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Cartels via the modiclus

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Abstract

We discuss market games or linear production games with finite sets of players. The representing distributions of initial assignments are assumed to have disjoint carriers. Thus, the agents decompose into finitely many disjoint groups each of which hold a corner of the market. In such a market traditional solution concepts like the core tend to favor the short side of the market excessively. We exhibit a solution concept which is more sensitive with respect to the preventive power of the long side. Thereby, profits of the long side are now feasible. This concept is the modified nucleolus or modiclus. Within certain limits, it predicts cartelization and assigns a "fair share" for cartels on the long side of the market. Also, it organizes the internal distribution for a specific cartel according to the "contested garment solution" of Aumann–Maschler.
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1. Introduction

Within this paper, we continue to explain the *endogenous* formation of cartels in large markets. The model is provided by a cooperative totally balanced game with a potentially large but finite set of players or agents. Within this framework, we discuss the formation of cartels predicted by a point-valued solution concept, the modified nucleolus or *modiclus*.

This concept respects the blocking power of a cartel: the result is not only influenced by what a coalition of traders can attain but also what they can prevent others to achieve. The modiclus formalizes the idea of the *preventive power* of a coalition.

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Formally, the tool to assess this preventive power of a coalition is the *dual game*. The dual game assigns to a coalition the complementary worth of the complementary coalition. Hence, if the complementary coalition is powerful then the original coalition is weak and vice versa.

Therefore, for market games with distinct separate corners, this concept assigns positive worth to the long side of the market.

Most solution concepts of cooperative game theory do not respect any bargaining achievement of the long side of the market, at least not when the game is large (i.e., in a replicated version or a nonatomic model).

We refer to a paper by Hart [3]. This author points out that the Walrasian equilibrium or the core are unable to predict the endogenous formation of cartels within corners of the market. Hart favors the vNM-stable set for his discussion. Indeed, this concept seems to be able to predict cartelization. A more recent result by Rosenmüller–Shitovitz [7] about the characterization of convex vNM-stable sets corroborates his analysis.

We believe that the success of the vNM-stable set is due to the external stability of this solution concept. External stability provides some preventive power for coalitions during the bargaining process.

In the present context the modiclus provides preventive forces for coalitions, because it involves the dual game. Let us shortly describe our concept. The framework is the one of cooperative game theory, which we introduce as follows.

Consider a coalitional *game* given by triple $(I, \underline{\underline{\mathbf{P}}}, \boldsymbol{v})$, here I is the (finite) set of *agents* or *players*, $\underline{\underline{\mathbf{P}}}$ the power set of I, called *system of coalitions* and

$$\boldsymbol{v} : \underline{\underline{\mathbf{P}}} \to \mathbb{R}, \quad \boldsymbol{v}(\emptyset) = 0,$$

a real-valued function on $\underline{\underline{\mathbf{P}}}$, the *coalitional function*. The *dual game* is given by

$$\boldsymbol{v}^{\star}(S) := \boldsymbol{v}(I) - \boldsymbol{v}(I - S) \quad (S \in \underline{\underline{\mathbf{P}}}) \tag{1}$$

and reflects the preventive power of coalitions.

The modiclus is a nucleolus type concept [10]. Recall the procedure that yields the *nucleolus*: for any *preimputation* $\boldsymbol{x}$ (i.e., $\boldsymbol{x} \in \mathbb{R}^I, \boldsymbol{x}(I) = \boldsymbol{v}(I)$), one lists the excesses

$$e(S, \boldsymbol{x}, \boldsymbol{v}) = \boldsymbol{v}(S) - \boldsymbol{x}(S)$$

(*reasons to complain*) in a (weakly) decreasing order, say

$$\theta(\boldsymbol{x}) := (\ldots, e(S, \boldsymbol{x}, \boldsymbol{v}), \ldots). \tag{2}$$

Then the *prenucleolus* $\boldsymbol{\nu}$ is the unique preimputation such that $\theta(\bullet)$ is lexicographically minimal, i.e.,

$$\theta(\boldsymbol{\nu}) \preceq_{\text{lexic}} \theta(\boldsymbol{x}) \quad \text{for all preimputations } \boldsymbol{x}. \tag{3}$$

In order to obtain the *modified nucleolus* or *modiclus* $\boldsymbol{\psi}$, one lists *bi-excesses*

$$e(S, \boldsymbol{x}, \boldsymbol{v}) - e(T, \boldsymbol{x}, \boldsymbol{v})$$

and proceeds accordingly. As differences of excesses ("*bi-excesses*") are sums of excesses of the primal and dual game, the modiclus represents achievement powers and preventive powers of coalitions alike.

For further intuitive insight, it is useful to construct another game which incorporates $\boldsymbol{v}$ and $\boldsymbol{v}^\star$ simultaneously. This game is the *dual cover*. Take two copies of the set of players or agents, say

$$I^{1,2} = I \times \{0,1\}$$

and define a game $\bar{\boldsymbol{v}}$: $\underline{\underline{\mathbf{P}}}^{1,2} \to \mathbb{R}$ on the coalitions of this set (the power sets are indexed canonically) via

$$\bar{\boldsymbol{v}}(S+T) := \max\{\boldsymbol{v}(S) + \boldsymbol{v}^\star(T), \boldsymbol{v}(T) + \boldsymbol{v}^\star(S)\} \quad (S \in \underline{\underline{\mathbf{P}}}^0, T \in \underline{\underline{\mathbf{P}}}^1) \tag{4}$$

(we use $+$ instead of $\cup$ for disjoint unions). The game $\bar{\boldsymbol{v}}$ takes pairs of coalitions into account, in one of them players act "constructively" and in the other one "preventively." This game reflects the combined influence of the game and its dual. Now, $\bar{\boldsymbol{v}}$ is defined on $I^{1,2}$. We obtain a concept defined on the original set of players by taking the *projection* of the prenucleolus of the *dual cover game* on the original player set I. As it turns out, (see [14]), this is the modiclus $\boldsymbol{\psi}$.

The analysis of the modiclus describes the exogenous or external bargaining process (between representatives of the cartels) as well as the endogenous (internal) bargaining process (inside a specific cartel). In [8] we consider the case of uniformly distributed initial assignments. This model already exhibits economic relevance in particular with respect to the external bargaining process. Technically, it admits of easy and more direct proofs.

The internal bargaining process inside a cartel is a much more complicated matter. In corners with uniformly distributed initial assignments, the symmetry properties of the modiclus yield a symmetric payoff. Hence, equal treatment prevails. However, in a corner with various initial assignments a serious problem arises: How should the internal bargaining process be captured? Therefore, we have to come up with a completely new approach which heavily rests on the (generalized) concept of *reduced games* (see Lemma 2.7).

Indeed, the modiclus is very sensitive towards the initial assignment. The internal bargaining process takes two "internal games" into account and carefully computes the resulting payoffs. One of these games is the reduced game which results from the distribution obtained by the external bargaining process in the sense of Davis–Maschler [2]. The second game is even more interesting: It turns out that one has to consider a "contested garment game" as discussed by Aumann–Maschler [1]. In this game, the various members of a specific cartel have certain claims which implicitly result from their ability to form efficient coalitions with players outside the cartel. These claims (like those in the contested garment game) are not totally realizable. The "estate," that is the assignment to the cartel by the external bargaining process, is limited and hence the coalitions worth is also limited by the size of the garment. It turns out that the contested garment solution, the reduced game and the external bargaining process provided by the modiclus have to be carefully knitted together in order to provide the internal share of a player according to the modiclus concept. For the details see Section 6.

The paper is organized as follows. In Section 2 we introduce the model, recall some important definitions and discuss simple properties of excesses. Section 3 exhibits the formation of cartels: the treatment of the various corners of the market is described for markets with a certain weak balancedness property. Under mild additional assumptions the corners of the long side of the market are treated equally and proportional to their total initial assignment. Further results of Section 4 shows that the nucleolus of a certain balanced game describes the amounts given to the players of the remaining corners of the short side.

Section 5 shows that the assumptions employed in the other sections are automatically satisfied, if the game is "sufficiently large." This can be ensured by e.g., by replication of the market.

Furthermore, Sections 6 and 7 exhibit the assignments to the various members of the cartels, reflecting the internal discussion within the cartels.

Finally, Section 8 contains examples and remarks.

2. Definitions, simple properties

A game, as explained in Section 1, is a triple $(I, \underline{\underline{\mathbf{P}}}, \boldsymbol{v})$ satisfying $\boldsymbol{v}(\emptyset) = 0$. It is not unusual to sloppily use the term just for the coalitional function and not always for the triple. We are predominantly interested in *market games* or *totally balanced games* which can be generated from exchange economies [11]. In order to represent such a game we use the representation as a minimum game. That is, $\boldsymbol{v}$ is the minimum of finitely many nonnegative additive set functions (distributions or *measures*), say $\boldsymbol{\lambda}^1, \ldots, \boldsymbol{\lambda}^r \in \mathbb{R}^I_+$, defined on $\underline{\underline{\mathbf{P}}}$ via $\boldsymbol{v}(S) = \min\{\boldsymbol{\lambda}^1(S), \ldots, \boldsymbol{\lambda}^r(S)\}$ $(S \in \underline{\underline{\mathbf{P}}})$. This we write conveniently

$$\boldsymbol{v} = \bigwedge\{\boldsymbol{\lambda}^1, \ldots, \boldsymbol{\lambda}^r\}. \tag{5}$$

According to Kalai–Zemel [4], every totally balanced game can be represented this way. Their interpretation is that $\boldsymbol{v}$ can be seen as a network game within which players command certain nodes of a network-flow setup. A traditional example is that of a glove game. Here, coalitions need to combine indispensable factors (right-hand and left-hand gloves) in order to acquire utility by selling the product (pairs of gloves) on some external market.

