \in X(N, v)$ and $\emptyset \neq S \subseteq N$. Then *x dominates y via S* if $x(S) \leq v(S)$ and $x_i > y_i$ for all $i \in S$. Moreover, *x dominates y* if x dominates y via some coalition $\emptyset \neq S \subseteq N$. The core of (N, v) is *stable* if for any $y \in X(N, v)$ such that $y_i \geq v(\{i\})$ (y is an *imputation*) for all $i \in N$, there exists $x \in \mathscr{C}(N, v)$ that dominates y. Note that core stability is invariant under adding null players.

3. The Modiclus for assignment games with a stable core

Shapley and Shubik (1972) introduced assignment games. For finite sets S and T an *assignment* of (S, T) is a bijection $b : S' \to T'$ such that $S' \subseteq S$, $T' \subseteq T$, and $|S'| = |T'| = \min\{|S|, |T|\}$. We shall identify b with $\{(i, b(i)) \mid i \in S'\}$. Let $\mathscr{B}(S, T)$ denote the set of assignments. A game (N, v) is an *assignment game* if there exist a partition $\{P, Q\}$ of N and a non-negative real matrix $A = (a_{ij})_{i \in P, j \in Q}$ such that

$$v(S) = \max_{b \in \mathscr{B}(S \cap P, S \cap Q)} \sum_{(i,j) \in b} a_{ij}.$$

Let (N, v) be an assignment game defined by the matrix A on $P \times Q$. As ψ satisfies the strong null player property and as core stability is invariant under adding null players, we assume in the sequel that $|P| = |Q| = p$. Also, we assume that $P = \{1, \ldots, p\}$ and $Q = \{1', \ldots, p'\}$. Finally, we assume that $v(N) = \sum_{i \in P} a_{ii'}$.

We say that A has a *dominant diagonal* if

$$a_{ii'} = \max_{j' \in Q} a_{ij'} = \max_{j \in P} a_{ji'} \quad \text{for all } i \in P.$$

Theorem 3.1. *If (N, v) is an assignment game with a stable core, then $\mathscr{MLC}(N, v) \subseteq \mathscr{C}(N, v)$ and hence the modiclus is in the core of the game.*

Proof: Let (N, v) be an assignment game defined by the $P \times Q$ matrix A and let $x \in X(N, v)$. Then $\mu(x, v) \geq 0$. Moreover,

$$\begin{aligned} e(P, x, v^*) + e(Q, x, v^*) &= v(N) - v(Q) - x(P) + v(N) - v(P) - x(Q) \\ &= 2v(N) - x(N) = v(N), \end{aligned}$$

hence $\mu(x, v^*) \geq v(N)/2$. As the core is stable for the assignment game (N, v), the matrix A has a dominant diagonal by Theorem 1 of Solymosi and Raghavan (2001). The proof is complete as soon as we have shown the following claim:

$$\mathscr{MLC}(N, v) = \{x \in \mathscr{C}(N, v) \mid \mu(x, v^*) = v(N)/2\}.$$

In order to prove our claim it suffices to find a preimputation $\widetilde{x} \in \mathscr{C}(N, v)$ that satisfies $\mu(\widetilde{x}, v^*) \leq v(N)/2$. Let $\widetilde{x} \in \mathbb{R}^N$ be defined by $\widetilde{x}_i = a_{ii'}/2 = \widetilde{x}_{i'}$ for all $i \in N$. Then $\widetilde{x} \in X(N, v)$. Let $S \subseteq N$. Since A has a dominant diagonal,

$$v(S) \leq \min\left\{\sum_{i \in S \cap P} a_{ii'}, \sum_{i' \in S \cap Q} a_{ii'}\right\}.$$

Hence $v(S) \leq \widetilde{x}(S)$ and $\widetilde{x} \in \mathscr{C}(N, v)$. Let

$$T = \{i \in S \cap P \mid i' \in S\} \cup \{i' \in S \cap Q \mid i \in S\}.$$

Then

$$v(S) \geq v(S \setminus T) + v(T) \geq v(S \setminus T) + \sum_{i \in T \cap P} a_{ii'}$$

and hence,

$$e(S, \widetilde{x}, v) \geq e(S \setminus T, \widetilde{x}, v) \geq -\widetilde{x}(S \setminus T) \geq -\sum_{i \in P} \frac{a_{ii'}}{2} = -\frac{v(N)}{2}.$$

We conclude that $0 \leq -e(S, \widetilde{x}, v) = e(N \setminus S, \widetilde{x}, v^*) \leq v(N)/2$. ■

Note that core stability was not used to prove that $\mu(\widetilde{x}, v^*) = v(N)/2$. In fact Sudhölter (2001) shows that $\mu(x, v^*) = v(N)/2$ for each member x of the modified least core of an arbitrary assignment game. While the groups P and Q of any assignment game are treated equally by each element of its modified least core, the nucleolus may assign different amounts to P and Q even if the game has a stable core, as the following example demonstrates.

