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Formation of Cartels in Glove Markets and the Modiclus

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We discuss market games or linear production games with a large but finite set of agents. The representing distributions of initial assignments are assumed to be uniform distributions with disjoint carriers. Thus, the agents decompose into finitely many disjoint groups where each holds a corner of the market. Following a paper of Hart we argue that the formation of cartels should be explained endogenously. Accordingly, we exhibit a solution concept that not only predicts cartelization but also explains the profits of the long side by its preventive power. This concept is the modified nucleolus or modiclus.

Keywords: cartels, coalitional game, glove game, solution concept, modiclus nucleolus.

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1 Introduction

In this paper, we attempt to explain the *endogenous* formation of cartels in large economies. We assume that contracts generating cartels are legally permissible and can be enforced. This situation can be observed; in Switzerland for instance, at some time antitrust legislative and executive measures were not accepted and as a result, cartelization of certain industries was observed. Moreover, if we consider a union representing a group of workers with (approximately) equal characteristics to be a cartel in the technical sense, then obviously at least parts of the society are legally cartelized in most western countries.

However, it would seem that general-equilibrium theory or related approaches via coalition formation in exchange economies are unable to predict the endogenous formation of cartels, even in clear-cut situations that call for joint action within sectors of the market. The simplest version of such a situation is represented by a glove market or a glove game.

In such an economy groups of traders occupy *corners* of the market, that is, sectors of different (non-overlapping) sets of initial assignments of indispensable goods. When contracts are feasible – and can be legally enforced – there seems to be a strong incentive for agents to form (at least intermediately) *cartels* by joining forces within these corners. These cartels may then act as players or agents themselves, so that the responsibility is delegated to representatives bargaining for a share of the market. This procedure points to a different game in which few players act to the benefit of those they represent. If we consider the game in which the various cartels act as players, then their bargaining power may increase and they may force (members of) opposing cartels to accept a distribution of profits (allocations, imputations) that is much more favorable to them. One reason for this achievement is the increasing blocking power of a cartel: it is not only of relevance what a coalition of traders can attain but also what they can prevent others to achieve.

In this situation the result of cooperation within cartels might be quite different from what is observed when agents show price-taking behavior. This points to the fact that equivalence theorems for large markets are incapable of representing cartelization.

The same is true, to mention one concept from Game Theory, for the Shapely value. Since it measures the marginal contribution of traders on average and since almost all coalitions in a large economy look rather similar to the grand coalition, the Shapley value represents (eventually) the marginal contribution of traders to the total market – which is zero for agents living in an excess supply corner.

In a paper "Formation of Cartels in Large Markets" Hart (1974) discusses this situation from the viewpoint of a different solution concept (the vNM–stable set). He argues that in markets with disjoint corners, the formation of cartels has to be a *result* of the solution concept employed, it should be an *endogenous* concept. And he points to the vNM–stable set which (for the non-atomic case and other than the core etc.) does indicate the power of cartels.

Hart's argument essentially is that there are vNM–stable sets which are obtained from finite vNM–stable sets in a symmetric way (treating all players of the same type alike). This, he goes on, shows that coalitions of *types* have been formed, acted as players (in the finite game) and distributed the profits obtained this way symmetrically among their members. As all the solutions in the continuous case are of this shape (his main result) he goes even further in holding that society *has* to organize itself this way.

Certainly one can argue that it is the way coalitions form that matters and not so much in which way they agree to distribute their profits. However, eventually this approach is not sufficiently explanatory.

First of all, vNM–stable sets are there in abundance – so which of them should be adopted and why? Some results about vNM–stable sets point to strange shapes of the solution concepts whose economic meaning may be questioned.

Secondly, we are dealing with a set-valued concept. Other than the core, it does not shrink down to a single-valued solution for large markets – so the claim that society organizes itself this way fails to explain the motivation resulting from a distribution of wealth to be expected from the bargaining process. After all it is the distribution of wealth that matters in the public discussion, not just the formation of cartels as such.

These problems may constitute grounds for economists to be not completely satisfied with vNM–stable sets.

Hart argues rightfully that approaches to use the core as the solution concept in the case of a multi-corner market (a glove game) fails to show some regard for the cartel power of the long side. These approaches generally assume the existence of cartels a priori. On the other hand, it may be difficult to argue in favor of the vNM-concept if an abundance of cartels is equally alike and in each of them it is not clear what its members can expect.

The situation has certainly improved since Hart's paper. There are now more results concerning vNM-stable sets which support his view. Einy et al. (1996) prove that, for certain totally balanced games, the core is a vNM–stable set (see also Einy and Shitovitz, 1996, and Einy et al., 1996, for the convex case).

Based on these results, in a recent paper Rosenmüller and Shitovitz (2000) are able to characterize all convex vNM–stable sets of a glove game with a continuum of traders representing disjoint corners of the market. They show that all convex vNM–stable sets are of a "standard" shape indicating indeed the formation of cartels within the different corners of the market. Cartels bargain by representatives. The (symmetric) distribution of the result of the "few player game" among the members of the cartels is organized in a most plausible fashion: on the short side, each trader gets a share exactly proportional to his holdings and on the long side, the share is not exceeding his holdings.

Nevertheless, the vNM-concept still admits a variety of distributions of the outcome.

In the finite context, Maschler (1976) points out a small class of market games with two corners that admit bargaining sets, some allocations of which reflect a natural way of cartelization. Based on this, Legros (1987) discussed the nucleolus of the same class.

We wish to point out another solution concept which also organizes society in cartels (that is, cartels of the various corners). Other than the vNM–stable sets, it is single valued and hence organizes not only the coalitions but also provides the allocations.

Moreover, in the context we are dealing with ("large but finite games") the concept really does what Hart claims should be done: it treats cartels as players in some different (small, finite) game and distributes accordingly *between cartels*. Moreover, it frequently assigns an equal share to each cartel.

Finally, our concept organizes *inside* each cartel: it provides symmetric allocations and respects the initial holdings of a single trader in a most sensitive way.

This concept is the *modified nucleolus* or *modiculus* as we call it, due to one of the authors (Sudhölter, 1997). Explicitly it derives its strength from the fact not taken into consideration by most concepts concerning coalitional power: not only the power of coalitions (cartels) to achieve gains is important but also their "preventive power". When organizing the society, arguments like "there is no way without us" may not be so cooperative, but they may be more convincing.

After all, a union representing a large group of workers usually does not argue that their members can achieve a great deal without the employees by organizing production and trading within their rank and files. On the contrary: the union threatens to organize a strike, at which time the opponents will not be able to achieve anything.

This idea is generally not captured by the core. Players and groups argue what they can achieve by cooperation within coalitions and eventually a set of generally accepted solutions appears. Which one to choose may depend on the situation. But in the cornered market, the glove game, it is just the bargaining power of a corner that fails: they can achieve little by cooperating on their own. A cartel representing a corner of the market develops coalitional power just because it prevents the opposing forces of the economy from organizing themselves successfully. This argument eventually leads to global agreements in which the cartels receive considerable shares. So, after all, coalitional power is not just reflected by the ability and the legal possiblities to organize a coalition but much more by the power of preventing organization elsewhere.

The formal tool to assess what a coalition can withhold from its opponents is the *dual game*. Formally, this is another coalitional function derived from the original one. It assigns the complementary worth of the complementary coalition. Thus, the dual game assigns little value to a coalition if the complementary coalition is powerful, and vice versa. We will have to elaborate this concept to some extent.

The framework we are dealing with is the coalitional function of Cooperative TU–Game Theory. This is indeed a serious assumption from the technical point of view. Hart argues within an NTU framework of exchange economy, the vNM–Concept he uses can be formulated in this context and vNM–Stable Sets may be regarded as sufficiently "ordinal". The modiclus initially is a cardinal concept. There are attempts to define nucleoli type concepts ordinally (e.g., Kalai, 1975). The definition of a modiclus based on the excess functions constructed within this context may eventually be successful.

Consider a coalition *game given by triple (I,* $\underline{\underline{\mathbf{P}}}$*,* $\boldsymbol{v}$*). Here I is the (finite) set of agents* or *players*, $\underline{\underline{\mathbf{P}}}$ the power set of I, called *system of coalitions*, and

$$\boldsymbol{v} : \underline{\underline{\mathbf{P}}} \to \mathbf{R}, \quad \boldsymbol{v}(\emptyset) = 0 \ ,$$

a real valued function on $\underline{\underline{\mathbf{P}}}$, the *coalitional function*. The *dual game* is given by

$$\boldsymbol{v}^{\star}(S) := \boldsymbol{v}(I) - \boldsymbol{v}(I - S)(S \in \underline{\underline{\mathbf{P}}}) \ . \tag{1.1}$$

This game reflects the preventive power of coalitions. We do not want, however, to solely rely on the dual game. Both the achievement power and the preventive power matter in the emergence of the final solution. Hence, we construct a device which incorporates $\boldsymbol{v}$ and $\boldsymbol{v}^{\star}$ simultaneously. This game is the *dual cover*. To this end we take two copies of the set of players or agents, say

$$I^{1,2} = I \times \{0, 1\} \ ,$$

and construct a game $\bar{\boldsymbol{v}}$ on the coalitions of this set (the power sets are indexed canonically), i.e., a coalitional function $\bar{\boldsymbol{v}} : \underline{\underline{\mathbf{P}}}^{1,2} \to \mathbf{R}$ defined by

$$\bar{\boldsymbol{v}}(S^0 \cup T^1) := \max\{\boldsymbol{v}(S^0) + \boldsymbol{v}^\star(T^1), \boldsymbol{v}(T^1) + \boldsymbol{v}^\star(S^0)\} \qquad (S^0 \in \underline{\underline{\mathbf{P}}}^0, T^1 \in \underline{\underline{\mathbf{P}}}^1) \ . \tag{1.2}$$

Here we have identified both copies of the original player set I with I, e.g., we do not distinguish between I and

$$I^0 = I \times \{0\} \subseteq I^{1,2} \ .$$

The game $\bar{\boldsymbol{v}}$ takes pairs of coalitions into account, in one of them players act "constructively" and in the other one "preventively". The roles are then reversed and one measures the maximal joint worth players could achieve by combining their forces this way. This game reflects the joint effects of the game and its dual. Note that it is defined for the "union" of both copies of the player set. Therefore, if we turn to the solution concept, we consider the projection of the result from the cover game onto the original player version. Then we obtain a concept that is defined for the original set of players.