We wish to concentrate on the *orthogonal* case, that is, the *carriers* of $\boldsymbol{\lambda}^\rho$, denoted by $C(\boldsymbol{\lambda}^\rho) = C^\rho$ $(\rho = 1, \ldots, r)$, are disjoint. Also we shall assume that $I = \sum_{\rho=1}^r C^\rho$ describes a partition of I (each player owns a quantity of one and only one factor). Finally, we assume that there are at least two measures (i.e., $r \geqslant 2$), because for $r = 1$ the game $\boldsymbol{v}$ is additive. Let us use the term *min-game* for a game that satisfies these requirement.

Orthogonality is certainly a restriction within the class of market games. The shape of a min-game appears more drastically, a coalition which completely lacks one factor receives no utility. Thus, players occupy r different *corners* of the market, each one defined by possession of a sole factor. The terms corner and carrier are synonyms in this view.

We use the abbreviation M^ρ in order to indicate the total mass of $\boldsymbol{\lambda}^\rho$, that is, the total initial assignment of goods in corner C^ρ, formally

$$M^\rho := \boldsymbol{\lambda}^\rho(I) = \boldsymbol{\lambda}^\rho(C^\rho) = \sum_{i \in C^\rho} \boldsymbol{\lambda}_i^\rho. \tag{6}$$

For convenience, the corners of the market are ordered according to total initial assignment, i.e., $M^1 \leqslant \cdots \leqslant M^r$ is satisfied. The min-game $\boldsymbol{v}$ given by (5) is not changed, if every weight λ_i^ρ $(\rho = 1, \ldots, r, i \in I)$ is replaced by the minimum of M^1 and this weight, thus $\lambda_i^\rho \leqslant M^1$ is generally assumed. Then the representation of the min-game is unique. Let

$$\sigma := |\{\rho \in \{1, \ldots, r\} | M^\rho = M^1\}|$$

denote the number of minimal corners.

Any coalition $S \in \underline{\underline{\mathbf{P}}}$ decomposes naturally into the coalitions of its *partners* in the various corners, this we write

$$S = \sum_{\rho=1}^{r} S^\rho \quad \text{with } S^\rho = S \cap C^\rho \ \ (\rho = 1, \ldots, r). \tag{7}$$

(We use $+$ instead of $\cup$ to indicate the union of two coalitions if and only if the coalitions are disjoint.)

A further important system of coalitions is provided by the *diagonal* which is formally given by

$$\underline{\underline{\mathbf{D}}} := \{S \in \underline{\underline{\mathbf{P}}} | \boldsymbol{\lambda}^\rho(S) = \boldsymbol{v}(S) \quad (\rho = 1, \ldots, r)\}. \tag{8}$$

A coalition $S \in \underline{\underline{\mathbf{D}}}$ is called a *diagonal coalition* because the image of S under the vectorvalued measure $(\boldsymbol{\lambda}^1, \ldots, \boldsymbol{\lambda}^r)$ is located on the diagonal of $\mathbb{R}^r$. Economically, diagonal coalitions are efficient, as there is no excess supply of factors available in order to generate $\boldsymbol{v}(S)$. Note that on diagonal sets, $\boldsymbol{v}$ behaves additively. As a consequence, it is not hard to see that any core element $\boldsymbol{x}$ equals the game on the diagonal $(\boldsymbol{x}(S) = \boldsymbol{v}(S)\ (S \in \underline{\underline{\mathbf{D}}}))$. In this sense, diagonal coalitions S are also *effective*: they can afford $\boldsymbol{x}(S)$ by their own productive power.

Within the diagonal we are particularly interested in *maximal* elements. These are diagonal coalitions S such that each corner assembles the maximal possible amount of goods and hence the coalition's worth is $\boldsymbol{v}(I)$. More precisely, such coalitions satisfy

$$\boldsymbol{\lambda}^1(S) = \cdots = \boldsymbol{\lambda}^r(S) = M^1. \tag{9}$$

The system of maximal coalitions is denoted by

$$\underline{\underline{\mathbf{D}}}^m := \{S \in \underline{\underline{\mathbf{P}}} \,|\, S \text{ satisfies } (9)\}. \tag{10}$$

The notion of *excess* is central to the discussion of nucleolus type solution concepts. Given a vector $\boldsymbol{x} \in \mathbb{R}^I$, recall that the excess of a coalition $S \in \underline{\underline{\mathbf{P}}}$ (cf. Section 1) is

$$e(S, \boldsymbol{x}, \boldsymbol{v}) = \boldsymbol{v}(S) - \boldsymbol{x}(S). \tag{11}$$

This quantity measures the amount by which coalition S misses its worth $\boldsymbol{v}(S)$, hence is dissatisfied with $\boldsymbol{x}$. The *maximal excess* of $\boldsymbol{v}$ at $\boldsymbol{x}$ is

$$\mu(\boldsymbol{x}, \boldsymbol{v}) := \max\{e(S, \boldsymbol{x}, \boldsymbol{v}) \mid S \in \underline{\underline{\mathbf{P}}}\}. \tag{12}$$

The task of computing excesses is a frequently imposed burden; we start out with some versions concerning min-games. An *imputation* $\boldsymbol{x}$ of a game $(I, \underline{\underline{\mathbf{P}}}, \boldsymbol{v})$ is a vector $\boldsymbol{x} \in \mathbb{R}^I$ satisfying *Pareto optimality* (i.e., $\boldsymbol{x}(I) = \boldsymbol{v}(I)$) and *individual rationality* (i.e., $x_i \geqslant \boldsymbol{v}(\{i\})$ $(i \in I)$). If $\boldsymbol{v}$ is the min-game given by (5) then an imputation $\boldsymbol{x}$ satisfies

$$x_i \geqslant 0 \quad (i \in I) \text{ and } \boldsymbol{x}(I) = M^1,$$

thus $x_i \leqslant \lambda_i^\rho$ holds true for any $i \in C^\rho$ and any corner C^ρ. This means that $\boldsymbol{x}$ can be written as

$$\boldsymbol{x} = M^1 \sum_{\rho=1}^{r} c_\rho \frac{\boldsymbol{\mu}^\rho}{M^\rho} \tag{13}$$

such that the $\boldsymbol{c} := (c_\rho)_{\rho=1,\ldots,r}$ is a vector of nonnegative coefficients summing up to 1 (the vector of *convexifying coefficients*) and $\boldsymbol{\mu}^\rho$ $(\rho = 1, \ldots, r)$ are *normalized* measures, i.e., measures with carriers C^ρ, having the same total mass $\boldsymbol{\mu}^\rho(C^\rho) = M^\rho$ as $\boldsymbol{\lambda}^\rho$. Conversely, any vector $\boldsymbol{c}$ of convexifying coefficients together with normalized measures $\boldsymbol{\mu}^\rho$ $(\rho = 1, \ldots, r)$ determines an imputation $\boldsymbol{x}$ by (13).

Here is the first simple Lemma:

Lemma 2.1. *Let* $\boldsymbol{v}$ *be a min-game given by* (5) *and* $\boldsymbol{c}$ *be a vector of convexifying coefficients. Let* $\boldsymbol{x}$ *be an imputation of the form*

$$\boldsymbol{x} = M^1 \sum_{\rho=1}^{r} c_\rho \frac{\boldsymbol{\mu}^\rho}{M^\rho} \tag{14}$$

satisfying $x_i \leqslant \lambda_i^\rho$ $(i \in C^\rho, \rho = \sigma + 1, \ldots, r)$ *and let* $S \in \underline{\underline{\mathbf{P}}}$ *be any coalition*:

(1) *The excess of* S *is given by*

$$e(S, \boldsymbol{x}, \boldsymbol{v}) = \boldsymbol{v}(S)\left(1 - \sum_{\rho=1}^{r} c_\rho \frac{M^1}{M^\rho}\right) - M^1 \sum_{\rho=1}^{r} c_\rho \frac{\boldsymbol{\mu}^\rho(S) - \boldsymbol{v}(S)}{M^\rho}. \tag{15}$$

(2) *For any* $\tau = 1, \ldots, \sigma$ *the dual excess of* S *satisfies*

$$e(S, \boldsymbol{x}, \boldsymbol{v}^\star) \leqslant \max\{M^1(1 - c_\rho) - \boldsymbol{x}(S - S^\rho) \mid \rho = \tau + 1, \ldots, r\} \tag{16}$$

or

$$e(S, \boldsymbol{x}, \boldsymbol{v}^\star) \leqslant \max\{\boldsymbol{\lambda}^\rho(S) - \boldsymbol{x}(S^\rho) - \boldsymbol{x}(S - S^\rho) \mid \rho = 1, \ldots, \tau\}. \tag{17}$$

Proof. The equation

$$
\begin{aligned}
e(S,\boldsymbol{x},\boldsymbol{v}) &= \boldsymbol{v}(S) - \boldsymbol{x}(S) \\
&= \boldsymbol{v}(S) - M^1 \sum_{\rho=1}^{r} c_\rho \frac{\boldsymbol{\mu}^\rho(S)}{M^\rho} \\
&= \boldsymbol{v}(S)\left(1 - \sum_{\rho=1}^{r} c_\rho \frac{M^1}{M^\rho}\right) - M^1 \sum_{\rho=1}^{r} c_\rho \frac{\boldsymbol{\mu}^\rho(S) - \boldsymbol{v}(S)}{M^\rho}
\end{aligned}
$$

shows (15).