Example 3.2. Let

$$A = \begin{pmatrix} 24 & 12 \\ 0 & 12 \end{pmatrix}$$

and let (N, v) be the assignment game defined by A. As A has a dominant diagonal, (N, v) has a stable core. Let $1, 2$ be the rows and let $3, 4$ be the columns of A. Then $x \in \mathbb{R}^4$ is an element of $\mathscr{C} = \mathscr{C}(N, v)$ if and only if $x \geq 0, x(N) = 36, x_1 + x_3 = 24, x_2 + x_4 = 12, x_1 + x_4 \geq 12$. Thus, $\mathscr{C}$ is the convex hull of the vectors

$$x^1 = (24, 12, 0, 0), x^2 = (24, 0, 0, 12), x^3 = (12, 12, 12, 0), x^4 = (0, 0, 24, 12).$$

Theorem 3.1 implies that $\mathscr{MLC}(N, v)$ is the convex hull of

$$z^1 = (12, 6, 12, 6) \text{ and } z^2 = (18, 0, 6, 12).$$

The nucleolus $\nu = \nu(N, v)$ and the modiclus $\psi = \psi(N, v)$ may be computed as

$$\nu = (15, 6, 9, 6) \text{ and } \psi = (14, 4, 10, 8).$$

Hence, $\nu(P) > \nu(Q)$. So, the row players prefer ν over ψ and the column players prefer ψ over ν. Both proposals belong to the relative interior of the core (see Figure 1). If the column players form a cartel and act as one player, that is, if the game $(\{1, 2, Q\}, v_1)$ defined by $v_1(S) = v(S)$ if $Q \notin S$ and $v_1(S) = v((S \setminus \{Q\}) \cup Q)$ if $Q \in S$, is considered, then both the nucleolus and the modiclus assign 18 to the syndicate Q. So, we may conclude that syndication is advantageous if the nucleolus is applied. If instead the modiclus is applied, then the column players do not have an incentive to form a cartel.

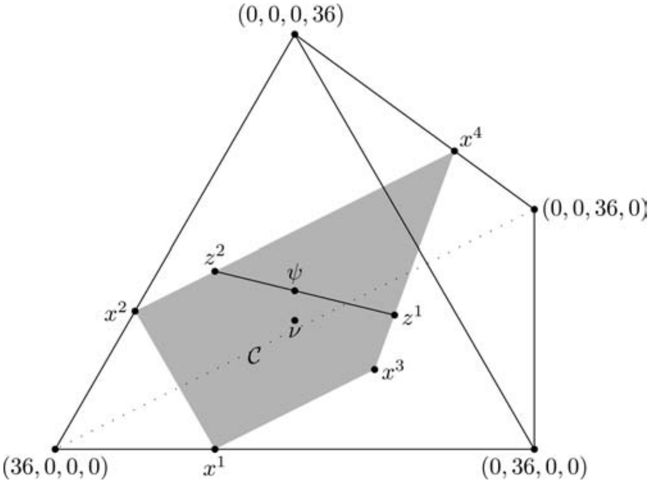

Fig. 1. $\mathscr{MLC}, \mathscr{C}, v$, and ψ in an Assignment Game

4. The Modiclus for exact orthogonal games

Let (N, v) be a game. Kalai and Zemel (1982) showed that (N, v) is totally balanced if and only if it is a minimum of finitely many additive games, that is, there exist a finite sequence $(\lambda^\rho)_{\rho=1,\ldots,r}$ such that $\lambda^\rho \in \mathbb{R}^N$, $\rho = 1, \ldots, r$, and

$$v(S) = \min_{\rho=1,\ldots,r} \lambda^\rho(S) \quad \text{for all } S \subseteq N. \tag{4.1}$$