Let us shortly describe the modiclus. On one hand, it is a nucleolus type concept (Schmeidlers, 1969). For the *prenucleolus*, one lists the excesses

$$e(S, \boldsymbol{x}, \boldsymbol{v}) = \boldsymbol{v}(S) - \boldsymbol{x}(S)$$

(*reasons to complain*) for any preimputation $\boldsymbol{x}$ (i.e., $\boldsymbol{x} \in \mathbf{R}^I, \boldsymbol{x}(I) = \boldsymbol{v}(I)$) in a (weakly) decreasing order, say

$$\theta(\boldsymbol{x}) := (\ldots, e(S, \boldsymbol{x}, \boldsymbol{v}), \ldots) \ . \tag{1.3}$$

Then the prenucleolus $\boldsymbol{\nu}$ is the unique imputation such that $\theta(\bullet)$ is lexicographically minimal, i.e.,

$$\theta(\boldsymbol{\nu}) \ \preceq_{\text{lexic}} \ \theta(\boldsymbol{x}) \quad \text{for all preimputations } \boldsymbol{x} \ . \tag{1.4}$$

The *modified nucleolus* or *modiclus* $\boldsymbol{\psi}$ lists *bi-excesses*

$$e(S, \boldsymbol{x}, \boldsymbol{v}) - e(T, \boldsymbol{x}, \boldsymbol{v})$$

and proceeds accordingly. Note that differences of excesses or bi-excesses can be seen as sum of excesses of the primal and dual game. Hence, the dual cover will eventually provide the appropriate interpretation. Under the regime of the modiclus, the pair of coalitions with respect to which agents exchange the most heated debate is, by a suitable agreement over an imputation, arranged as best as possible. Thereafter, the second pair in conflict is taken care of, and so on (lexicographically).

One may think that, at first sight, this concept complicates the already involved procedure the nucleolus is asking for. Technically, this is certainly true and possibly constitutes a barrier against intensive treatment. From the point of view of interpretative power however, the concept surpasses the nucleolus. As it turns out, it takes care of the dual game (the "preventive power of coalitions") in the most natural way – and allows for all interpretations the nucleolus is capable of. For, as it turns out (see Sudhölter, 1997), the modiclus ψ is the *projection* of the prenucleolus of the *dual cover game* $\bar{\mathfrak{v}}$ defined on $I^{1,2}$ on the original player set I.

Hence, if one wants to represent the constructive and preventive power of coalitions simultaneously, then one should turn to the modiclus. Indeed, the modiclus shows a surprising capacity in situations where the preventive power is predominant – that is, in markets with corners, which call for the endogenous formation of cartels. This is what we will demonstrate in the following sections.

In this context, the modiclus not only generates cartels, it frequently generates the "natural" or most "ideal" distributions of payoff. It respects *types*, that is, traders of equal characteristics obtain the same share. Thus, inside each corner, the argumentative strength of a trader in the coalition formation process is respected and cartels are seen to emerge.

However, in addition, the modiclus also respects the impact of cartelization referring to the "representative game" behind the scene. For if cartels are of equal power, then they are treated equally. That is, after cartelization the long side of the market has the same strength as the short side and the symmetries of the "representative game" are respected as well. Therefore, we believe that the complex procedure of distributing according to the modiclus justifies consideration.

The paper is organized as follows. In Sect. 2, we introduce the model and discuss the behavior of excesses. We explain the decisive behavior of the excess function on "diagonal coalitions" – which are in a well specified sense efficient and effective. In Sect. 3, we classify the behavior of the modiclus within the framework of the "ocean game" which represents uniform distribution of initial assignment. For this game we achieve a complete description of the modiclus. This section exhibits the formation of cartels. The treatment of the various corners of the market is described for “glove markets”.

In a subsequent paper (Rosenmüller and Sudhölter, 2000), we are dealing with a more general framework which permits nonuniform distribution of the initial assignment. When large chunks of a commodity are assigned to one cartel, then the internal bargaining process of this cartel becomes more involved. This requires the introduction of the notion of a reduced game (Davis and Maschler, 1965). The emphasis of the argument is shifted towards the discussion of the “internal” behavior of the modiclus inside the corners and as a consequence, one finds a surprising relation to the “contested garment solution” discussed in Aumann and Maschler (1985).

2 The Model, Excesses, and the Diagonal

A game, as explained in Sect. 1, is a triple $(I, \underline{\underline{\mathbf{P}}}, \boldsymbol{v})$ satisfying $\boldsymbol{v}(\emptyset) = 0$. Frequently we use the term also for the coalitional function and not always for the triple. We are predominantly interested in *market games* or *totally balanced games* which can be generated from exchange economies (Shapley and Shubik, 1969). In order to represent such a game we use the representation as a minimum game. That is, $\boldsymbol{v}$ is the minimum of finitely many nonnegative additive set functions (distributions or *measures*), say $\boldsymbol{\lambda}^1, \dots, \boldsymbol{\lambda}^r$, defined on $\underline{\underline{\mathbf{P}}}$ via $\boldsymbol{v}(S) = \min\{\boldsymbol{\lambda}^1(S), \dots, \boldsymbol{\lambda}^r(S)\}$ $(S \in \underline{\underline{\mathbf{P}}})$. This we write conveniently as follows:

$$\boldsymbol{v} = \bigwedge\{\boldsymbol{\lambda}^1, \dots, \boldsymbol{\lambda}^r\} \ . \tag{2.1}$$

According to Kalai–Zemel (1982), every totally balanced game can be represented in this way. A traditional version is the *glove game*. In this game coalitions need to combine indispensable factors (right-hand and left-hand gloves) in order to acquire profits by selling the product (pairs of gloves) on some external market.

We wish to concentrate on the *orthogonal* case, that is, there are separate (disjoint) *corners* of the market represented by the carriers of $\boldsymbol{\lambda}^\rho$, denoted by $C(\boldsymbol{\lambda}^\rho) = C^\rho (\rho = 1, \dots, r)$. Eventually (cf. Sect. 3) we shall assume that each player owns a quantity of one and only one factor, hence $I = \bigcup_{\rho=1}^r C^\rho$ describes a partition of I. The initial assignments introduced this way are *uniformly distributed*. Consequently, each $\boldsymbol{\lambda}^\rho$ is described by $\boldsymbol{\lambda}^\rho(S) = |S \cap C^\rho| (S \in \underline{\underline{\mathbf{P}}})$. In this case we call the “multi-sided glove game" $(I, \underline{\underline{\mathbf{P}}}, \boldsymbol{v})$ an *ocean game*. (For $r = 2$ this amounts to a glove game.)

The first study on glove games with two corners was Shapley (1959). However, the term as such is only introduced in Shapley (1966). Milnor and Shapley (1978) use the term "oceanic game" in a slightly different context in order to refer to a large (in their case continuous) homogeneous set of players. We wish to use the term "ocean game" for a multi-sided glove game because Shapley initiated the work on such games and in our opinion these games reflect many features that are generally attached to large games with homogeneous players.

Orthogonality is certainly a restriction within the class of market games. It implies that a coalition which completely lacks one factor receives no profit. Thus, each of the r different corners of the market is defined by the possession of a sole factor.

We use the abbreviation M^ρ in order to indicate the total mass of $\boldsymbol{\lambda}^\rho$, that is, the total initial assignment of goods in corner C^ρ. Thus

$$M^\rho := \boldsymbol{\lambda}^\rho(I) = \boldsymbol{\lambda}^\rho(C^\rho) = \sum_{i \in C^\rho} \boldsymbol{\lambda}^\rho(\{i\}) = |C^\rho| \ (\rho = 1, \ldots, r) \quad (2.2)$$

is satisfied. For convenience, the corners of the market are ordered according to total initial assignment, i.e., $M^1 \leq \cdots \leq M^r$ holds true.