Choose ρ_0 satisfying $\boldsymbol{v}^\star(S) = M^1 - \boldsymbol{\lambda}^{\rho_0}(I-S)$. If $\rho_0 > \tau$ is valid, then the observation

$$
\begin{aligned}
e(S,\boldsymbol{x},\boldsymbol{v}^\star) &= M^1 - \boldsymbol{\lambda}^{\rho_0}(I-S) - \boldsymbol{x}(S^{\rho_0}) - \boldsymbol{x}(S - S^{\rho_0}) \\
&\leqslant M^1 - \boldsymbol{x}(C^{\rho_0}) - \boldsymbol{x}(S - S^{\rho_0}) \qquad (\text{because } x_i \leqslant \lambda_i^{\rho_0}\ (i \in C^{\rho_0})) \\
&= M^1(1 - c_{\rho_0}) - \boldsymbol{x}(S - S^{\rho_0})
\end{aligned}
$$

implies (16). If $\rho_0 \leqslant \tau$, then (17) is implied by the equation

$$
\begin{aligned}
e(S,\boldsymbol{x},\boldsymbol{v}^\star) &= M^1 - \boldsymbol{\lambda}^{\rho_0}(I-S) - \boldsymbol{x}(S) \\
&= \boldsymbol{\lambda}^{\rho_0}(S) - \boldsymbol{x}(S) \quad (\text{because } \tau \leqslant \sigma). \qquad \square
\end{aligned}
$$

The first part of the lemma emphasizes the rôle of the diagonal, in particular that of the maximal diagonal, in the case that the imputation is a convex combination of the underlying measures. Indeed, it directly implies the following result.

Corollary 2.2. *Let $\boldsymbol{v}$ and $\boldsymbol{c}$ satisfy the assumptions of Lemma* 2.1 *and let $\boldsymbol{x}$ be the imputation given by*

$$
\boldsymbol{x} = M^1 \sum_{\rho=1}^{r} c_\rho \frac{\boldsymbol{\lambda}^\rho}{M^\rho}. \tag{18}
$$

If $S \in \underline{\underline{\mathbf{P}}}$ is a coalition and $\tilde{S} \in \underline{\underline{\mathbf{D}}}$ is a diagonal coalition satisfying $\boldsymbol{v}(\tilde{S}) \geqslant \boldsymbol{v}(S)$, then

$$
e(\tilde{S},\boldsymbol{x},\boldsymbol{v}) \geqslant e(S,\boldsymbol{x},\boldsymbol{v}) \tag{19}
$$

holds true.

Proof. The inequalities $M^1 \leqslant M^\rho$ $(\rho = 1,\ldots,r)$ directly imply

$$
\delta := \left(1 - \sum_{\rho=1}^{r} c_\rho \frac{M^1}{M^\rho}\right) \geqslant 0,
$$

thus we obtain

$$e(\tilde{S},\boldsymbol{x},\boldsymbol{v}) - e(S,\boldsymbol{x},\boldsymbol{v}) = (\boldsymbol{v}(\tilde{S}) - \boldsymbol{v}(S))\delta + M^1 \sum_{\rho=1}^{r} c_\rho \frac{\boldsymbol{\lambda}^\rho(S) - \boldsymbol{v}(S)}{M^\rho} \geqslant 0. \qquad \square$$

Due to the results of Kohlberg [5] there is a closed connection between a nucleolus type concept and the balanced systems of coalitions it generates via the various levels of excesses. Let us shortly introduce our notion of balancedness. We use a slightly more general version which refers to collections of vectors (and induces the notions for systems of coalitions).

Let $S \in \underline{\underline{\mathbf{P}}}$, $S \neq \emptyset$ be a coalition. A finite nonempty collection of vectors $\mathscr{X} \subseteq \mathbb{R}^S$ is said to be *balanced* with respect to $z \in \mathbb{R}^S$, (or just "balances $\boldsymbol{z}$") if there is a sequence of *balancing coefficients* $(b_{\boldsymbol{x}})_{\boldsymbol{x}\in\mathscr{X}}$ satisfying

$$b_{\boldsymbol{x}} > 0 \quad \text{and} \quad \sum_{x\in\mathscr{X}} b_{\boldsymbol{x}}\boldsymbol{x} = \boldsymbol{z}. \tag{20}$$

Moreover, we shall say that $\mathscr{X}$ is just *balanced*, if it is balanced with respect to $(1,\ldots,1)\in\mathbb{R}^S$. Switching to systems of coalitions means to refer to the indicator function. Thus, If $\underline{\underline{\mathbf{S}}} \subseteq \underline{\underline{\mathbf{P}}}$ is a nonempty system of coalitions such that $S \subseteq T$ $(S \in \underline{\underline{\mathbf{S}}})$ is true for some $T \in \underline{\underline{\mathbf{P}}}$, then we say that $\underline{\underline{\mathbf{S}}}$ is balanced with respect to T, if the collection $\{1_S \mid S \in \underline{\underline{\mathbf{S}}}\}$ balances 1_T. This amounts to the traditional notion. However, in the context of the modiclus, systems of *pairs* of coalitions are relevant. Indeed, we shall say that a nonempty system $\tilde{\underline{\underline{\mathbf{S}}}} \subseteq \underline{\underline{\mathbf{P}}} \times \underline{\underline{\mathbf{P}}}$ of pairs of coalitions is balanced w.r.t. some coalition U, if the collection $\{1_R + 1_T | (R,T) \in \tilde{\underline{\underline{\mathbf{S}}}}\}$ balances 1_U. Of course we say that a system of coalitions or a system of pairs of coalitions, respectively, is *balanced*, if the system balances the grand coalition I.

We are particularly interested in balanced systems that span the corresponding subspace generated by the indicator functions. This is based on the following remark which is due to Sudhölter [14, Remark 2.7].

Remark 2.3. Let $\mathscr{X} \subseteq \mathbb{R}^I$ be a finite collection of vectors and let $\boldsymbol{z} \in \mathbb{R}^I$. Assume that $\mathscr{X}$ balances $\boldsymbol{z}$. Also, let $\mathscr{Y} \subseteq \mathbb{R}^I$ be a finite collection which contains $\mathscr{X}$. If $\mathscr{Y}$ is contained in the linear span of $\mathscr{X}$, then $\mathscr{Y}$ balances $\boldsymbol{z}$ as well.

Clearly this remark greatly increases the possibilities of recognizing a system or collection as balanced. For, usually a system we are dealing with is rather large and unaccessible, so the construction of balancing coefficients is quite out of the question. However, the general technique is to single out a subsystem which is balanced *and spanning* in the above sense. Then the above remark does the job.

The notion of nondegeneracy is introduced as follows (cf. [9]). A finite collection $\mathscr{X} \subseteq \mathbb{R}^I$ is *nondegenerate*, if it spans $\mathbb{R}^I$. Analogously, a system $\underline{\underline{\mathbf{S}}}$ of coalitions or a system $\tilde{\underline{\underline{\mathbf{S}}}}$ of pairs of coalitions, respectively, is said to be *nondegenerate* w.r.t. some coalition T, if the collection of corresponding indicators or sums of pairs of indicators, respectively, spans $\mathbb{R}^T$ and T is the union of all coalitions involved.

Occasionally, we shall also deal with weakly balanced collections. We say that $\mathscr{X}$ is *weakly* balanced, if it allows for a set $(b_{\boldsymbol{x}})_{\boldsymbol{x}\in\mathscr{X}}$ of *weakly* balancing coefficients, i.e., the condition $b_{\boldsymbol{x}} > 0$ in (20) is replaced by $b_{\boldsymbol{x}} \geqslant 0$.

Now, as we have mentioned above, some *pre*imputation $\boldsymbol{x}$ (a Pareto optimal vector) of some game $\boldsymbol{v}$ generates certain balanced system via the various levels of excesses. In connection with the modiclus, it turns out that the relevant definitions are useful also when biexcesses are involved.

For $\alpha \in \mathbb{R}$ and any vector $\boldsymbol{x} \in \mathbb{R}^I$ define the system of coalitions with excess at least α which is

$$\underline{\underline{\mathbf{S}}}(\alpha, \boldsymbol{x}, \boldsymbol{u}) := \{S \in \underline{\underline{\mathbf{P}}} \mid e(S, \boldsymbol{x}, \boldsymbol{u}) \geqslant \alpha\}. \tag{21}$$

Now, as we want to deal with the modiclus, it is actually the notion of biexcesses which matters most. We approach this idea by the analogous definition as follows:

$$\tilde{\underline{\underline{\mathbf{S}}}}(\alpha, \boldsymbol{x}, \boldsymbol{v}) := \{(R, T) \in \underline{\underline{\mathbf{P}}} \times \underline{\underline{\mathbf{P}}} \mid e(R, \boldsymbol{x}, \boldsymbol{v}) + e(T, \boldsymbol{x}, \boldsymbol{v}^{\star}) \geqslant \alpha\}. \tag{22}$$

We are now in the position to discuss our solution concept the modified nucleolus or *modiclus*. The definition has been indicated in the introduction: the modiclus of a game $\boldsymbol{v}$, denoted by $\boldsymbol{\psi}(\boldsymbol{v})$, is the unique preimputation that lexicographically minimizes the (ordered) vector of biexcesses. Note that the modiclus is an *imputation* in the case that it is applied to a min-game. Indeed it must be individually rational by Corollary 2.6 of [14], because a min-game is *zero-monotonic*, i.e., $\boldsymbol{v}(S \cup \{i\}) - \boldsymbol{v}(S) \geqslant 0 = \boldsymbol{v}\{i\}$ $(S \in \underline{\underline{\mathbf{P}}}, i \in I)$ holds true.

Equivalently, it is the projection of the prenucleolus of the dual cover game onto the set of primal players. For the details see [14].