(Recall that $\lambda^\rho(S) = \sum_{i \in S} \lambda_i^\rho$.) Since all of our solutions are covariant under strategic equivalence, we may assume that $\min_{\rho=1,\ldots,r} \lambda_i^\rho = 0$ for all $i \in N$, that is, (N, v) is 0-normalized. A 0-normalized totally balanced game (N, v), defined by (4.1), is *orthogonal* if the *carriers* of the finite measures λ^ρ, $\rho = 1, \ldots, r$, are mutually disjoint, that is, if $\lambda_i^\rho > 0$ implies $\lambda_i^\sigma = 0$ for all $i \in N$ and all $\sigma, \rho \in \{1, \ldots, r\}$, $\sigma \neq \rho$ (see Figure 2).

So, for any $\rho = 1, \ldots, r$, we may select $N^\rho \subseteq N$ such that $\{i \in N \mid \lambda^\rho > 0\} \subseteq N^\rho$ and such that $\{N^\rho \mid \rho = 1, \ldots, r\}$ is a partition of N. Defining $\lambda = \sum_{\rho=1}^r \lambda^\rho$,

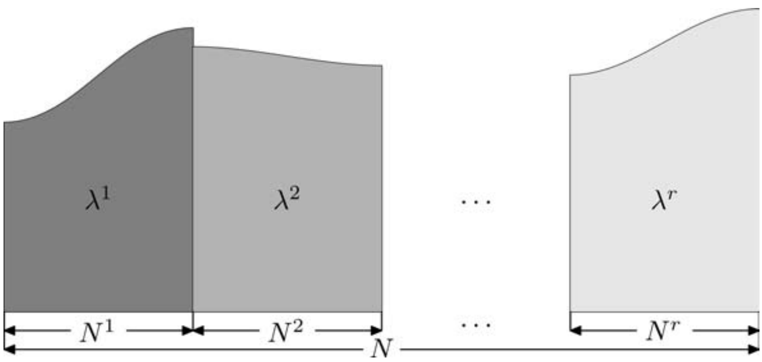

Fig. 2. An Orthogonal Game

we have $\lambda_i^\rho = \lambda_i$ for $i \in N^\rho$ and $\lambda_j^\rho = 0$ for $j \in N \setminus N^\rho$. Hence, (N, v) is an orthogonal game if and only if there is a partition $\{N^\rho \mid \rho = 1, \ldots, r\}$ of N and $\lambda \in \mathbb{R}_+^N$ such that $v(S) = \min_{\rho=1,\ldots,r} \lambda(S \cap N^\rho)$ for all $S \subseteq N$. In the orthogonal case we shall always assume that $\lambda(N^1) \leq \cdots \leq \lambda(N^r)$. Also, we may assume without loss of generality that, for every $i \in N$, $\lambda_i \leq \lambda(N^1) = v(N)$. The pair $(\{N^\rho \mid \rho = 1, \ldots r\}, \lambda)$ is called a representation of (N, v).

Orthogonal games are not just mathematical constructs but carry nice economic import. In a market, economic subjects may control different corners of the market (we treat corners synonymous with carriers) by the possession of a sole factor. While core tends to favor the short sides of the market excessively, the modiclus in such games is sensitive to the possible formation of cartels of the long side, by assigning fair share for the formation of such cartels. For details see Rosenmüller and Sudhölter (2004).

A representation of an orthogonal game is "almost" unique. Indeed, λ is uniquely determined. Moreover, if (N, v) is not the flat game (that is $v(S) = 0$ for all $S \subseteq N$), then the partition is uniquely determined except that a null player may be a member of any element of the partition.

Let (N, v) be an orthogonal game and let $(\{N^\rho | \rho = 1, \ldots, r\}, \lambda)$ be a representation of (N, v). The following two lemmata are useful.