Any coalition $S \in \underline{\underline{\mathbf{P}}}$ decomposes naturally into the coalitions of its *partners* in the various corners, which can be written in the following way:

$$S = \bigcup_{\rho=1}^{r} S^\rho \text{ with } S^\rho = S \cap C^\rho \ (\rho = 1, \ldots, r) \ . \quad (2.3)$$

An important system of coalitions is provided by the *diagonal* which is formally given by

$$\underline{\underline{\mathbf{D}}} := \{S \in \underline{\underline{\mathbf{P}}} \mid \boldsymbol{\lambda}^\rho(S) = \boldsymbol{v}(S) \ (\rho = 1, \ldots, r)\} \ . \quad (2.4)$$

A coalition $S \in \underline{\underline{\mathbf{D}}}$ is called a *diagonal coalition* because, geometrically, the image of S under the vector-valued measure $(\boldsymbol{\lambda}^1, \ldots, \boldsymbol{\lambda}^r)$ is located on the diagonal of $\mathbf{R}^r$. Economically, diagonal coalitions are *efficient*, as there is no excess supply of factors available in order to generate a utility of $\boldsymbol{v}(S)$.

An *imputation* of $\boldsymbol{v}$ is a distribution of the total wealth of the grand coalition that is also individually rational, i.e., an element of

$$\mathscr{J}(\boldsymbol{v}) := \{\boldsymbol{x} \in \mathbf{R}^I \mid \boldsymbol{x}(I) = \boldsymbol{v}(I),\ x_i \geq \boldsymbol{v}(\{i\}) \quad (i \in I)\} \ .$$

Similarly, the ***core*** of $\boldsymbol{v}$ is given by

$$\mathscr{C}(\boldsymbol{v}) := \{\boldsymbol{x} \in \mathbf{R}^I \mid \boldsymbol{x}(I) = \boldsymbol{v}(I),\ \boldsymbol{x}(S) \geq \boldsymbol{v}(S) \quad (S \in \underline{\underline{\mathbf{P}}})\} \ .$$

Whenever we are dealing with uniformly distributed initial assignments (i.e., with ocean games), the core is the convex hull of the measures $\boldsymbol{\lambda}^\rho$ with minimal $M^\rho = M^1$ $(\rho = 1, \ldots, r)$.

Note that, on diagonal sets, $\boldsymbol{v}$ behaves additively. As a consequence, it is not hard to see that any core element $\boldsymbol{x}$ equals the game on the diagonal system $\underline{\underline{\mathbf{D}}}$ (i.e., $\boldsymbol{x}(S) = \boldsymbol{v}(S) \quad (S \in \underline{\underline{\mathbf{D}}})$). In a sense, a diagonal coalition S is also *effective*: it can afford $(x_i)_{i \in S}$ by its own productive possibilities.

Within the diagonal we are particularly interested in *maximal* elements. These are diagonal coalitions S such that each corner assembles the maximal possible amount of goods and hence the coalition's worth is $\boldsymbol{v}(I)$. More precisely, such coalitions satisfy

$$\boldsymbol{\lambda}^1(S) = \ \cdots \ = \boldsymbol{\lambda}^r(S) = M^1 \ , \tag{2.5}$$

i.e., the same number of agents M^1 joins from each corner. The system of maximal coalitions is denoted by

$$\underline{\underline{\mathbf{D}}}^m := \{S \in \underline{\underline{\mathbf{P}}} \mid S \text{ satisfies } (2.5)\} \ . \tag{2.6}$$

Note that this system is nonempty in view of the uniform distribution generally assumed.

The notion of *excess* is central to the discussion of nucleolus-type solution concepts. Given a preimputation $\boldsymbol{x}$, recall that the excess of a coalition $S \in \underline{\underline{\mathbf{P}}}$ (cf. Sect. 1) is given by

$$e(S, \boldsymbol{x}, \boldsymbol{v}) = \boldsymbol{v}(S) - \boldsymbol{x}(S) \ . \tag{2.7}$$

This quantity measures the amount by which coalition S misses its worth $\boldsymbol{v}(S)$, hence it is dissatisfied with $\boldsymbol{x}$. The *maximal excess* at $\boldsymbol{x}$ is

$$\mu(\boldsymbol{x}, \boldsymbol{v}) := \max\{e(S, \boldsymbol{x}, \boldsymbol{v}) \mid S \in \underline{\underline{\mathbf{P}}}\} \ . \tag{2.8}$$

The task of computing excesses appears frequently; we begin with some versions concerning glove markets. Here is the first simple Lemma:

Lemma 2.1: Let $\boldsymbol{v}$ be an orthogonal game and let $\boldsymbol{x}$ be an imputation of $\boldsymbol{v}$. Let $S \in \underline{\underline{\mathbf{D}}}$ be a diagonal set and let $S \in \underline{\underline{\mathbf{P}}}$ be such that $\bar{S} \subseteq S$ holds true. Assume that $S^\sigma = \bar{S}^\sigma$ is true for at least one $\sigma \in \{1, \ldots, r\}$. Then it follows that

$$e(S, \boldsymbol{x}, \boldsymbol{v}) \leq e(\bar{S}, \boldsymbol{x}, \boldsymbol{v}) \tag{2.9}$$

is true.

In other words, if all corners (apart from the smallest ones) get rid of excess agents (providing excess supply of commodities), then the reason to complain increases (i.e., the excess $e(\bullet,\ \boldsymbol{x},\ \boldsymbol{v})$ increases with more efficient and effective coalitions). Or, geometrically, we could say that "moving towards the diagonal" from above increases the excess. Diagonal coalitions have a tendency to most effectively phrase opposition against imputations proposed.

Proof: Let $\boldsymbol{v}$ be represented via (2.1). As $\bar{S}$ is a diagonal coalition, we have

$$\boldsymbol{v}(\bar{S}) = \boldsymbol{\lambda}^\rho(\bar{S}) = \boldsymbol{\lambda}^\sigma(\bar{S}) \quad (\rho = 1, \ldots, r) \ . \tag{2.10}$$

Moreover, as $\bar{S} \subseteq S$ and $\boldsymbol{\lambda}^\sigma(S) = \boldsymbol{\lambda}^\sigma(\bar{S})$, we obtain $\boldsymbol{v}(S) = \boldsymbol{v}(\bar{S})$, thus

$$e(S, \boldsymbol{x}, \boldsymbol{v}) = \boldsymbol{v}(S) - \boldsymbol{x}(S) = \boldsymbol{v}(\bar{S}) - \boldsymbol{x}(S) \leq \boldsymbol{v}(\bar{S}) - \boldsymbol{x}(\bar{S}) = e(\bar{S}, \boldsymbol{x}, \boldsymbol{v}) \ . \tag{2.11}$$

□

The next lemma shows that, with stronger conditions, we can "move towards the diagonal" from below with still increasing excess. In other words, it is also true that coalitions can improve their effectiveness/efficiency by recruiting agents from outside. To see this, let $\boldsymbol{v}$ again be orthogonal and represented as in (2.1). Note that any imputation $\widehat{\boldsymbol{x}}$ can be represented in a standardized version respecting the corners of the market. Precisely, this means that $\widehat{\boldsymbol{x}}$ can be written as

$$\widehat{\boldsymbol{x}} = M^1 \sum_{\rho=1}^{r} c_\rho \frac{\boldsymbol{\mu}^\rho}{M^\rho} \tag{2.12}$$

such that $\boldsymbol{c} = (c_\rho)_{\rho=1,\dots,r}$ is a vector of nonnegative coefficients summing up to 1 ("convexifying coefficients") and $(\boldsymbol{\mu}^\rho)_{\rho=1,\dots,r}$ are measures with carriers C^ρ, having the same total mass $\boldsymbol{\mu}^\rho(C^\rho) = M^\rho$ as $\boldsymbol{\lambda}^\rho$. Now we have

Lemma 2.2: Let $\boldsymbol{v}$ be an orthogonal game represented via (2.1). Let $\widehat{\boldsymbol{x}}$ be an imputation and let $\boldsymbol{\mu}^1, \dots, \boldsymbol{\mu}^r$ be the corresponding representing measures according to (2.12). Let $S \in \underline{\underline{\mathbf{P}}}$ and $\bar{S} \in \underline{\underline{\mathbf{D}}}$ be such that

$$\boldsymbol{v}(S) \leq \boldsymbol{v}(\bar{S}) \tag{2.13}$$

and

$$\frac{\boldsymbol{\lambda}^\rho(\bar{S}) - \boldsymbol{\lambda}^\rho(S)}{M^1} \geq \frac{\boldsymbol{\mu}^\rho(\bar{S}) - \boldsymbol{\mu}^\rho(S)}{M^\rho} \tag{2.14}$$

holds true for $\rho = 1, \dots, r$.

Then

$$e(S, \boldsymbol{x}, \boldsymbol{v}) \leq e(\bar{S}, \boldsymbol{x}, \boldsymbol{v}) \tag{2.15}$$

is true.

That is, moving towards the diagonal from below increases the excess. Moreover, the excess increases with increasing sets on the diagonal. Consequently, if (2.14) is globally valid for pairs of coalitions satisfying (2.13), then the maximal excess appears at coalitions of $\underline{\underline{\mathbf{D}}}^m$, provided this system is nonempty.