Theorem 2.4. *Let $\boldsymbol{v}$ be a game and let $\boldsymbol{x}$ be a preimputation of this game. Then $\boldsymbol{x} = \boldsymbol{\psi}(\boldsymbol{v})$ holds true, if and only if $\tilde{\underline{\underline{\mathbf{S}}}}(\alpha, \boldsymbol{x}, \boldsymbol{v})$ is balanced whenever this system is nonempty.*

For a proof of Theorem 2.4 see [14, Theorem 2.2].

Remark 2.5. Note that Theorem 2.4 is the analog of Kohlberg's [5] well-known result which characterizes the (pre)nucleolus by balanced systems of coalitions.

A further technique to be employed frequently is provided by the idea of the *derived game*, which is a relative of the reduced game à la Davis–Maschler [2]. Recall that the *reduced game* $v^{S,\boldsymbol{x}}$ of a game $(I, \underline{\underline{\mathbf{P}}}, \boldsymbol{v})$ is defined on the powerset of S for any nonempty coalition $\emptyset \neq S \subseteq I$ and a any vector $x \in \mathbb{R}^I$ by

$$\boldsymbol{v}^{S,\boldsymbol{x}}(R) = \begin{cases} 0 & \text{if } R = \emptyset, \\ \boldsymbol{v}(I) - \boldsymbol{x}(I - S) & \text{if } R = S, \\ \max_{Q \subseteq I - S} \boldsymbol{v}(R + Q) - \boldsymbol{x}(Q) & \text{if } \emptyset \neq R \subsetneq S. \end{cases}$$

But in the vicinity of the modiclus, the appropriate reduction takes into account both, the game and its dual. Define the *derived game* with respect to S and $\boldsymbol{x}$ to be the game $\boldsymbol{v}_{S,x}$ on the powerset of S given by

$$\boldsymbol{v}_{S,x}(R) := \begin{cases} \boldsymbol{v}^{S,x}(R) & \text{if } R \in \{\emptyset, S\}, \\ \max\{\boldsymbol{v}^{S,\boldsymbol{x}}(R) - \mu, (\boldsymbol{v}^\star)^{S,\boldsymbol{x}}(R) - \mu^\star\} & \text{otherwise.} \end{cases} \tag{23}$$

Here we use the abbreviations $\mu = \mu(\boldsymbol{x}, \boldsymbol{v})$ and $\mu^\star = \mu(\boldsymbol{x}, \boldsymbol{v}^\star)$.

Remark 2.6. Let $(I, \underline{\underline{\mathbf{P}}}, \boldsymbol{v})$ be a game:

(1) If $\boldsymbol{x}$ is a preimputation, then its projection to any nonempty coalition S belongs to the core of the derived game $\boldsymbol{v}_{S,\boldsymbol{x}}$. Indeed, for any $R \subseteq S$ with $\emptyset \neq R \neq S$ the inequalities

$$e(T, \boldsymbol{x}_S, \boldsymbol{v}^{S,\boldsymbol{x}}) = \max_{Q \subseteq I-S} e(T+Q, \boldsymbol{x}, \boldsymbol{v}) \leqslant \mu$$

and

$$e(T, \boldsymbol{x}_S, (\boldsymbol{v}^\star)^{S,\boldsymbol{x}}) = \max_{Q \subseteq I-S} e(T+Q, \boldsymbol{x}, \boldsymbol{v}^\star) \leqslant \mu^\star$$

are valid by the definition of the reduced game. Moreover, the equation $\boldsymbol{v}^{S,\boldsymbol{x}}(S) = \boldsymbol{x}(S)$ holds true by Pareto optimality of $\boldsymbol{x}$.

(2) If $\boldsymbol{v}^t$ is the game which arises from $\boldsymbol{v}$ by adding the constant $t \in \mathbb{R}$ to the worth of every nontrivial coalition, i.e., if $\boldsymbol{v}^t$ is defined by

$$\boldsymbol{v}^t(S) := \begin{cases} \boldsymbol{v}(S) & \text{if } S \in \{\emptyset, I\} \\ \boldsymbol{v}(S) + t & \text{otherwise} \end{cases} \quad (S \in \underline{\underline{\mathbf{P}}}),$$

then the prenucleoli of $\boldsymbol{v}$ and $\boldsymbol{v}^t$ coincide (see Lemma 4.5 in [13]).

(3) The prenucleolus satisfies the *reduced game property* (see [12] or [6]): The projection of the prenucleolus of a game coincides with the prenucleolus of the corresponding reduced game. Of course reduction has to be taken with respect to the prenucleolus.

(4) It is well known that the prenucleolus and the nucleolus coincide, when applied to a game with a nonempty core.

The following lemma will be used in several proofs and can be regarded as an adequate modification of the reduced game property.

Lemma 2.7. *Let $\boldsymbol{v}$ be a game and let $\hat{\boldsymbol{x}} := \boldsymbol{\psi}(\boldsymbol{v})$ be its modiclus. Furthermore, let $S \in \underline{\underline{\mathbf{P}}}$ be a nonempty coalition. Then the nucleolus $\boldsymbol{x} := \boldsymbol{\nu}(\boldsymbol{v}_{S,\hat{x}})$ of the derived game coincides with the projection of the modiclus, i.e., $\boldsymbol{x} = \hat{\boldsymbol{x}}_S$ holds true.*

Proof. We abbreviate $\mu := \mu(\hat{\boldsymbol{x}}, \boldsymbol{v})$ and $\mu^\star := \mu(\hat{\boldsymbol{x}}, \boldsymbol{v}^\star)$. The modiclus of $\boldsymbol{v}$ is the projection to I of the prenucleolus of the dual cover $\bar{\boldsymbol{v}}$ as defined in (4) of Section 1.

Let $\bar{\boldsymbol{x}}$ denote the prenucleolus of $\bar{\boldsymbol{v}}$. Proposition 1.4 in [13] shows that

$$\mu(\bar{\boldsymbol{x}}, \bar{\boldsymbol{v}}) = \mu + \mu^\star$$

and

$$\bar{\boldsymbol{v}}^{I,\bar{x}}(S) = \begin{cases} 0 & \text{if } S = \emptyset, \\ \boldsymbol{v}(I) & \text{if } S = I, \\ \max\{\boldsymbol{v}(S) + \mu^\star, \boldsymbol{v}^\star(S) + \mu\} & \text{otherwise} \end{cases} \tag{24}$$

hold true. Let $\boldsymbol{w} := \bar{\boldsymbol{v}}^{I,\bar{x}}$ denote this reduced game. By the reduced game property the modiclus of $\boldsymbol{v}$ coincides with the prenucleolus of $\boldsymbol{w}$. Let $\boldsymbol{u} := \boldsymbol{w}^{S,\hat{x}}$ denote the reduced game with respect to S. With $t := -(\mu + \mu^\star)$ we obtain $\boldsymbol{u}^t = \boldsymbol{v}_{S,\hat{x}}$, thus Remark 2.6 completes the proof. □

3. The treatment of corners

During this section let $\boldsymbol{v} = \bigwedge\{\boldsymbol{\lambda}^1, \ldots, \boldsymbol{\lambda}^r\}$ be a min-game. We claim that the modiclus represents the formation of cartels within the various corners of the market. These cartels—or may be their representatives—bargain about their share of the total worth M^1 of the grand coalition. Let $\boldsymbol{x}$ be an imputation represented as in formula (13) of Section 2. As $\boldsymbol{x}(C^\rho) = c_\rho M^1$ holds true, the convexifying coefficients c_ρ indicate the *share* the various corners obtain at $\boldsymbol{x}$. Similarly, the normalized measure $\boldsymbol{\mu}^\rho$ indicates the *internal distribution* according to $\boldsymbol{x}$ inside a corner ρ.

Within this section, we begin to clarify the shape of the coefficient vector $\boldsymbol{c}$ of the modiclus. It turns out that there are basically three situations depending on the relations of the total initial assignments in the corners in a peculiar way. Accordingly, in the two extreme cases, the modiclus assigns the same share to all corners or just to the minimal ones. In the intermediate case, the modiclus chooses a carefully constructed combination of the two extremes.

The maximal diagonal coalitions play a crucial rôle (cf. (10) of Section 2). If we focus on a corner, we should consider the partners of such coalitions, i.e., the system

$$\underline{\underline{\mathbf{D}}}^{m\rho} := \{S \cap C^\rho \mid S \in \underline{\underline{\mathbf{D}}}^m\}. \tag{25}$$

We shall impose some conditions (e.g. balancedness) upon this system which allow the computation of maximal excesses and, later on, the determination of the coefficient vector $\boldsymbol{c}$. This condition is of interest in its own right, however, we shall see in a later section that it is satisfied for "large games," i.e., for replicated versions or games with "sufficiently many" small players.

Lemma 3.1. *Assume that $\underline{\underline{\mathbf{D}}}^{m\rho}$ is weakly balanced w.r.t. C^ρ for every $\rho \in \{\sigma+1, \ldots, r\}$. Also, let $\boldsymbol{x}$ be an imputation. Define a further imputation $\tilde{\boldsymbol{x}}$ by*

$$\tilde{\boldsymbol{x}} := M^1 \sum_{\rho=1}^{r} \frac{\boldsymbol{x}(C^\rho)}{M^1} \frac{\boldsymbol{\lambda}^\rho}{M^\rho}, \tag{26}$$

such that $\tilde{c}_\rho := \boldsymbol{x}(C^\rho)/M^1$ $(\rho = 1,\ldots,r)$ *constitute convexifying coefficients. Then*

$$\mu(\boldsymbol{x},\boldsymbol{v}) \geqslant \mu(\tilde{\boldsymbol{x}},\boldsymbol{v}) = M^1\left(1-\sum_{\rho=1}^{r}\tilde{c}_\rho\frac{M^1}{M^\rho}\right) \tag{27}$$

and

$$\mu(\boldsymbol{x},\boldsymbol{v}^\star) \geqslant \mu(\tilde{\boldsymbol{x}},\boldsymbol{v}^\star) = M^1\left(1-\min_\rho \tilde{c}_\rho\right) \tag{28}$$

holds true. If equation prevails in (20), *then* $\boldsymbol{x}(S)=\tilde{\boldsymbol{x}}(S)$ *holds true for all coalitions* S *of any balanced system in* $\underline{\underline{D}}^{m\rho}$ $(\rho = s+1,\ldots,r)$.