Lemma 4.1. *The orthogonal game (N, v) is exact if and only if $\lambda(N^\rho) = v(N)$ for every $\rho = 1, \ldots, r$.*

Proof: If $\lambda(N^\rho) = v(N)$ for all $\rho = 1, \ldots, r$, then (N, v) is exact. In order to show the opposite direction let (N, v) be exact and let $\rho \in \{1, \ldots, r\}$. Then there exists $x \in \mathscr{C}(N, v)$ such that $x(N \setminus N^\rho) = v(N \setminus N^\rho) = 0$. Hence $x_i = 0$ for all $i \in N \setminus N^\rho$ and $x_j = \lambda_j$ for all $j \in N^\rho$. ■

Lemma 4.2. *If an orthogonal game has a stable core, then it is exact.*

Proof: Let $(\{N^\rho \mid \rho = 1, \ldots, r\}, \lambda)$ be the representation of an orthogonal game (N, v). If (N, v) is not exact, then $\lambda(N^r) > v(N)$. Let $\alpha = \frac{v(N)}{\lambda(N^r)}$ and let $y \in \mathbb{R}^N$ be defined by $y_{N \setminus N^r} = 0 \in \mathbb{R}^{N \setminus N^r}$ and $y_i = \alpha\lambda_i$ for all $i \in N^r$. Then $y(N) = v(N)$ and $y_j \geq 0$ for all $j \in N$. Hence y is an imputation. Also, if $i \in N^r$ with $\lambda_i > 0$, then $v(\{i\} \cup (N \setminus N^r)) = \lambda_i > y_i$ and we conclude that $y \notin \mathscr{C}(N, v)$.

Now assume, on the contrary, that y is dominated by some $x \in \mathscr{C}(N, v)$ by some nonempty coalition S. Then $\lambda(S \cap N^\rho) > 0$ for every $\rho = 1, \ldots, r$, because otherwise $v(S) = 0$. Let $S^r = S \cap N^r$. Then $x(S^r) > y(S^r) = \alpha\lambda(S^r)$. Thus, $v(S) > \alpha\lambda(S^r)$. Two cases may be distinguished. If $v(N \setminus S^r) = v(N)$, then

$$v(N) = x(N) = x(N \setminus S^r) + x(S^r) > v(N) + \alpha\lambda(S^r) > v(N),$$

which is impossible. If $v(N \setminus S^r) < v(N)$, then $v(N \setminus S^r) = \lambda(N^r \setminus S^r)$. Thus,

$$v(N) = x(N) = x(N \setminus S^r) + x(S^r) > \lambda(N^r \setminus S^r) + \alpha\lambda(S^r) \geq \alpha\lambda(N^r) = v(N),$$

which is also impossible. ■

The following example presents an orthogonal exact game that does not have a stable core.

Example 4.3. Let $N = \{1, \ldots, 5\}$, let $\lambda = (2, 1, 1, 1, 1)$, let $N^1 = \{1, 2\}$, $N^2 = \{3, 4, 5\}$, and let (N, v) be the orthogonal game represented by $(\{N^1, N^2\}, \lambda)$. Then (N, v) is exact. Moreover,

$$\mathscr{C}(N, v) = \text{convh}\{(2, 1, 0, 0, 0), (0, 0, 1, 1, 1)\},$$

where "convh" denotes "convex hull". Let $y = (1, 1, 0, 1/2, 1/2)$. Then y is an imputation. Also, $e(S, y, v) > 0$ just for $S = \{1, 3, 4\}$ and for $S = \{1, 3, 5\}$. Therefore, y is not dominated by any member of the core.

It should be noted that Example 4.3 may be generalized (see Biswas, Parthasarathy and Potters (1999), p. 6).

Theorem 4.4. *If (N, v) is an exact orthogonal game, then $\mathscr{MLC}(N, v) \subseteq \mathscr{C}(N, v)$.*

Proof: Let (N, v) be represented by $(\{N^\rho \mid \rho = 1, \ldots, r\}, \lambda)$ and let $x \in X(N, v)$. Then $\mu(x, v) \geq 0$. Moreover,

$$\sum_{\rho=1}^{r} e(N^k, x, v^*) = rv(N) - v(N) = (r-1)v(N) =: r\mu^*.$$

Hence $\mu(x, v^*) \geq \mu^*$. We claim that

$$\mathscr{MLC}(N, v) = \{x \in \mathscr{C}(N, v) \mid \mu(x, v^*) = \mu^*\}. \tag{4.2}$$