Proof: Choose τ such that

$$\boldsymbol{v}(S) = \boldsymbol{\lambda}^\tau(S) \leq \boldsymbol{\lambda}^\rho(S) \quad (\rho = 1, \dots, r) \tag{2.16}$$

holds true, then we obtain the following set of equations and inequalities:

$$\begin{aligned} \boldsymbol{\lambda}^\tau(\bar{S}) - \boldsymbol{\lambda}^\tau(S) &= \boldsymbol{\lambda}^\rho(\bar{S}) - \boldsymbol{\lambda}^\tau(S) \quad (\text{as } \bar{S} \in \underline{\underline{\mathbf{D}}}) \\ &\geq \boldsymbol{\lambda}^\rho(\bar{S}) - \boldsymbol{\lambda}^\rho(S) \quad (\text{by } (2.16)) \end{aligned} . \tag{2.17}$$

Here, the left-hand side term is nonnegative because it equals $\boldsymbol{v}(\bar{S}) - \boldsymbol{v}(S)$. Therefore, using 2.14 we obtain:

$$\frac{\boldsymbol{\lambda}^{\tau}(\bar{S}) - \boldsymbol{\lambda}^{\tau}(S)}{M^1} \geq \frac{\boldsymbol{\lambda}^{\rho}(\bar{S}) - \boldsymbol{\lambda}^{\rho}(S)}{M^1} \geq \frac{\mu^{\rho}(\bar{S}) - \mu^{\rho}(S)}{M^{\rho}} \tag{2.18}$$

for $\rho = 1, \ldots, r$ and hence, using the convexifying coefficients involved in $\widehat{\boldsymbol{x}}$ via (2.12),

$$\frac{\boldsymbol{\lambda}^{\tau}(\bar{S}) - \boldsymbol{\lambda}^{\tau}(S)}{M^1} \geq \sum_{\rho=1}^{r} c_p \frac{\boldsymbol{\mu}^{\rho}(\bar{S}) - \boldsymbol{\mu}^{\rho}(S)}{M^{\rho}} \quad . \tag{2.19}$$

Reshuffling the terms yields:

$$\frac{\boldsymbol{\lambda}^{\tau}(\bar{S})}{M^1} - \sum_{\rho=1}^{r} c_p \frac{\boldsymbol{\mu}^{\rho}(\bar{S})}{M^{\rho}} \geq \frac{\boldsymbol{\lambda}^{\tau}(S)}{M^1} - \sum_{\rho=1}^{r} c_p \frac{\boldsymbol{\mu}^{\rho}(S)}{M^{\rho}} \quad , \tag{2.20}$$

which reads

$$\boldsymbol{v}(\bar{S}) - \widehat{\boldsymbol{x}}(\bar{S}) \geq \boldsymbol{v}(S) - \widehat{\boldsymbol{x}}(S) \quad . \tag{2.21}$$

$\square$

Remark 2.3: In the situation described by Lemma 2.2, the condition (2.13) is certainly implied if $S \subseteq \bar{S}$ prevails. If so, (2.14) is satisfied if the two measures involved satisfy

$$\frac{\boldsymbol{\lambda}^{\rho}}{M^1} \geq \frac{\boldsymbol{\mu}^{\rho}}{M^{\rho}} \quad (\rho = 1, \ldots, r) \quad . \tag{2.22}$$

Thus, there is a bound on the relative deviation of an imputation from the initial assignment inside which the excess increases towards the diagonal.

3 The Modiclus of the Ocean Game

In this section, we treat the modiclus of the ocean game ("multi-sided glove game")

$$\boldsymbol{v} = \bigwedge \{\boldsymbol{\lambda}^1, \ldots, \boldsymbol{\lambda}^r\} \tag{3.1}$$

as defined in Sect. 2. Recall that each measure $\boldsymbol{\lambda}^{\rho}$ on its carrier C^{ρ} can be viewed as vector $\boldsymbol{\lambda}^{\rho} = (1, \ldots, 1)$. Let us now introduce the long side and short side of the market. We assume that the corners are ordered according to size. Now, let the first σ groups be of equal (minimal) size. That is, define $\sigma \in \{1, \ldots, r\}$ by the requirement

$$M^1 = \cdots = M^\sigma < M^{\sigma+1} \leq \cdots \leq M^r \ . \tag{3.2}$$

Then these first σ corners have to be completely present in any coalition achieving the total worth $\boldsymbol{v}(I)$. They represent the short side of the market. With respect to this game, we are in the position to completely compute our solution concept, i.e., the modified nucleolus or *modiclus*.

The definition of this concept has been indicated in the introduction: the modiclus of a game, denoted by $\boldsymbol{\psi}(\boldsymbol{v})$, is the unique imputation that lexicographically minimizes the (ordered) vector of bi-excesses. (Indeed, note that the modiclus must be individually rational by Corollary 2.6 of Sudhölter (1997) because an orthogonal game is *zero-monotonic*: $\boldsymbol{v}(S \cup \{i\}) - \boldsymbol{v}(S) \geq 0 = \boldsymbol{v}(\{i\})$ $(S \in \underline{\underline{\mathbf{P}}}, i \in I))$ Equivalently, it is the projection of the prenucleolus of the dual cover game onto the set of primal agents. For the details, see Sudhölter (1997).

As it turns out, the modiclus is quite sensitive with respect to the relative size of the corners. If the long side of the market exceeds the short side just moderately, then the long side has sufficiently much bargaining power. By the formation of cartels, each corner on the long side can achieve some gains. The modiclus assigns the *ideal point*, that is, equal treatment prevails with respect to the corners as well as inside each corner. Formally, the ideal point is the assignment $\bar{\boldsymbol{x}}$ given by

$$\bar{x}_i := \frac{M^1}{rM^\rho} \ (i \in C^\rho, \rho = 1, \ldots, r) \ . \tag{3.3}$$

If there are excessively many agents on the long side, then the modiclus reacts as the core, the (limiting) Shapley value, and the Walrasian payoff distribution: agents with excess supply of commodity receive zero utility. Indeed, the core is the convex hull of the short side assignments $\boldsymbol{\lambda}^1, \ldots, \boldsymbol{\lambda}^\sigma$ while the Walrasian payoff is the center of the core (see e.g., Shapley, 1969). The Shapley value in the limit approaches the center of the core as well; the proof appears in Shapely (1964) and (for $r = 2$) in Shapley (1969).

The modiclus assigns the *center of the core*, denoted by $\mathring{\boldsymbol{x}}$ and given by

$$\mathring{x}_i = \begin{cases} \frac{1}{\sigma}, & \text{if } i \in C^\tau \ (\tau = 1, \ldots, \sigma) \\ 0, & \text{if } i \in C^\rho \ (\rho = \sigma + 1, \ldots, r) \end{cases} . \tag{3.4}$$

However, the modiclus yields this result only for large excess supply on the large side while all the other concepts behave this way independently on the relative size of the corners. Finally, there is a borderline case that is particularly involved and at which the modiclus measures the influence of both the short and the long side in a most detailed fashion.

Theorem 3.1: Let $\boldsymbol{v}$ be an ocean game.
(1) If $\boldsymbol{\lambda}$ satisfies

$$1 + \sum_{\rho=1}^{r} \frac{M^1}{M^\rho} > r \ , \tag{3.5}$$

then the modiclus is the ideal point, i.e.,

$$\boldsymbol{\psi}(\boldsymbol{v}) = \bar{\boldsymbol{x}} = \frac{1}{r} M^1 \sum_{\rho=1}^{r} \frac{\boldsymbol{\lambda}^\rho}{M^\rho} \ . \tag{3.6}$$

(2) If $\boldsymbol{\lambda}$ satisfies

$$1 + \sum_{\rho=1}^{r} \frac{M^1}{M^\rho} < r \ , \tag{3.7}$$

then the modiclus is the center of the core, i.e.,

$$\boldsymbol{\psi}(\boldsymbol{v}) = \mathring{\boldsymbol{x}} = \frac{1}{\sigma} M^1 \sum_{\rho=1}^{\sigma} \frac{\boldsymbol{\lambda}^\rho}{M^\rho} = \frac{1}{\sigma} \sum_{\rho=1}^{\sigma} \boldsymbol{\lambda}^\rho \ . \tag{3.8}$$

(3) Finally, if

$$1 + \sum_{\rho=1}^{r} \frac{M^1}{M^\rho} = r \ , \tag{3.9}$$

is the case, then the modiclus is given by

$$\boldsymbol{\psi}(\boldsymbol{v}) = \frac{\sigma}{\sigma + rM^r} \mathring{\boldsymbol{x}} + \frac{rM^r}{\sigma + rM^r} \bar{\boldsymbol{x}} \ , \tag{3.10}$$

i.e.,

$$\psi_i(v) = \begin{cases} \frac{M^r+1}{\sigma+rM^r}, & \text{if } i \in C^\tau \quad (\tau = 1, \ldots, \sigma), \\ \frac{M^r}{\sigma+rM^r}\frac{M^1}{M^\rho}, & \text{if } i \in C^\rho \quad (\rho = \sigma+1, \ldots, r) \ . \end{cases} \tag{3.11}$$