Proof. By the weak balancedness of $\underline{\underline{\mathbf{D}}}^{m\rho}$ the system $\underline{\underline{\mathbf{D}}}^{m}$ of maximal diagonal coalitions is nonempty. Corollary 2.2 implies that the maximal excess with respect to the primal game at $\tilde{\boldsymbol{x}}$ is attained by the coalitions of the system $\underline{\underline{\mathbf{D}}}^{m}$. Inserting any coalition of this system into (15) of Lemma 2.1 yields that this excess is indeed the one listed in formula (27) for $\tilde{\boldsymbol{x}}$.

Furthermore, an inspection of Lemma 2.1 ((16) and (17)) shows that the maximal excess with respect to the dual game at $\tilde{\boldsymbol{x}}$ is attained at those carriers which have minimal total weight. This shows indeed the equation in formula (27). Of course these carriers have the same weight at $\boldsymbol{x}$ as they have at $\tilde{\boldsymbol{x}}$. Thus the statement of (28) is verified.

Now in order to compare the maximal excess at $\boldsymbol{x}$ and the maximal excess at $\tilde{\boldsymbol{x}}$ we proceed as follows. As $\underline{\underline{\mathbf{D}}}^{m\rho}$ is weakly balanced for all ρ, we fix some ρ and choose balancing coefficients $(c_R)_{R\in\underline{\underline{\mathbf{D}}}^{m\rho}}$. Then we obtain the equations

$$\begin{aligned}\sum_{R\in\underline{\underline{\mathbf{D}}}^{m\rho}} c_R\boldsymbol{x}(R) &= \boldsymbol{x}\left(\sum_{R\in\underline{\underline{\mathbf{D}}}^{m\rho}} c_R 1_R\right) = \boldsymbol{x}(C^\rho)\\ &= \tilde{\boldsymbol{x}}(C^\rho) = \tilde{\boldsymbol{x}}\left(\sum_{R\in\underline{\underline{\mathbf{D}}}^{m\rho}} c_R 1_R\right) = \sum_{R\in\underline{\underline{\mathbf{D}}}^{m\rho}} c_R\tilde{\boldsymbol{x}}(R)\\ &= \sum_{R\in\underline{\underline{\mathbf{D}}}^{m\rho}} c_R\frac{M^1}{M^\rho}\tilde{\boldsymbol{x}}(C^\rho).\end{aligned}$$

Hence, for some $S^\rho \in \underline{\underline{\mathbf{D}}}^{m\rho}$ we have

$$\boldsymbol{x}(S^\rho) \leqslant \frac{M^1}{M^\rho}\boldsymbol{x}(C^\rho) = \tilde{\boldsymbol{x}}(S^\rho).$$

Thus, the excess of $S := \sum_{\rho=1}^{r} S^\rho$ at $\boldsymbol{x}$ exceeds the one at $\tilde{\boldsymbol{x}}$, i.e.,

$$e(S,\boldsymbol{x},\boldsymbol{v}) \geqslant e(S,\tilde{\boldsymbol{x}},\boldsymbol{v}) = \mu(\tilde{\boldsymbol{x}},\boldsymbol{v}).$$

The final assertion is as well implied by these considerations. □

Remark 3.2. It is the aim of the modiclus to minimize the maximal dual excess simultaneously with the maximal excess. With the dual game, the "preventive power" of coalitions enters the scene. Now, in view of formula (28) (and the subsequent proof), it is seen that the maximal dual excess (hence the maximal force of complaints) is attained at the corners with minimal coefficient (share) c_ρ. While this is presently proved with respect to $\tilde{\boldsymbol{x}}$, it will also be true with respect to the modiclus. Clearly, this indicates "the formation of cartels" in the various corners of the market.

Analogously, the fact that the maximal excess is attained at maximal diagonal coalitions points to the maximal "achievement power" of this type of coalitions. This is a consequence of the fact that these coalitions are efficient as well as effective in the maximal possible fashion.

Lemma 3.3. *Assume that* $\underline{\underline{\mathbf{D}}}^{m\rho}$ *is nonempty for every* $\rho \in \{1,\dots,r\}$. *Also, let* $\boldsymbol{x} = M^1 \sum_{\rho=1}^{r} c_\rho \boldsymbol{\lambda}^\rho / M^\rho$ *be an imputation. Choose convexifying coefficients* $(d_\rho)_{\rho=1,\dots,r}$ *satisfying*

$$d_\tau \geqslant d_{\sigma+1} = \cdots = d_r = \min\{c_\rho \mid \rho = 1,\dots,r\} \quad (\tau = 1,\dots,\sigma)$$

and put $\boldsymbol{y} := M^1 \sum_{\rho=1}^{r} d_\rho \boldsymbol{\lambda}^\rho / M^\rho$.

Then

$$\mu(\boldsymbol{x},\boldsymbol{v}) \geqslant \mu(\boldsymbol{y},\boldsymbol{v}) \tag{29}$$

and

$$\mu(\boldsymbol{x},\boldsymbol{v}^\star) = \mu(\boldsymbol{y},\boldsymbol{v}^\star) \tag{30}$$

holds true. Moreover, equation prevails in formula (29) *if and only if*

$$c_\tau \geqslant c_{\sigma+1} = \cdots = c_r = \min\{c_\rho \mid \rho = 1,\dots,r\} \ (\tau = 1,\dots,\sigma)$$

holds true.

Proof. Formula (30) is a direct consequence of Lemma 3.1.

Now we turn to formula (29). Recall that the maximal excess is attained at the elements of $\underline{\underline{\mathbf{D}}}^m$ (Corollary 2.2) which is assumed to be nonempty. In fact, this excess at $\boldsymbol{x}$ is given by (15) of Lemma 2.1, that is, we have

$$\mu(\boldsymbol{x},\boldsymbol{v}) = M^1 \left(1 - \sum_{\rho=1}^{r} c_\rho \frac{M^1}{M^\rho}\right). \tag{31}$$

The same formula holds true *mutatis mutandis* for $\boldsymbol{y}$. But as the coefficients defining $\boldsymbol{y}$ are of the special shape indicated, the formula reduces at once. We introduce

$$c_0 := \min\{c_\rho \mid \rho = 1,\dots,r\}$$

and obtain

$$\mu(\boldsymbol{y},\boldsymbol{v}) = M^1\left((r-\sigma)c_0 - c_0\sum_{\rho=\sigma+1}^{r}\frac{M^1}{M^\rho}\right)$$

$$= M^1 c_0\left(r-\sigma-\sum_{\rho=\sigma+1}^{r}\frac{M^1}{M^\rho}\right) = M^1 c_0\left(r-\sum_{\rho=1}^{r}\frac{M^1}{M^\rho}\right). \tag{32}$$

Now the reader has to convince himself that this expression is smaller then the one referring to $\boldsymbol{x}$ (cf. (31)), as the smallest coefficients are attached to the smallest quotients of weights. □

Theorem 3.4. *Suppose that* $\underline{\underline{\mathbf{D}}}^{m\rho}$ *is weakly balanced w.r.t.* C^ρ *for every* $\rho\in\{\sigma+1,\ldots,r\}$. *Then the following holds true*:

(1) *If* $\boldsymbol{\lambda}^1,\ldots,\boldsymbol{\lambda}^r$ *satisfy*

$$1+\sum_{\rho=1}^{r}\frac{M^1}{M^\rho} > r, \tag{33}$$

then the modiclus treats all corners equally, i.e., $\boldsymbol{\psi}$ *is of the form*

$$\boldsymbol{\psi}(\boldsymbol{v}) = M^1\sum_{\rho=1}^{r}\frac{1}{r}\frac{\boldsymbol{\mu}^\rho}{M^\rho}. \tag{34}$$

with a suitable family of normalized measures $\boldsymbol{\mu}^\rho$.

(2) *If* $\boldsymbol{\lambda}^1,\ldots,\boldsymbol{\lambda}^r$ *satisfy*

$$1+\sum_{\rho=1}^{r}\frac{M^1}{M^\rho} < r, \tag{35}$$

then the modiclus is of the form

$$\boldsymbol{\psi}(\boldsymbol{v}) = M^1\sum_{\rho=1}^{\sigma}c_\rho\frac{\boldsymbol{\mu}^\rho}{M^\rho} = \sum_{\rho=1}^{\sigma}c_\rho\boldsymbol{\mu}^\rho \tag{36}$$

with convexifying coefficients c_ρ $(\rho=1,\ldots,\sigma)$. *In particular, the modiclus is located in the core.*

(3) *Finally, if*

$$1+\sum_{\rho=1}^{r}\frac{M^1}{M^\rho} = r \tag{37}$$

is the case, then the modiclus treats all nonminimal corners equally, and the minimal corners at least as well, i.e.,

$$\boldsymbol{\psi}(\boldsymbol{v}) = M^1\sum_{\rho=1}^{r}c_\rho\frac{\boldsymbol{\mu}^\rho}{M^\rho}. \tag{38}$$

Here $c_{\sigma+1}=\cdots=c_r\leqslant c_\rho$ $(\rho=1,\ldots,\sigma)$.