In order to prove our claim it suffices to find $\widehat{x} \in \mathscr{C}(N, v)$ such that $\mu(\widehat{x}, v^*) = \mu^*$. Let $\widehat{x} = \frac{1}{r}\lambda$. Then $\widehat{x} \in \mathscr{C}(N, v)$. Let $S \subseteq N$. It remains to show that $e(S, \widehat{x}, v) \geq -\mu^*$. Let $\widehat{\rho} \in \{1, \ldots, r\}$ be such that $v(S) = \lambda(S \cap N^{\widehat{\rho}})$. Let $T = (N \setminus N^{\widehat{\rho}}) \cup (S \cap N^{\widehat{\rho}})$. As $v(T) = v(S)$ and $S \subseteq T$, $e(S, \widehat{x}, v) \geq e(T, \widehat{x}, v)$. However,

$$\begin{aligned} e(T, \widehat{x}, v) &= \lambda(S \cap N^{\widehat{\rho}}) - \widehat{x}(T) = \lambda(S \cap N^{\widehat{\rho}}) - \frac{r-1}{r}v(N) - \frac{1}{r}\lambda(S \cap N^{\widehat{\rho}}) \\ &\geq -\frac{r-1}{r}v(N) = -\mu^*. \end{aligned}$$

■

Remark 4.5. Let (N, v) be an exact orthogonal game represented by $(\{N^1, N^2\}, \lambda)$ and let $\widehat{x} \in \mathbb{R}^N$ be defined as in the proof of Theorem 4.4, that is, $\widehat{x} = \lambda/2$. Let $S \subseteq N$. Then $min_{\rho=1,2}\lambda(S \cap N^\rho) + \max_{\rho=1,2} \lambda(S \cap N^\rho) = \lambda(S)$. Also,

$$e(N \setminus S, \widehat{x}, v) = \min_{\rho=1,2}(\lambda(N^\rho) - \lambda(S \cap N^\rho)) - \widehat{x}(N) + \widehat{x}(S) = \widehat{x}(S) - \max_{\rho=1,2}\lambda(S \cap N^\rho).$$

Hence $e(S, \widehat{x}, v) - e(N \setminus S, \widehat{x}, v) = 2\widehat{x}(S) - \lambda(S) = 0$. Hence, the excess of any coalition coincides with the excess of the complement coalition. It is straightforward to deduce that this fact implies that $v(N, v) = \psi(N, v) = \widehat{x}$ (see Sudhölter (2001)).

In view of the foregoing remark an exact orthogonal game has three types of players if the modiclus and the nucleolus do not coincide.

Remark 4.6. Every exact assignment game has a stable core (Solymosi and Raghavan (2001)). Hence, for every assignment game and every orthogonal game the modiclus selects a member of the core provided the game is exact or it has a stable core. In Section 5 it is shown that these results cannot be generalized to arbitrary exact games with a stable core.

5 The modified least core of an exact 16-person game

In this section we shall construct an exact TU game whose core is stable and does not contain the modiclus.

The key ideas behind the construction of the counterexample are as follows. An exact game with player set N is easily constructed by taking the minimum of finitely many additive games that assign a common worth to the grand coalition N. In this case the additive games belong to the core of the constructed exact game. Moreover, any nonempty polyhedral subset of $\mathbb{R}^N$ defined by inequalities of the form $x(S) \geq \alpha_S$ for $\emptyset \neq SN$ and by the equation $x(N) = \alpha_N$ is the core of the exact game given by the minimum of the finitely many extreme points of the polyhedral set. If the core constructed in this way is also large (see below), then the game is not only exact but has the core as its unique stable set (see Sharkey (1982)). We shall choose $\alpha_N = 0$ in our example. Finally, our aim will be achieved if the game (N, v) so constructed satisfies the following properties: (a) The null vector z does not belong to its core and (b) for any element x of its core, $\mu(x, v^*) = \mu(x, v) + \mu(x, v^*)$ exceeds $\mu(z, v) + \mu(z, v^*)$. In what follows we shall construct a game with the foregoing properties.

Example 5.1. Let $N = \{1, \ldots, 16\}$ and let

$$N^1 = \{1, 2, 3\}, \quad N^2 = \{4, 5, 6\}, \quad N^3 = \{7, \ldots, 16\},$$
$$S^1 = \{1, 2, 4\}, \quad S^2 = \{1, 3, 5\}, \quad S^3 = \{2, 3, 6\}.$$