Proof: 1st step: We use the abbreviation $\widehat{\boldsymbol{x}} := \boldsymbol{\psi}(\boldsymbol{v})$. In view of the fact that agents of the same corner are "equals", the symmetry properties (e.g., the equal treatment property, see Corollary 2.6 of Sudhölter, 1996) of the modiclus are relevant. They imply that there exists a vector $\widehat{\mathbf{c}}$ of "convexifying" coefficients satisfying

$$\widehat{c}_\rho \geq 0 \ (\rho = 1, \ldots, r) \ \text{and} \ \sum_{\rho=1}^{r} \widehat{c}_\rho = 1 \tag{3.12}$$

such that

$$\widehat{\boldsymbol{x}} = M^1 \sum_{\rho=1}^{r} \widehat{c}_\rho \frac{\boldsymbol{\lambda}^\rho}{M^\rho} \tag{3.13}$$

holds true. Any two corners of the same cardinality can be exchanged without changing the game (i.e., for C^ρ, C^τ $(\tau = 1, \ldots, \sigma)$ with $|C^\rho| = |C^\tau|$ there is a permutation of the agent set I which maps C^ρ onto C^τ such that the "permuted" game coincides with $\boldsymbol{v}$). Hence, by anonymity (see Remark 1.2 of Sudhölter, 1997) these corners are treated equally. That is, the coefficients $\widehat{c}_\rho$ $(\rho = 1, \ldots, r)$ of the corners satisfy

$$\widehat{c}_1 = \cdots = \widehat{c}_\sigma \ \text{and} \ \widehat{c}_\tau = \widehat{c}_\rho \ (\tau, \rho = \sigma+1, \ldots, r) \ \text{if} \ |C^\tau| = |C^\rho| \ . \tag{3.14}$$

If $\sigma = r$ holds true, then $\boldsymbol{v}$ is an exact game. In an exact game any coalition is effective with respect to some core element. In this case equation (3.14) already shows the theorem, because (3.5) is satisfied. In view of this fact we assume that $r > \sigma$ from now on.

2nd step: Now we consider an arbitrary convex combination of the $\boldsymbol{\lambda}^\rho$ $(\rho = 1, \ldots, r)$. Let $\boldsymbol{c}$ denote a vector of convexifying coefficients, i.e., $\boldsymbol{c}$ is assumed to satisfy

$$c_\rho \geq 0 \ (\rho = 1, \ldots, r) \ \text{and} \ \sum_{\rho=1}^{r} c_\rho = 1 \ , \tag{3.15}$$

and define

$$\boldsymbol{x}^{\boldsymbol{c}} = \boldsymbol{x} := M^1 \sum_{\rho=1}^{r} c_\rho \frac{\boldsymbol{\lambda}^\rho}{M^\rho} \quad . \tag{3.16}$$

Using this notation the modiclus can be expressed by $\widehat{\boldsymbol{x}} = \boldsymbol{x}^{\hat{\boldsymbol{c}}}$. For any coalition $S \in \underline{\underline{\mathbf{P}}}$, the excess is given by

$$\begin{aligned}
e(S, \boldsymbol{x}, \boldsymbol{v}) &= \boldsymbol{v}(S) - \boldsymbol{x}(S), \\
&= \boldsymbol{v}(S) - M^1 \sum_{\rho=1}^{r} c_\rho \frac{\boldsymbol{\lambda}^\rho(S^\rho)}{M^\rho} \\
&= \boldsymbol{v}(S) \left(1 - \sum_{\rho=1}^{r} c_\rho \frac{M^1}{M^\rho} \right) - M^1 \sum_{\rho=1}^{r} c_\rho \frac{\boldsymbol{\lambda}^\rho(S) - \boldsymbol{v}(S)}{M^\rho}, \\
&\leq \boldsymbol{v}(S) \left(1 - \sum_{\rho=1}^{r} c_\rho \frac{M^1}{M^\rho} \right) \quad .
\end{aligned} \tag{3.17}$$

Because of (3.15) we conclude that

$$\sum_{\rho=1}^{r} c_\rho \frac{M^1}{M^\rho} < \sum_{\rho=1}^{r} c_\rho = 1$$

and hence the excess increases with increasing S on the diagonal $\underline{\underline{\mathbf{D}}}$. It follows that the maximal excess is attained by the maximal diagonal coalitions, i.e., by coalitions $D_0 \in \underline{\underline{\mathbf{D}}}^m$ which are of the shape

$$D_0 = \bigcup_{\rho=1}^{r} D_0^\rho \tag{3.18}$$

such that

$$D_0^\tau = C^\tau \ (\tau = 1, \ldots, \sigma), \ \boldsymbol{\lambda}^\rho(D_0^\rho) = |D_0^\rho| = M^1 \ (\rho = \sigma + 1, \ldots, r) \tag{3.19}$$

is satisfied. By (3.17) the value of this maximal excess is given by

$$\mu(\boldsymbol{x}, \boldsymbol{v}) = e(D_0, \boldsymbol{x}, \boldsymbol{v}) = M^1 \left(1 - \sum_{\rho=1}^{r} c_\rho \frac{M^1}{M^\rho} \right) \quad . \tag{3.20}$$

3rd step: Now we turn to the *maximal dual* excess at $\boldsymbol{x} = \boldsymbol{x}^{\boldsymbol{c}}$. In view of (3.14) we shall assume from now on that $\boldsymbol{c}$ satisfies (3.15) and

$$c_1 = \cdots = c_\sigma \text{ and } c_\tau = c_\rho \quad (\tau, \rho = \sigma + 1, \ldots, r) \text{ if } |C^\tau| = |C^\rho| \ . \tag{3.21}$$

Let $S \in \underline{\underline{\mathbf{P}}}$ be any coalition. Choose $\tau \in \{1, \ldots, r\}$ such that $\boldsymbol{v}(S)$ is attained by $\boldsymbol{\lambda}^\tau(S)$. Then we obtain the inequalities

$$\begin{array}{lll} e(S, \boldsymbol{x}, \boldsymbol{v}) & = \boldsymbol{\lambda}^\tau(S) - \boldsymbol{x}(S) & \geq \boldsymbol{\lambda}^\tau(S) - \boldsymbol{x}(S^\tau) - \boldsymbol{x}(I - C^\tau) \\ & \geq -\boldsymbol{x}(I - C^\tau) & = e(I - C^\tau, \boldsymbol{x}, \boldsymbol{v}) \ . \end{array} \tag{3.22}$$

Hence the *minimal primal* excess is attained by coalitions of the shape

$$S = C^1 \cup \cdots \cup \emptyset \cup \cdots \cup C^\tau = I - C^\tau \ .$$

Therefore, the minimal primal excess is attained by complements of those carriers C^τ such that c_τ is minimizing, i.e., such that

$$c_\tau = \min\{c_\rho \mid \rho = 1, \ldots, r\}$$

holds true. Consequently, the maximal dual excess appears for coalitions of the shape

$$T = \emptyset \cup \cdots \cup \emptyset \cup C^\tau \cup \emptyset \cup \ldots \cup \emptyset = C^\tau$$

and the value of this maximal dual excess is

$$\begin{aligned} \mu(\boldsymbol{x}, \boldsymbol{v}^\star) &= -e(I - T, \boldsymbol{x}, \boldsymbol{v}) \\ &= -\left(0 - M^1 \sum_{\rho \neq \tau}^{r} c_\rho\right) \\ &= M^1(1 - c_\tau) \ . \end{aligned} \tag{3.23}$$

The maximal dual excess is, thus, obtained as

$$\mu(\boldsymbol{x}, \boldsymbol{v}^\star) = M^1(1 - \min_{\tau = 1, \ldots, r} c_t) \ , \tag{3.24}$$

and the maximal bi-excess

$$\tilde{\mu}(\boldsymbol{x}, \boldsymbol{v}) = \mu(\boldsymbol{x}, \boldsymbol{v}) + \mu(\boldsymbol{x}, \boldsymbol{v}^\star)$$

is described by

$$\tilde{\mu}(\boldsymbol{x}, \boldsymbol{v}) = M^1 \left[\left(1 - \sum_{\rho=1}^{r} c_\rho \frac{M^1}{M^\rho} \right) + \left(1 - \min_{\tau=1,\ldots,r} c_\tau \right) \right] . \tag{3.25}$$

Minimizing this expression amounts to solving the problem suggested by

$$\max \left\{ \sum_{\rho=1}^{r} c_\rho \frac{M^1}{M^\rho} + \min_{\tau=1,\ldots,r} c_\tau \,\middle|\, \boldsymbol{c} \text{ satisfies } (3.15) \text{ and } (3.21) \right\} . \tag{3.26}$$