Proof. Put $\hat{\boldsymbol{x}} := \boldsymbol{\psi}(\boldsymbol{v})$. By weak balancedness of $\underline{\underline{\mathbf{D}}}^{m\rho}$ $(\rho = \sigma + 1, \ldots, r)$ both, Lemmas 3.1 and 3.3, may be applied. Indeed, the modiclus is an imputation which minimizes the maximal biexcess. Therefore, we obtain

$$\hat{\boldsymbol{x}}(C^\rho) \geqslant \hat{\boldsymbol{x}}(C^{\sigma+1}) = \cdots = \hat{\boldsymbol{x}}(C^r) =: \alpha \geqslant 0 \quad (\rho = 1, \ldots, \sigma).$$

Thus,

$$M^1 - (r-\sigma)\alpha = \hat{\boldsymbol{x}}\left(\sum_{\rho=1}^{\sigma} C^\rho\right) \geqslant \sigma\alpha$$

is valid by Pareto optimality. We conclude that $\alpha \leqslant M^1/r$ holds true. It remains to prove that $\alpha = M^1/r$ or $\alpha = 0$, respectively, holds in the case that (33) or (35), respectively, is satisfied. In view of (27) and (28), the maximal excesses can be expressed by the two formulae

$$\begin{aligned}\mu(\hat{\boldsymbol{x}}, \boldsymbol{v}) &= M^1\left(1 - \sum_{\rho=1}^{r} \frac{\hat{\boldsymbol{x}}(C^\rho)}{M^\rho}\right) \\ &= M^1\left(1 - \frac{M^1 - (r-\sigma)\alpha}{M^1} - \alpha \sum_{\rho=\sigma+1}^{r} \frac{1}{M^\rho}\right) \\ &= \alpha\left(r - \sum_{\rho=1}^{r} \frac{M^1}{M^\rho}\right) \end{aligned} \tag{39}$$

and

$$\mu(\hat{\boldsymbol{x}}, \boldsymbol{v}^\star) = M^1\left(1 - \frac{\alpha}{M^1}\right). \tag{40}$$

Hence, the maximal biexcess is given by

$$\mu(\hat{\boldsymbol{x}}, \boldsymbol{v}) + \mu(\hat{\boldsymbol{x}}, \boldsymbol{v}^\star) = M^1 + \alpha\left(r - 1 - \sum_{\rho=1}^{r} \frac{M^1}{M^\rho}\right). \tag{41}$$

By the definition of the modiclus this maximal biexcess must be as small as possible. If (33) or (35), respectively, is satisfied, then the expression in the brackets is negative or positive respectively. Hence α has to be maximal (i.e., $\alpha = M^1/r$ holds) in the first case and it has to be minimal (i.e., $\alpha = 0$ holds) in the latter case. □

This way we have now clarified the distribution of wealth between the cartels as suggested by the modiclus. It depends crucially on the masses of the initial assignments: if the excess supply on the long side of the market is just moderate (in the sense of formula (33)), then the modiclus treats all corners equally and this is essentially a result of the preventive powers the cartels can exercise (Remark 3.2). If the excess supply on the long side is overwhelming, the modiclus falls into the core (and the primal maximal excesses are the important quantities). The intermediate case mixes both ingredients.

The determination of the coefficient vector $\boldsymbol{c}$ (i.e., the shares of the cartels) is not yet complete. The next section continues treating this task. It turns out that the modiclus is determined by the nucleolus of a suitable derived game (Section 2) defined on the playerset $\sum_{\rho=1}^{\sigma} C^{\rho}$, i.e., on the short side.

4. The derived game on the short side

During this section we fix a min-game $\boldsymbol{v} = \bigwedge\{\boldsymbol{\lambda}^1, \ldots, \boldsymbol{\lambda}^r\}$ and continue to discuss the treatment of corners. It turns out that a suitable derived game (cf. (23) of Section 2) defined on the short side $\tilde{S} := \sum_{\rho=1}^{\sigma} C^{\rho}$ of the market allows to further specify the coefficient vector $\boldsymbol{c}$ attached to the modiclus. Since the derived game is a relative of the reduced game and reflects the projection from the dual cover game down onto the original player set, one might expect that the nucleolus enters the scene (recall our explanations in Section 1). Indeed, it is seen that the modiclus can be described employing the nucleolus of a suitable balanced game on the short side $\tilde{S}$.

Motivated by Theorem 3.4 we introduce the notion of the *index of powers* which is the quantity

$$\imath(\boldsymbol{v}) := 1 + \sum_{\rho=1}^{r} \frac{M^1}{M^{\rho}}. \tag{42}$$

This index depends on $\boldsymbol{v}$ only as the representation is unique (cf. Section 2).

Theorem 3.4 also suggests the classification of min-games as follows. We say that $\boldsymbol{v}$ has a *strong long side* or a *strong short side*, if (33) or (35) of Theorem 3.4, respectively, is satisfied, i.e., if

$$\imath(\boldsymbol{v}) > r \tag{43}$$

or

$$\imath(\boldsymbol{v}) < r, \tag{44}$$

respectively, holds true. In the remaining case, i.e., if

$$\imath(\boldsymbol{v}) = r \tag{45}$$

holds true, we say that $\boldsymbol{v}$ has *balanced sides*.

We start out with a strong short side.

Theorem 4.1. *Let $\boldsymbol{v}$ have a strong short side. If $\underline{\underline{\mathbf{D}}}^{m\rho}$ $(\rho = \sigma + 1, \ldots, r)$ is weakly balanced w.r.t. C^{ρ} then the modiclus coincides with the nucleolus, i.e., $\boldsymbol{\psi}(\boldsymbol{v}) = \boldsymbol{\nu}(\boldsymbol{v})$ holds true.*

Proof. Let $\hat{\boldsymbol{x}} := \boldsymbol{\psi}(\boldsymbol{v})$ and $\boldsymbol{x} := \boldsymbol{\nu}(\boldsymbol{v})$ denote the modiclus and nucleolus of the game $\boldsymbol{v}$. Note that $\hat{x}_i = x_i = 0$ holds true for $i \in I - \tilde{S}$ by Theorem 3.4 and the fact that the nucleolus is a member of the core. In view of Remark 2.6 and Lemma 2.7 it suffices to show that the corresponding reduced and derived games coincide, i.e., that

$$\boldsymbol{v}_{\tilde{S},\hat{\boldsymbol{x}}} = \boldsymbol{v}^{\tilde{S},\boldsymbol{x}} =: \boldsymbol{w}$$

holds true. Note that $\boldsymbol{w}$ coincides with the reduced game with respect to $\hat{\boldsymbol{x}}$, because $\boldsymbol{x}_{I-\tilde{S}} = \hat{\boldsymbol{x}}_{I-\tilde{S}}$ holds true. Since both vectors show zero coordinates outside of $\tilde{S}$ the computation of the reduced game is particularly easy and yields

$$\boldsymbol{w}(R) = \begin{cases} 0 & \text{if } R = \emptyset, \\ M^1 & \text{if } R = \tilde{S} \quad (R \subseteq \tilde{S}), \\ \min_{\rho=1,\dots,\sigma} \boldsymbol{\lambda}^\rho(R) & \text{otherwise.} \end{cases} \tag{46}$$

Note that $\mu(\hat{\boldsymbol{x}}, \boldsymbol{v}) = 0$ and $\mu(\hat{\boldsymbol{x}}, \boldsymbol{v}^\star) = M^1$ hold true. In view of (23) of Section 2 and by (46) it suffices to show that the inequality

$$(\boldsymbol{v}^\star)^{\tilde{S},\hat{\boldsymbol{x}}}(R) - M^1 \leqslant 0 \quad (\leqslant \boldsymbol{w}(R))$$

is correct for any nontrivial coalition $R \subseteq \tilde{S}$. This inequality follows immediately from (16) and (17) (see Section 2) applied to $\tau = \sigma$. □

Now the case of balanced sides is considered. We shall show that, under some additional assumptions, the convexifying coefficients c_ρ occurring in (38) of Theorem 3.4 can be determined.

We have to introduce the following concept.

Definition 4.2. Let $\boldsymbol{v} = \bigwedge\{\boldsymbol{\lambda}^1, \dots, \boldsymbol{\lambda}^r\}$ be a min game. We write $\lambda^{\min} := \min_{\rho=1,\dots,r} \min_{i \in C^\rho} \lambda_i^\rho$. Furthermore, we say that the long side *shows small players* if some corner ρ with maximal weight $M^\rho = M^r$ contains a player with minimal (positive) weight $\lambda^{\min}$.

Now we have the following theorem.

Theorem 4.3. *Let $\boldsymbol{v}$ have balanced sides and let the long side show small players. If $\underline{\underline{\mathbf{D}}}^{m\rho}$ is nondegenerate and balanced w.r.t. C^ρ for every $\rho \in \{\sigma+1, \dots, r\}$, then the modiclus is of the form*

$$\boldsymbol{\psi}(\boldsymbol{v}) = M^1 \left(\sum_{\rho=1}^{\sigma} \frac{M^r + \lambda^{\min}}{\sigma\lambda^{\min} + rM^r} \frac{\boldsymbol{\mu}^\rho}{M^\rho} + \sum_{\rho=\sigma+1}^{\rho} \frac{M^r}{\sigma\lambda^{\min} + rM^r} \frac{\boldsymbol{\lambda}^\rho}{M^\rho} \right) \tag{47}$$

with a suitable family of normalized measures $\boldsymbol{\mu}^\rho$ $(\rho = 1, \dots, \sigma)$.