We shall now define the nonempty compact polyhedral set $\mathscr{C} \subseteq \mathbb{R}^N$ which will turn out to be the core of our exact game: Let $x \in \mathbb{R}^N$. Then $x \in \mathscr{C}$ iff

$$x(S) \geq -27 \text{ for all } S \subseteq N, \tag{5.3}$$

$$x_i \geq -1 \text{ for all } i \in N^1, \tag{5.4}$$

$$x_j \geq -3 \text{ for all } j \in N^2, \tag{5.5}$$

$$x(S^k) \geq 1 \text{ for all } k \in N^1, \text{ and} \tag{5.6}$$

$$x(N) = 0. \tag{5.7}$$

Indeed, $\mathscr{C}$ is a closed polyhedral set. Moreover, it is compact (see (5.3) and (5.7)) and nonempty. Let r be the number of extreme points of $\mathscr{C}$ and let λ^ρ, $\rho = 1, \ldots, r$, denote the extreme points. Then $\mathscr{C} = \text{convh}(\{\lambda^\rho \mid \rho = 1, \ldots, r\})$. Define (N, v) by

$$v(S) = \min_{\rho=1,\ldots,r} \lambda^\rho(S) \quad \text{for all } S \subseteq N.$$

Then (N, v) is exact (by (5.7)) and $\mathscr{C}(N, v) = \mathscr{C}$. In order to show that (N, v) has a stable core it suffices to verify that $\mathscr{C}$ is *large*, that is, if $y \in \mathbb{R}^N$ satisfies $y(S) \geq v(S)$ for all $S \subseteq N$, then there exists $\lambda \in \mathscr{C}$ such that $\lambda \leq y$. Indeed, according to Sharkey (1982) the core of a game is stable if it is large.

Lemma 5.2. *The game (N, v) of Example 5.1 has a large core.*

Proof: Let $y \in \mathbb{R}^N$ satisfy $y(S) \geq v(S)$ for all $S \subseteq N$. Let X denote the set of vectors $x \in \mathbb{R}^N$ that satisfy (5.3) – (5.6), $x(N) \geq 0$, and $x \leq y$. Then X is nonempty (because $y \in X$) and polyhedral. Hence X is compact. Let $\widehat{x} \in X$ be such that

$$\widehat{x}(N) \leq x(N) \quad \text{for all } x \in X. \tag{5.8}$$

It remains to show that $\widehat{x} \in \mathscr{C}$, that is, $\widehat{x}(N) = 0$. Assume, on the contrary, that $\widehat{x}(N) > 0$. Denote $N^- = \{i \in N \mid \widehat{x}_i \leq 0\}$. We first claim that

$$N^3 \subseteq N^- \text{ and } \widehat{x}(N^-) = -27. \tag{5.9}$$

Eq. (5.9) is shown by contradiction. If $\ell \in N^3 \setminus N^-$, then there exists $\epsilon > 0$ such that $\widehat{x} - \epsilon\chi_{\{\ell\}} \in X$. ($\chi_S \in \mathbb{R}^N$ denotes the indicator function of $S \subseteq N$.) If $\widehat{x}(N^-) > -27$, then there exists $\epsilon > 0$ such that $\widehat{x} - \epsilon\chi_{\{\ell\}} \in X$ for every $\ell \in N^3$. Hence, both cases are in contrast to Eq. (5.8).

Now the proof can be completed. By Eq. (5.9) and the assumption that $\widehat{x}(N) > 0$, there exists $S \subseteq N^1 \cup N^2$ such that $\widehat{x}(S) > 27$. We now claim that $\widehat{x}_i \leq 5$ for all $i \in N^1$ and $\widehat{x}_j \leq 3$ for all $j \in N^2$. Indeed, if $\widehat{x}_i > 5$ for some $i \in N^1$, then, in view of (5.4) – (5.6), there exists $\epsilon > 0$ such that $\widehat{x} - \epsilon\chi_{\{i\}} \in X$. A similar argument is valid if $\widehat{x}_j > 3$ for some $j \in N^2$. Both cases contradict Eq. (5.8). Hence, $\widehat{x}(S) \leq 3 \cdot 5 + 3 \cdot 3 = 24 < 27$ for all $S \subseteq N$. ■