We are now going to show that a maximizer of (3.26) has to assign constant weights to the non-minimal corners. Define a vector $\tilde{\boldsymbol{c}}$ by

$$\tilde{c}_1 = \cdots = \tilde{c}_\sigma = \frac{1 - (r - \sigma) \min_{\tau=1,\ldots,r} c_\tau}{\sigma}, \; \tilde{c}_{\sigma+1} = \cdots = \tilde{c}_\tau = \min_{\tau=1,\ldots,r} c_\tau$$

and observe that $\tilde{\boldsymbol{c}}$ satisfies (3.15) and (3.21). Also, $\min_{\tau=1,\ldots,r} \tilde{c}_\tau = \min_{\tau=1,\ldots,r} c_\tau$ holds true. Moreover we find

$$\begin{aligned}
& \sum_{\rho=1}^{r} \tilde{c}_\rho \frac{M^1}{M^\rho} + \min_{\tau=1,\ldots,r} \tilde{c}_\tau - \left(\sum_{\rho=1}^{r} c_\rho \frac{M^1}{M^\rho} + \min_{\tau=1,\ldots,r} c_\tau \right) \\
&= \sigma \frac{1 - (r - \sigma) \min_{\tau=1,\ldots,r} c_\tau}{\sigma} + \sum_{\rho=\sigma+1}^{r} \left(\min_{\tau=1,\ldots,r} c_\tau \right) \frac{M^1}{M^\tau} + \min_{\tau=1,\ldots,r} \tilde{c}_\tau \\
&\quad - \left(\sum_{\rho=1}^{r} c_\rho \frac{M^1}{M^\rho} + \min_{\tau=1,\ldots,r} c_\tau \right) \\
&= 1 - (r - \sigma) \min_{\tau=1,\ldots,r} c_\tau + \sum_{\rho=\sigma+1}^{r} (\min_{\tau=1,\ldots,r} c_\tau) \frac{M^1}{M^\rho} \\
&\quad - \left(\sum_{\rho=1}^{\sigma} c_\rho + \sum_{\rho=\sigma+1}^{r} c_\rho \frac{M^1}{M^\rho} \right) \\
&= \sum_{\rho=\sigma+1}^{r} c_\rho - (r - \sigma) \min_{\tau=1,\ldots,r} c_\tau + \sum_{\rho=\sigma+1}^{r} \left(\min_{\tau=1,\ldots,r} c_\tau - c_\rho \right) \frac{M^1}{M^\rho} \\
&\geq \sum_{\rho=\sigma+1}^{r} c_\rho - (r - \sigma) \min_{\tau=1,\ldots,r} c_\tau + \sum_{\rho=\sigma+1}^{r} \left(\min_{\tau=1,\ldots,r} c_\tau - c_\rho \right) = 0 \; .
\end{aligned}$$

Now consider the above inequality. The term in parenthesis is strictly negative unless $\boldsymbol{c} = \tilde{\boldsymbol{c}}$ holds true. Therefore, a maximizer $\boldsymbol{c}$ of (3.26) has to satisfy $c_1 \geq \cdots \geq c_r$. By (3.21) this vector $\boldsymbol{c}$ satisfies

$$c_1 = \cdots = c_\sigma \geq c_{\sigma+1} = \cdots = c_r =: \alpha^{\boldsymbol{c}}, \quad \sum_{\rho=1}^{r} c_\rho = 1 \ . \tag{3.27}$$

Note that for any $\alpha \in \mathbf{R}$ satisfying $0 \leq \alpha \leq 1$ there is a unique *underlying* vector $\boldsymbol{c}$ satisfying $\alpha^{\boldsymbol{c}} = \alpha$. Indeed, the vector $\boldsymbol{c}$ is defined by the requirement

$$c_1 = \cdots c_\sigma = \frac{1-(r-\sigma)\alpha}{\sigma}, \quad c_{\sigma+1} = \cdots = c_r = \alpha \ . \tag{3.28}$$

For convenience we use the expressions $\boldsymbol{x}^\alpha$ and $\boldsymbol{x}^{\boldsymbol{c}}$ synonymously. Then, e.g., $\hat{\boldsymbol{x}} = \boldsymbol{x}^{\hat{\alpha}}$ holds true and $\boldsymbol{x}^\alpha = r\alpha\bar{\boldsymbol{x}} + (1-r\alpha)\mathring{\boldsymbol{x}}$ is valid for any $\alpha \in \mathbf{R}$ in general.

Hence, the modiclus or rather the real number $\widehat{\alpha} := \alpha^{\hat{\boldsymbol{c}}}$ has to constitute a solution of the problem indicated by

$$\max\left\{ \left(\frac{1-(r-\sigma)\alpha}{\sigma}\right) \sum_{\rho=1}^{\sigma} \frac{M^1}{M^\rho} + \alpha \sum_{\rho=\sigma+1}^{r} \frac{M^1}{M^\rho} + \alpha \,\middle|\, 0 \leq \alpha \leq \frac{1}{r} \right\} \tag{3.29}$$

or by

$$\max\left\{ 1-(r-\sigma)\alpha + \alpha \sum_{\rho=\sigma+1}^{r} \frac{M^1}{M^\rho} + \alpha \,\middle|\, 0 \leq \alpha \leq \frac{1}{r} \right\} \ . \tag{3.30}$$

As $\sigma = \sum_{\tau=1}^{\sigma} \frac{M^1}{M^\tau}$ holds true by definition of σ, we have to determine

$$\operatorname{argmax}\left\{ \alpha\left(1 + \sum_{\rho=1}^{r} \frac{M^1}{M^\rho} - r\right) \,\middle|\, 0 \leq \alpha \leq \frac{1}{r} \right\} \ . \tag{3.31}$$

4th step: Inspection of the maximizing problem posed by (3.31) shows the following: Let $\bar{\alpha}$ be a maximizer of (3.31). If (3.5) or (3.7) respectively is satisfied, then $\bar{\alpha} = \frac{1}{r}$ or $\bar{\alpha} = 0$, respectively, must be true. The arising convexifying coefficients correspond to the case that $\widehat{\boldsymbol{x}} = \bar{\boldsymbol{x}}$ or $\widehat{\boldsymbol{x}} = \mathring{\boldsymbol{x}}$ respectively happens to be true. Hence, the first two assertions of the theorem are proved.

5th step: It remains to consider the case that (3.9) is true. Then the maximal bi-excess is the same for all convex combinations of $\bar{\boldsymbol{x}}$ and $\mathring{\boldsymbol{x}}$, i.e., the set of the maximizers given by (3.31) is in fact $\{\alpha \in \mathbf{R} \mid 0 \leq \alpha \leq \frac{1}{r}\}$. Now we observe that for any α satisfying $0 < \alpha < 1$ the maximal excess $\mu(\boldsymbol{x}^{\alpha}, \boldsymbol{v})$ is *only* attained by the coalitions in $\underline{\underline{\mathbf{D}}}^m$, i.e.,

$$\{S \in \underline{\underline{\mathbf{P}}} \mid e(S, \boldsymbol{x}^{\alpha}, \boldsymbol{v}) = \mu(\boldsymbol{x}^{\alpha}, \boldsymbol{v})\} = \underline{\underline{\mathbf{D}}}^m$$

holds true. Indeed, let S be a coalition attaining the maximal excess. By (3.17) we have

$$\boldsymbol{\lambda}^{\rho}(S) = \boldsymbol{v}(S) \quad (\rho = 1, \ldots, r) \text{ and } \boldsymbol{v}(S) = \boldsymbol{v}(I) \ .$$

If, in addition, $\alpha < \frac{1}{r}$, then the maximal dual excess $\mu(\boldsymbol{x}^{\alpha}, \boldsymbol{v}^{\star})$ is *only* attained by the corners C^{ρ} $(\rho = \sigma + 1, \ldots, r)$. Let S be a coalition of minimal primal excess such that $\boldsymbol{v}(S) = \boldsymbol{\lambda}^{\tau}(S)$ holds true for some τ. Then, by (3.22), $S = I - C^{\tau}$. Moreover, our condition $\alpha < \frac{1}{r}$ implies $\tau > \sigma$.

On the other hand

$$e(R, \mathring{\boldsymbol{x}}, \boldsymbol{v}) = \mu(\mathring{\boldsymbol{x}}, \boldsymbol{v}) = 0 \ ,$$

if R contains a maximal diagonal coalition and

$$e(C^{\tau}, \bar{\boldsymbol{x}}, \boldsymbol{v}^{\star}) = \mu(\bar{\boldsymbol{x}}, \boldsymbol{v}^{\star}) \quad (\tau = 1, \ldots, \sigma)$$

hold true. Thus, the set of pairs of coalitions that attain the maximal biexcess at $\boldsymbol{x}^0 = \mathring{\boldsymbol{x}}$ as well as at $\boldsymbol{x}^{\frac{1}{r}} = \bar{\boldsymbol{x}}$ strictly contains the set of pairs of coalitions attaining the maximal bi-excess at $\boldsymbol{x}^{\alpha}$ $(0 < \alpha < \frac{1}{r})$. The modiclus $\boldsymbol{x}^{\hat{\alpha}}$ is the imputation that lexicographically minimizes the vector of all bi-excesses. Hence, first of all it minimizes the maximal bi-excess and secondly it minimizes the number of pairs attaining this maximal bi-excess. We conclude that $0 < \widehat{\alpha} < \frac{1}{r}$ must hold.

Therefore, we shall consider the second highest bi-excess now. Let $0 < \alpha < \frac{1}{r}$ and $\boldsymbol{x} = \boldsymbol{x}^{\alpha}$.

6th step: First we consider the second highest primal excess. In view of (3.17) this excess can be attained either at the second largest diagonal coalitions (containing $M^1 - 1$ members of every corner) or at the coalition immediately "on top of $\underline{\underline{\mathbf{D}}}^m$" (containing $M^1 + 1$ members of some corner C^ρ satisfying $M^\rho = M^r$, and M^1 members of any other corner). By (3.17) and (3.24) we have:

$$0 + M^1 = \mu(\boldsymbol{x}^0, \boldsymbol{v}) + \mu(\boldsymbol{x}^0, \boldsymbol{v}^\star) = \tilde{\mu}(\boldsymbol{x}^\alpha, \boldsymbol{v}) = \mu(\boldsymbol{x}, \boldsymbol{v}) + M^1(1-\alpha) \ ,$$

thus (3.17) implies that the excess of the former coalitions coincides with $\mu(\boldsymbol{x}, \boldsymbol{v}) - \alpha$ and the excess of the latter coalitions coincides with $\mu(\boldsymbol{x}, \boldsymbol{v}) - \frac{M^1}{M^r}\alpha$.