Proof. *Step* 1: Let $\hat{\boldsymbol{x}}$ denote the modiclus of $\boldsymbol{v}$. By Theorem 3.4 there are normalized measures $\boldsymbol{\mu}^\rho$ and convexifying coefficients c_ρ $(\rho = 1, \dots, r)$ satisfying

$$c_\rho \geqslant c_{\sigma+1} = \cdots = c_r =: \gamma$$

such that

$$\hat{\boldsymbol{x}} = M^1 \left(\sum_{\rho=1}^{\sigma} c_\rho \frac{\boldsymbol{\mu}^\rho}{M^\rho} + \sum_{\rho=\sigma+1}^{\rho} \gamma \frac{\boldsymbol{\mu}^\rho}{M^\rho} \right) \tag{48}$$

holds true. By Lemma 3.1 the maximal excesses are given by the expressions

$$\mu(\hat{\boldsymbol{x}}, \boldsymbol{v}) = M^1\gamma \quad \text{and} \quad \mu(\hat{\boldsymbol{x}}, \boldsymbol{v}^\star) = M^1 - M^1\gamma \tag{49}$$

and they are attained by all maximal diagonal coalitions. Hence, nondegeneracy and balancedness of the $\underline{\underline{\mathbf{D}}}^{m\rho}$ $(\rho = \sigma+1, \ldots, \sigma)$ implies that $\boldsymbol{\mu}^\rho = \boldsymbol{\lambda}^\rho$ holds true.

Step 2: Define

$$d_1 = \cdots = d_\sigma := \frac{M^r + \lambda^{\min}}{\sigma\lambda^{\min} + rM^r} \quad \text{and} \quad d_{\sigma+1} = \cdots = d_\rho = \delta := \frac{M^r}{\sigma\lambda^{\min} + rM^r} \tag{50}$$

and put

$$\boldsymbol{x} := M^1 \left(\sum_{\rho=1}^{\sigma} d_\rho \frac{\boldsymbol{\lambda}^\rho}{M^\rho} + \sum_{\rho=\sigma+1}^{\rho} \delta \frac{\boldsymbol{\lambda}^\rho}{M^\rho} \right). \tag{51}$$

Then the d_ρ are convexifying coefficients and by Lemmas 3.1 and 3.3 the maximal excesses are given by the expressions

$$\mu(\boldsymbol{x}, \boldsymbol{v}) = M^1\delta \quad \text{and} \quad \mu(\boldsymbol{x}, \boldsymbol{v}^\star) = M^1 - M^1\delta. \tag{52}$$

Hence, the maximal biexcesses at $\hat{\boldsymbol{x}}$ and at $\boldsymbol{x}$ coincide and can be computed as

$$\mu(\hat{\boldsymbol{x}}, \boldsymbol{v}) + \mu(\hat{\boldsymbol{x}}, \boldsymbol{v}^\star) = M^1 = \mu(\boldsymbol{x}, \boldsymbol{v}) + \mu(\boldsymbol{x}, \boldsymbol{v}^\star). \tag{53}$$

The next two steps serve to determine the second highest excesses at $\boldsymbol{x}$.

Step 3: Let $S \in \underline{\underline{\mathbf{P}}} - \underline{\underline{\mathbf{D}}}^m$ be any coalition which is not a maximal diagonal coalition. We are going to prove that

$$e(S, \boldsymbol{x}, \boldsymbol{v}) \leqslant \mu(\boldsymbol{x}, \boldsymbol{v}) - \frac{\delta\lambda^{\min}M^1}{M^r} =: \mu_2 \tag{54}$$

holds true. As $S \notin \underline{\underline{\mathbf{D}}}^m$ two cases may occur:

(1) If $S \in \underline{\underline{\mathbf{D}}} - \underline{\underline{\mathbf{D}}}^m$ holds true, then $e(S, \boldsymbol{x}, \boldsymbol{v}) \leqslant (M^1 - \lambda^{\min})\delta = \mu(\boldsymbol{x}, \boldsymbol{v}) - \delta\lambda^{\min}$ is valid by (15) of Section 2.
(2) In the remaining case there exist $\rho, \tau \in \{1, \ldots, r\}$ with $\rho \neq \tau$ such that $\boldsymbol{\lambda}^\rho(S) \geqslant \boldsymbol{\lambda}^\tau(S) + \lambda^{\min}$ holds true. In this case we conclude via (15) of Lemma 2.1 that $e(S, \boldsymbol{x}, \boldsymbol{v}) \leqslant \mu(\boldsymbol{x}, \boldsymbol{v}) - \delta\lambda^{\min}M^1/M^r$ holds true.

Step 4: Let $S \in \underline{\underline{\mathbf{P}}} - \{C^\rho \mid \rho = s+1, \ldots, r\}$ be any coalition which is not a nonminimal corner. We are going to prove that

$$e(S, \boldsymbol{x}, \boldsymbol{v}^\star) \leqslant \mu(\boldsymbol{x}, \boldsymbol{v}^\star) - \frac{\delta\lambda^{\min}M^1}{M^r} =: \mu_2^\star \tag{55}$$

holds true. We distinguish two cases:

(1) If S is contained in C^τ for some $\tau = \sigma+1,\ldots,r$, then we $S \neq C^\tau$ holds true by the assumption. Therefore, the dual excess is given by

$$e(S,\boldsymbol{x},\boldsymbol{v}^\star) = M^1 - \min\{M^1, \boldsymbol{\lambda}^\tau(I-S)\} - \boldsymbol{x}(S). \tag{56}$$

If the minimum is M^1, then (55) follows from the fact that $\mu_2^\star \geqslant 0$ holds true. In the remaining case we obtain

$$e(S,\boldsymbol{x},\boldsymbol{v}^\star) = M^1 - \boldsymbol{\lambda}^\tau(I-S) - \boldsymbol{x}(S) = M^1 - \boldsymbol{\lambda}^\tau(I-S) - \delta\,\frac{M^1}{M^\tau}\boldsymbol{\lambda}^\tau(S)$$

holds true. By (50) this expression yields

$$e(S,\boldsymbol{x},\boldsymbol{v}^\star) = \mu(\boldsymbol{x},\boldsymbol{v}^\star) - \left(1 - \frac{\delta M^1}{M^\tau}\right)\boldsymbol{\lambda}^t(I-S).$$

The fact that

$$1 - \frac{\delta\lambda^{\min}M^1}{M^\tau} = \frac{M^\tau - \delta M^1}{M^\tau} > \frac{\delta M^1}{M^r}$$

holds true implies (55) in the current case.

(2) If S is not contained in any nonminimal corner, then by Lemma 2.1 ((16) or (17) applied to $\tau=\sigma$) it suffices to show that

$$\mu_2^\star \geqslant \max_{\rho=\sigma+1,\ldots,\rho} M^1(1-\delta) - \boldsymbol{x}(S - C^\rho) \tag{57}$$

and

$$\mu_2^\star \geqslant \max_{\rho=1,\ldots,\sigma} \boldsymbol{\lambda}^\rho(S) - \boldsymbol{x}(S) \tag{58}$$

hold true. By the assumption $S - C^\rho$ is nonempty, thus the inequalities

$$\boldsymbol{x}(S - C^\rho) \geqslant \min_{i\in I} x_i \geqslant \frac{M^1}{M^r} \geqslant \frac{\delta M^1}{M^r}$$

show (57). Moreover, the observation that

$$\max_{\rho=1,\ldots,\sigma} \boldsymbol{\lambda}^\rho(S) - \boldsymbol{x}(S) = \max_{\rho=1,\ldots,\sigma} \boldsymbol{\lambda}^\rho(S)(1-d_\rho) \leqslant M^1(1-d_1) = \mu_2^\star$$

holds true directly shows (58).

Step 5: In view of the fact that $\mu(\boldsymbol{x},\boldsymbol{v}) - \mu_2 = \mu(\boldsymbol{x},\boldsymbol{v}^\star) - \mu_2^\star$ we conclude that

$$e(R,\boldsymbol{x},\boldsymbol{v}) + e(T,\boldsymbol{x},\boldsymbol{v}^\star) \leqslant M^1 - \frac{\delta M^1}{M^r} \tag{59}$$

holds true for any pair of coalitions such that $R \notin \underline{\underline{\mathbf{D}}}^m$ or $T \notin \{C^\rho \mid \rho = \sigma+1,\ldots,r\}\}$ is satisfied. By (53) the same property must be satisfied for $\hat{\boldsymbol{x}}$. Indeed, the modiclus lexicographically minimizes the biexcesses. Let $\tau \in \{1,\dots,\sigma\}$ be such that $\hat{\boldsymbol{x}}(C^\tau)$ is minimal. Moreover, let $i \in C^{\rho_0}$ satisfy $\lambda_i^r = 1$. Eq. (45) shows that $M^r > M^1$ holds true. By balancedness of $\underline{\underline{\mathbf{D}}}^{m\rho_0}$ there is a coalition $T \in \underline{\underline{\mathbf{D}}}^m$ such that $i \notin T$ is valid. Put $\hat{R} := T \cup \{i\}$. Then we have the equations

$$e(\hat{R}, \hat{\boldsymbol{x}}, \boldsymbol{v}) = \mu(\hat{\boldsymbol{x}}, \boldsymbol{v}) - \gamma \frac{M^1}{M^r} \tag{60}$$

and

$$e(C^\tau, \hat{\boldsymbol{x}}, \boldsymbol{v}^\star) = \mu(\hat{\boldsymbol{x}}, \boldsymbol{v}^\star) - c_\tau M^1. \tag{61}$$

These equations imply $\gamma \geqslant \delta$ and $c_\tau \geqslant d_\tau$. The coefficients d_ρ and the coefficients c_r are convexifying coefficients, thus $c_\rho = d_\rho$ $(\rho = 1,\dots,r)$ holds true. $\square$

In order to describe the modiclus via the nucleolus of a certain game with playerset $\tilde{S}$ in the case that the min-game $\boldsymbol{v}$ has a strong long side or balanced sides an additional assumption is needed.