In order to determine the worth of some coalitions, we define 42 vectors of $\mathscr{C}$ as follows. For every $k \in N^1$ and $\ell \in N^3$ let $T^k = (S^k \cap N^1) \cup (N^2 \setminus S^k)$ and let

$$\lambda^0, \lambda^1, \lambda^\ell (10 \text{ elements }), \lambda^{k\ell} (30 \text{ elements })$$

be defined by

$$\lambda^0 = \Big(-1, -1, -1, 3, 3, 3, \underbrace{-\frac{3}{5}, \ldots, -\frac{3}{5}}_{10 \text{ times}}\Big),$$

$$\lambda^1 = (1, 1, 1, -1, -1, -1, \underbrace{0, \ldots, 0}_{10 \text{ times}}),$$

$$\lambda^\ell_i = 9 \quad \text{for } i \in N^2, \lambda^\ell_\ell = -27, \text{ and } \lambda^\ell_i = 0, \text{ otherwise },$$

$$\lambda^{k\ell}_i = \frac{27}{2}, \quad \textit{if } i \in (N^1 \cup N^2) \setminus T^k, \lambda^{k\ell}_\ell = -27, \text{ and } \lambda^{k\ell}_i = 0, \text{ otherwise.}$$

It is straightforward to check that these 42 vectors are elements of $\mathscr{C}$.

Lemma 5.3. *If $S \subseteq N$, $k \in N^1$, and $\ell \in N^3$, then*

$$v(S) \leq 1; \tag{5.10}$$

$$v(T^k \cup \{\ell\}) = -27; \tag{5.11}$$

$$v(N^1 \cup \{\ell\}) = -27. \tag{5.12}$$

Proof: Let $S \subseteq N$, let $k \in N^1$, and let $\ell \in N^3$. If $N^1 \setminus S \neq \emptyset$ and $S \cap N^2 \neq \emptyset$, then $v(S) \leq \lambda^1(S) \leq 1$. If $N^1 \subseteq S$ and $|S \cap N^2| \leq 1$, then $v(S) \leq \lambda^0(S) \leq 0$. If $|S \cap N^2| \geq 2$, then $v(S) \leq \lambda^1(S) \leq 1$. Also,

$$v(N^1 \cup \{\ell\}) \leq \lambda^\ell(N^1 \cup \{\ell\}) = -27 = \lambda^{k\ell}(T^k \cup \{\ell\}) \geq v(T^k \cup \{\ell\})$$

and (5.3) completes the proof. ■

Lemma 5.4. *Let $x \in \mathscr{C}$ and let $\ell \in N^3$ satisfy $x_\ell = \max_{j \in N^3} x_j$. If $x(N^1 \cup \{\ell\}) \leq 1$, then there exists $k \in N^1$ such that $x(T^k \cup \{\ell\}) > 1$.*

Proof: Assume the contrary. Then $x(N^1) + x_\ell \leq 1$ and

$$x(T^k) + x_\ell \leq 1 \text{ for all } k \in N^1. \tag{5.13}$$

Summing the 3 equations of (5.13) yields $2x(N^1 \cup N^2) + 3x_\ell \leq 3$. As $x_\ell \geq \frac{x(N^3)}{10}$ and as $x(N) = 0$, we receive $x_\ell \geq -\frac{x(N^1 \cup N^2)}{10}$. Hence we may deduce that

$$9x(N^1) - x(N^2) \leq 10, \tag{5.14}$$

$$17x(N^1) + 17x(N^2) \leq 30. \tag{5.15}$$

Multiplying (5.14) by 17, multiplying (5.15) by 11, and summing up yields

$$340x(N^1) + 170x(N^2) \leq 500$$

or, equivalently, $2x(N^1) + x(N^2) \leq \frac{50}{17} < 3$. On the other hand, by (5.6), $2x(N^1) + x(N^2) = x(S^1) + x(S^2) + x(S^3) \geq 3$, hence the desired contradiction has been obtained. ■

It remains to verify the following result.

Corollary 5.5. $\mathscr{C}(N, v) \cap \mathscr{MLC}(N, v) = \emptyset$.

Proof: Let $z = 0 \in \mathbb{R}^N$. By (5.3) and by Lemma 5.3, $\mu(z, v) = 1$ and $\mu(z, v^*) = 27$ (see (2.1)). For any $x \in \mathscr{C}$, $\mu(x, v) = v(N) - x(N) = 0$ and, by Lemma 5.4 together with (5.11) or (5.12) respectively, $\mu(x, v^*) > 28$. Thus, $\mu(x, v) + \mu(x, v^*) > \mu(z, v) + \mu(z, v^*)$ and $x \notin \mathscr{MLC}(N, v)$ by (2.2). ■

Note that $z = 0$ is, in fact, the unique element of the modified least core. For a proof of this statement see Raghavan and Sudhölter (2003).

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