Summarizing these facts the second highest excess $\mu_2(\alpha)$ at $\boldsymbol{x}^\alpha$ satisfies the expression:

$$\mu_2(\alpha) = \max\{e(S, \boldsymbol{x}^\alpha, \boldsymbol{v}) \mid S \in \underline{\underline{\mathbf{P}}} - \underline{\underline{\mathbf{D}}}^m\} = \mu(\boldsymbol{x}^\alpha, \boldsymbol{v}) - M^1 \frac{\alpha}{M^r} \ . \tag{3.32}$$

7th step: Now we turn to the second highest dual excess. Let $T \in \underline{\underline{\mathbf{P}}}$ be given. For any $\tau = 1, \ldots, r$ satisfying $\boldsymbol{v}(I - T) = \boldsymbol{\lambda}^\tau(I - T)$ (i.e., $\boldsymbol{v}^\star(T) = M^1 - \boldsymbol{\lambda}^\tau(I - T)$) formula (3.22) yields:

$$e(I - T, \boldsymbol{x}^\alpha, \boldsymbol{v}) > \begin{cases} e((I - T) \cup \{i\}, \boldsymbol{x}^\alpha, \boldsymbol{v}), & \text{if } i \in T - C^\tau \\ e((I - T) - \{j\}, \boldsymbol{x}^\alpha, \boldsymbol{v}), & \text{if } j \in (I - T) \cap C^\tau \end{cases} ,$$

thus

$$e(T, \boldsymbol{x}^\alpha, \boldsymbol{v}^\star) > \begin{cases} e(T - \{i\}, \boldsymbol{x}^\alpha, \boldsymbol{v}^\star), & \text{if } i \in T - C^\tau \\ e(T \cup \{j\}, \boldsymbol{x}^\alpha, \boldsymbol{v}^\star), & \text{if } j \in (I - T) \cap C^\tau \end{cases} . \tag{3.33}$$

Hence, the second highest dual excess can either be attained by all coalitions $C^\tau \quad (\tau = 1, \ldots, \sigma)$ or by coalitions consisting of one complete carrier and one additional agent. In the latter case the dual excess is maximal if the coalition can be written as $C^\rho \cup \{i\}$ such that $i \in C^\tau$ for some τ satisfying $M^\tau = M^r$ (i.e., agent i is a member of some maximal corner C^τ) and $\rho \in \{\sigma + 1, \ldots, r\} - \{\tau\}$. That is, the corner C^ρ is not of minimal size and it is not C^τ. In fact, (3.9) implies $r \geq \sigma + 2$, thus the coalitions $C^{r-1} \cup \{i\} \quad (i \in C^r)$ have the required shape. For such a coalition we have:

$$e(C^\rho \cup \{i\}, \boldsymbol{x}^\alpha, \boldsymbol{v}^\star) = \mu(\boldsymbol{x}^\alpha, \boldsymbol{v}^\star) - \alpha \frac{M^1}{M^r}. \tag{3.34}$$

In order to determine the dual excess in the former case let $\boldsymbol{c}$ be the underlying vector defined by (3.28). Then the dual excess is given by:

$$\begin{aligned} e(C^\tau, \boldsymbol{x}^\alpha, \boldsymbol{v}^\star) &= M^1(1 - c_1) = M^1(1 - \alpha) - M^1(c_1 + \alpha), \\ &= \mu(\boldsymbol{x}^\alpha, \boldsymbol{v}^\star) - M^1\left(\frac{1 - (r - \sigma)\alpha}{\sigma} + \alpha\right), \\ &= \mu(\boldsymbol{x}^\alpha, \boldsymbol{v}^\star) - M^1\left(\frac{1 - r\alpha}{\sigma}\right) \ . \end{aligned} \tag{3.35}$$

By Eqs. (3.34) and (3.35) the second highest dual excess $\mu_2^\star(\alpha)$ at $\boldsymbol{x}^\alpha$ is given by the expression:

$$\mu_2^\star(\alpha) = \mu(\boldsymbol{x}^\alpha, \boldsymbol{v}^\star) - M^1 \min\left\{\frac{\alpha}{M^r}, \frac{1 - r\alpha}{\sigma}\right\} \ . \tag{3.36}$$

8th step: The second highest bi-excess $\tilde{\mu}_2(\alpha)$ at $\boldsymbol{x}^\alpha$ satisfies the expression

$$\tilde{\mu}_2(\alpha) = \max\{\mu(\boldsymbol{x}^\alpha, \boldsymbol{v}) + \mu_2^\star(\alpha), \mu(\boldsymbol{x}^\alpha, \boldsymbol{v}^\star) + \mu_2(\alpha)\} \ .$$

By (3.36) and (3.32), we obtain:

$$\tilde{\mu}_2(\alpha) = \mu(\boldsymbol{x}^\alpha, \boldsymbol{v}) + \mu(\boldsymbol{x}^\alpha, \boldsymbol{v}^\star) - M^1 \min\left\{\frac{\alpha}{M^r}, \frac{1 - r\alpha}{\sigma}\right\} \ .$$

Hence, by the definition of the modiclus, $\widehat{\alpha}$ maximizes the expression

$$\min\left\{\frac{\alpha}{M^r}, \frac{1 - r\alpha}{\sigma}\right\} \ .$$

This expression possesses a unique maximizer, thus $\widehat{\alpha}$ is implicitly given by the requirement

$$\frac{\widehat{\alpha}}{M^r} = \frac{1 - r\widehat{\alpha}}{\sigma} \ . \tag{3.37}$$

The solution of (3.37) immediately yields (3.10) or (3.11). □

The previous proof deserves some explanation. The decisive quantity is given by

$$1 + \sum_{\rho=1}^{r} \frac{M^1}{M^\rho} \quad ,$$

which measures the relations of total initial assignments.

Obviously condition (3.5) is satisfied if the total assignments $M^\rho \quad (\rho = 1, \ldots, r)$ exceed the total assignment M^1 only moderately. Therefore, the conditions rendering the modiclus to be either the center of the core or the ideal point are interpreted in a most natural way: the ideal point appears when the long side of the market is not excessively large compared to the short side. Otherwise the modiclus will fall into the core and will be its central point. The intermediate situation represented by (3.9) is the borderline case in which the relation between the two extreme cases is carefully balanced.

Example 3.2: Let $\boldsymbol{v} = \bigwedge\{\boldsymbol{\lambda}^1, \ldots, \boldsymbol{\lambda}^\tau\}$ be an ocean game.

(1) If $\sigma \geq r - 1$ holds true, then formula (3.5) is automatically satisfied and, thus, the modiclus coincides with the ideal point in this case. In particular, this is true for $r = 2$.

(2) The same is true if the weights M^ρ only differ moderately, e.g., if $(r-1)M^1 > (r-2)M^r$ holds true.

(3) An example for the border case is obtained for $r \geq 3$ by the requirement $\sigma = r - 2$ and $M^r = M^{r-1} = 2M^1$. In this case the modiclus $\widehat{\boldsymbol{x}} = \boldsymbol{\psi}(\boldsymbol{v})$ can be computed via (3.11) which yields:

$$\hat{x}_i = \begin{cases} \frac{2M^1+1}{r-2+2rM^1}, & \text{if } i \in C^\tau \quad (\tau = 1, \ldots, r-2) \\ \frac{M^1}{r-2+2rM^1}, & \text{if } i \in C^\rho \quad (\rho = r-1, r) \end{cases} .$$

Hence, the quotient $\hat{x}_i(\boldsymbol{v})/\hat{x}_j(\boldsymbol{v})$ approaches 2 for $i \in \bigcup_{\rho=1}^{r-2} C^\rho$, $j \in C^{r-1} \cup C^r$ whenever M^1 approaches ∞. This means that equal treatment of the corners is "approximately" satisfied for a huge total minimal weight M^1.

It has now become clear how the bargaining power of the various cartels arises. The vector $\boldsymbol{c}$ of convexifying coefficients represents the shares of the corners. This is the result of the external bargaining process between the (representatives of the) various cartels. How is the modiclus capable of assigning positive coefficients to the long side of the market?

The third step provides the answer: the dual game $\boldsymbol{v}^\star$ represents the preventive power of coalitions and the maximal dual excess is attained at the corners C^ρ. Thus, the cartels consisting of the various corners muster the maximal preventive power. In particular, formula (3.24) shows that the smallest corners (cartels) command the maximal preventive power.

Accordingly, formula (3.25) shows the appropriate mixture of achievement power and preventive power which has to be minimized in order to achieve distribution according to the modiclus. The resulting minimizing problem (after reversing the sign a maximizing problem) is suggested by (3.28) and (3.31).

This shows that the external (or, from the view point of the agents inside a cartel exogenous) bargaining procedure between the cartels is reflected by the maximizing procedure that determines the coefficients leading to the modiclus.

The internal (or endogenous) discussion between the members of a cartel is a less complicated matter within the present framework. The symmetry properties of the modiclus ensure that the assignment to each member of a cartel is the same in view of the uniform distribution of initial assignments. In other words, since all members of a particular corner of the market look alike, there is little room for exercising an internal bargaining power.

When the initial assignment provides big chunks of a commodity to single agents (multiple gloves for one agent), then these matters are more involved. The endogenous bargaining process is much more difficult to capture. The mathematical intricacies increase rapidly and the economic relevance of the endogenous procedure has to be studied within the framework of an additional solution concept. Indeed, the "contested garment solution" introduced by Aumann-Maschler (1985) appears on the scene. It turns out that this concept (which is the nucleolus of an appropriately defined derived game) determines the shares of the players inside a cartel. For the details, we refer the reader to Rosenmüller and Sudhölter (2000).

4 Conclusions

We have discussed the behavior of our solution concept, the modiclus or modified nucleolus, for ocean games. These games are multi–sided glove games with uniformly distributed initial assignments in each corner of the market. Theorem 3.1 explains this in detail. If corners $\rho = 1, \ldots, \sigma$ denote the short side of the market, then the core of the game is the convex hull of the involved measures (initial assignments) $\boldsymbol{\lambda}^1, \ldots, \boldsymbol{\lambda}^\sigma$, i.e.,

$$\mathscr{C}(\boldsymbol{v}) = \left\{ \sum_{\rho=1}^{\sigma} c_\rho \boldsymbol{\lambda}^\rho \,\middle|\, c_\rho \geq 0 \ (\rho = 1, \ldots, \sigma), \sum_{\rho=1}^{\sigma} c_\rho = 1 \right\} . \tag{4.1}$$

As for the various concepts that show a different behavior we have discussed the Shapley value and the Walrasian equilibrium in the introduction. The nucleolus is a core allocation, hence by symmetry it equals the center of the core i.e.,

$$\boldsymbol{\nu}(\boldsymbol{v}) = \mathring{\boldsymbol{x}} = \frac{1}{\sigma} \sum_{\rho=1}^{\sigma} \boldsymbol{\lambda}^\rho \ . \tag{4.2}$$

Hence, the nucleolus coincides with the modiclus when the long side of the market exceeds the short side excessively (see (3.9)) and hence the bargaining power of the former one is rather limited. On the other hand, this shows that the nucleolus is *not* capable of explaining cartelization endogenously.

For the per capita nucleolus the same result holds true with the same reasoning.

It may be of interest to remark that the "preventive power" of coalitions may be introduced with respect to some of the above concepts. That is, given a solution concept one can study the dual-cover game (see (1.2)). One can then compute the solution of the dual-cover game and consider its projection onto the original player set. This procedure is parallel to the construction of the modiclus based on the nucleolus.

Note that the mechanism applied to the core concept does not yield a reasonable solution: the dual cover of a balanced game is not balanced unless the game is additive (inessential).

For the per capita nucleolus our procedure makes sense. However, the complete description of the "per capita modiclus" is a formidable task that requires additional research and is well beyond the scope of this paper.

As for the Shapley value, we obtain a result – but not a new concept. This we are going to explain in passing.

Let $\Phi(\boldsymbol{v})$ denote the Shapley value of a game $\boldsymbol{v}$. With $s = |S|$ for $S \subseteq I$ and $n = |I|$ we have (cf. Shapley, 1953)

$$\Phi_i(\boldsymbol{v}) = \sum_{S \subseteq I - \{i\}} \frac{s!(n-s-1)!}{n!} (\boldsymbol{v}(S \cup \{i\}) - \boldsymbol{v}(S)) \ (i \in I) \ , \quad (4.3)$$

thus

$$\Phi_i(\boldsymbol{v}) = \sum_{S \subseteq I - \{i\}} \frac{s!(n-s-1)!}{n!} (\boldsymbol{v}(I - S) - \boldsymbol{v}((I-S) - \{i\})) \ (i \in I) \ . \quad (4.4)$$

Equations (4.3) and (4.4) yield for all $i \in I$

$$2\Phi_i(\boldsymbol{v}) = \sum_{S \subseteq I - \{i\}} \frac{s!(n-s-1)!}{n!} (\boldsymbol{v}(S \cup \{i\}) + \boldsymbol{v}(I-S) - \boldsymbol{v}(S) - \boldsymbol{v}((I-S) - \{i\})) \ . \quad (4.5)$$

Remark 4.1: Let $S \subseteq I$. Then $\boldsymbol{v}(S) \geq \boldsymbol{v}^\star(S)$ if and only if $\boldsymbol{v}(I - S) \geq \boldsymbol{v}^\star(I - S)$.

Lemma 4.2: Let $\boldsymbol{u}$ be the game defined by:

$$\boldsymbol{u}(S) := \max\{\boldsymbol{v}(S), \boldsymbol{v}^\star(S)\} \ (S \in \underline{\underline{\mathbf{P}}}) \ .$$

Then $\Phi(\boldsymbol{u}) = \Phi(\boldsymbol{v})$ holds true.

Proof: Let $i \in I$ and $S \subseteq I - \{i\}$. By (4.5) it suffices to show that

$$\begin{aligned} \alpha : &= \boldsymbol{v}(S \cup \{i\}) + \boldsymbol{v}(I-S) - \boldsymbol{v}(S) - \boldsymbol{v}((I-S) - \{i\}) \\ &= \boldsymbol{u}(S \cup \{i\}) + \boldsymbol{u}(I-S) - \boldsymbol{u}(S) - \boldsymbol{u}((I-S) - \{i\}) =: \beta \ . \end{aligned} \quad (4.6)$$

Four cases may be distinguished:

(1) $\boldsymbol{u}(S \cup \{i\}) = \boldsymbol{v}(S \cup \{i\})$ and $\boldsymbol{u}(I - S) = \boldsymbol{v}(I - S)$: by Remark 4.1, $\boldsymbol{u}(S) = \boldsymbol{v}(S)$ and $\boldsymbol{u}((I - S) - \{i\}) = \boldsymbol{v}((I - S) - \{i\})$, thus (4.6) is valid.

(2) $\boldsymbol{u}(S \cup \{i\}) = \boldsymbol{v}(S \cup \{i\})$ and $\boldsymbol{u}(I - S) = \boldsymbol{v}^*(I - S)$: by Remark 4.1, $\boldsymbol{u}(S) = \boldsymbol{v}^*(S)$ and $\boldsymbol{u}((I - S) - \{i\}) = \boldsymbol{v}((I - S) - \{i\})$, thus

$$\begin{aligned}\beta &= \boldsymbol{v}(S \cup \{i\}) + \boldsymbol{v}^*(I - S) - \boldsymbol{v}^*(S) - \boldsymbol{v}((I - S) - \{i\}) \\ &= \boldsymbol{v}(S \cup \{i\}) + \boldsymbol{v}(I) - \boldsymbol{v}(S) - \boldsymbol{v}(I) + \boldsymbol{v}(I - S) - \boldsymbol{v}((I - S) - \{i\}) = \alpha \ .\end{aligned}$$

(3) The case $\boldsymbol{u}(S \cup \{i\}) = \boldsymbol{v}^*(S \cup \{i\})$ and $\boldsymbol{u}(I - S) = \boldsymbol{v}(I - S)$ can be treated analogously to case 2 by interchanging the roles of $S \cup \{i\}$ and $I - S$.

(4) $\boldsymbol{u}(S \cup \{i\}) = \boldsymbol{v}^*(S \cup \{i\})$ and $\boldsymbol{u}(I - S) = \boldsymbol{v}^*(I - S)$: by Remark 4.1, $\boldsymbol{u}(S) = \boldsymbol{v}^*(S)$ and $\boldsymbol{u}((I - S) - \{i\}) = \boldsymbol{v}^*((I - S) - \{i\})$, thus the definition of the dual game again implies $\beta = \boldsymbol{v}^*(S \cup \{i\}) + \boldsymbol{v}^\star(I - S) - \boldsymbol{v}^*(S) - \boldsymbol{v}^*((I - S) - \{i\}) = \alpha$. □

Corollary 4.3: The projection of the Shapley value of the dual game onto the primal player set coincides with the Shapley value, formally

$$\Phi(\bar{\boldsymbol{v}})_{|I} = \Phi(\boldsymbol{v}) \ .$$

Proof: The proof is based on Lemma 1.7 of Sudhölter (1998). Let $\bar{\boldsymbol{v}}$ be the dual cover of (this is a game on the player set $I \times \{0, 1\} = I^{1,2}$). Also, let $\bar{\boldsymbol{v}}$ be defined by

$$\bar{\boldsymbol{v}}(S^0 \cup T^1) = \boldsymbol{v}(S^0) + \boldsymbol{v}^\star(T^1) \ (S^0 \subseteq I^0, T^1 \subseteq I^1) \ ;$$

this game lives on the same player set. By Lemma 4.2 we obtain $\Phi(\tilde{\boldsymbol{v}}) = \Phi(\tilde{\boldsymbol{v}})$. Hence Lemma 1.7 of Sudhölter (1997) completes the proof. □

The above presentation shows that our procedure applied to the Shapley preserves the Shapley value. The Shapley value (in all versions) *does* respect the bargaining power of the long side of a market, but less and less with increasing size of the player set.

Our result is, therefore, that the modiclus is somehow unique in its behavior: given a moderate relative size of the corners of an ocean game it preserves the bargaining power of the long side (resulting from cartelization) for arbitrarily large replications of the game.

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