Definition 4.4. Let $\boldsymbol{v} = \bigwedge\{\boldsymbol{\lambda}^1,\dots,\boldsymbol{\lambda}^r\}$ be a game satisfying $\boldsymbol{\iota}(\boldsymbol{v}) \geqslant r$.

1. $\boldsymbol{v}$ or $(\boldsymbol{\lambda}^1,\dots,\boldsymbol{\lambda}^r)$ *allows matches*, if the following condition is satisfied:

$$\forall \tau = 1,\dots,\sigma \;\; \forall S \in C^\tau \;\; \forall \rho = \sigma+1,\dots,r \;\; \exists T \in C^\rho\colon\; \boldsymbol{\lambda}^\tau(S) = \boldsymbol{\lambda}^\rho(T) \tag{62}$$

2. Define

$$\gamma := \gamma(\boldsymbol{v}) := \begin{cases} \frac{1}{r} & \text{if } \boldsymbol{\iota}(\boldsymbol{v}) > r \text{ is true,} \\ \frac{M^r}{\sigma\lambda^{\min} + rM^r} & \text{if } \boldsymbol{\iota}(\boldsymbol{v}) = r \text{ is true.} \end{cases} \tag{63}$$

3. Let

$$F := M^1\gamma, G := 1 - \gamma \sum_{\rho=\sigma+1}^{r} \frac{M^1}{M^\rho}, \quad H := M^1(1-(r-\sigma)\gamma). \tag{64}$$

Theorem 4.5. *Let $\boldsymbol{v}$ have either a strong long side or balanced sides. Let the long side show small players and let $(\boldsymbol{\lambda}^1,\dots,\boldsymbol{\lambda}^r)$ allow matches. Let the game $\boldsymbol{w}$ on the short side $\tilde{S}$ be defined by*

$$\boldsymbol{w}(R) := \max\left\{H - G \max_{\rho=1,\dots,\sigma} \boldsymbol{\lambda}^\rho(\tilde{S} - R),\; F - \min_{\rho=1,\dots,\sigma} \boldsymbol{\lambda}^\rho(\tilde{S} - R), 0\right\}. \tag{65}$$

Let $\boldsymbol{x} := \boldsymbol{\nu}(\boldsymbol{w})$ be the nucleolus of $\boldsymbol{w}$. Then the modiclus $\hat{\boldsymbol{x}} := \boldsymbol{\psi}(\boldsymbol{v})$ is given by

$$\hat{\boldsymbol{x}}_{\tilde{S}} = \boldsymbol{x} \quad \textit{and} \quad \hat{x}_i = F\frac{\lambda_i^\rho}{M^\rho} \quad (i \in C^\rho,\; \rho = \sigma+1,\dots,r). \tag{66}$$

In other words, the modiclus coincides with $\boldsymbol{\nu}(\boldsymbol{w})$ on $\tilde{S}$ and with the measure $F\sum_{\rho=\sigma+1}^{r} \boldsymbol{\lambda}^\rho/M^\rho$ on $I - \tilde{S}$.

Proof. In view of balancedness and nondegeneracy of the $\underline{\underline{\mathbf{D}}}^{m\rho}$ Theorems 3.4, 4.3 and Lemma 3.3 show that the modiclus has the desired shape on $I - \tilde{S}$.

In view of Lemma 2.7 it suffices to show that the derived game $\boldsymbol{v}_{\tilde{S},\hat{\boldsymbol{x}}}$ coincides with $\boldsymbol{w}$. For the "trivial" coalitions, i.e., for $\tilde{S}$ and $\emptyset$, coincidence is certainly true. Let $R \subseteq \tilde{S}, \emptyset \neq R \neq \tilde{S}$ be a nontrivial coalition and let $\boldsymbol{u}_1 := \boldsymbol{v}^{\tilde{S},\hat{\boldsymbol{x}}}$ and $\boldsymbol{u}_2 := (\boldsymbol{v}^{\star})^{\tilde{S},\hat{\boldsymbol{x}}}$ be the corresponding reduced games. In view of (27) and (28) of Section 3 we obtain

$$\mu := \mu(\hat{\boldsymbol{x}}, \boldsymbol{v}) = F\left(r - \sum_{\rho=1}^{r} \frac{M^1}{M^\rho}\right), \quad \mu^\star := \mu(\hat{\boldsymbol{x}}, \boldsymbol{v}^\star) = M^1 - F.$$

In order to show that

$$\boldsymbol{u}_1(R) - \mu = H - G \max_{\rho=1,\dots,\sigma} \boldsymbol{\lambda}^\rho(\tilde{S} - R) \tag{67}$$

is satisfied, let $Q \subseteq I - \tilde{S}$. An application of (15) of Section in 2 yields

$$\boldsymbol{v}(R + Q) - \hat{\boldsymbol{x}}(Q) \leqslant \min_{\rho=1,\dots,\sigma} \boldsymbol{\lambda}^\rho(R) \left(1 - F \sum_{\rho=\sigma+1}^{r} \frac{1}{M^r}\right),$$

thus

$$\begin{aligned} \boldsymbol{v}(R + Q) - \hat{\boldsymbol{x}}(Q) - \mu &\leqslant \left(M^1 - \max_{\rho=1,\dots,\sigma} \boldsymbol{\lambda}^\rho(\tilde{S} - R)\right) G - \mu \\ &= H - G \max_{\rho=1,\dots,\sigma} \boldsymbol{\lambda}^\rho(\tilde{S} - R). \end{aligned} \tag{68}$$

On the other hand the measures allow matches. Take coalitions $Q^\rho \subseteq C^\rho$ $(\rho = \sigma + 1, \dots, r)$ satisfying $\boldsymbol{\lambda}^\rho(Q^\rho) = \min_{\tau=1,\dots,\sigma} \boldsymbol{\lambda}^\tau(R)$, define $Q := \sum_{\rho=\sigma+1}^{r} Q^\rho$ and note that (68) is now, in fact, an equation. We conclude that (67) is satisfied.

Moreover, we want to show that

$$F - \min_{\rho=1,\dots,\sigma} \boldsymbol{\lambda}^\rho(\tilde{S} - R) \leqslant \boldsymbol{u}_2(R) - \mu^\star \tag{69}$$

and

$$\max\left\{F - \min_{\rho=1,\dots,\sigma} \boldsymbol{\lambda}^\rho(\tilde{S} - R), 0\right\} \geqslant \boldsymbol{u}_2(R) - \mu^\star \tag{70}$$

hold true. Indeed, an application of (16) and (17) of Section 2 in the case $\tau = \sigma$ yields

$$\boldsymbol{v}^\star(R + Q) - \hat{\boldsymbol{x}}(Q) \leqslant M^1 - F$$

or

$$\boldsymbol{v}^\star(R + Q) - \hat{\boldsymbol{x}}(Q) \leqslant \max_{\rho=1,\dots,\sigma} \boldsymbol{\lambda}^\rho(R),$$

thus

$$\boldsymbol{v}^{\star}(R+Q) - \hat{\boldsymbol{x}}(Q) - \mu^{\star} \leqslant \max\left\{F - \min_{\rho=1,\dots,\sigma} \boldsymbol{\lambda}^{\rho}(\tilde{S} - R), 0\right\}. \tag{71}$$

On the other hand we have

$$\boldsymbol{v}^{\star}(R) - \mu^{\star} = F - \min_{\rho=1,\dots,\sigma} \boldsymbol{\lambda}^{\rho}(\tilde{S} - R).$$

We conclude that (69) and (70) are satisfied.

If $r > \sigma$ holds true, then the equation

$$\boldsymbol{v}^{\star}(R + C^r) - \hat{\boldsymbol{x}}(C^r) - \mu^{\star} = M^1 - F - \mu^{\star} = 0$$

is satisfied. Hence, the derived game coincides with $\boldsymbol{w}$ in this case.

A game is *exact*, if any coalition is effective with respect to some core element. Clearly a min-game is exact, iff $\sigma = r$ holds true. For an exact min-game inequality (43) is necessarily satisfied; formally we have a strong long side. In the exact case we obtain $G = 1$ and $H = M^1$, thus

$$H - G \max_{\rho=1,\dots,r} \boldsymbol{\lambda}^{\rho}(I - R) = \min_{\rho=1,\dots,r} \boldsymbol{\lambda}^{\rho}(R) \geqslant 0$$

is satisfied. Therefore, $\boldsymbol{w}(R)$ is given by

$$\boldsymbol{w}(R) = \max\left\{H - G \max_{\rho=1,\dots,\sigma} \boldsymbol{\lambda}^{\rho}(\tilde{S} - R),\ F - \min_{\rho=1,\dots,\sigma} \boldsymbol{\lambda}^{\rho}(\tilde{S} - R)\right\}$$

and the proof is again finished by (67), (69), and (70). □

Note that the proof of the theorem, when applied to min-games with a strong long side only, does not require the assumption that some maximal corner contains a player of minimal weight.

The internal discussion inside each cartel determines the shape of the solution or rather the shape of each $\boldsymbol{\mu}^{\rho}$. This goal we approach in Section 6. Within the next section, we explain that the assumptions about balancedness employed so far follow from requirements concerning the size of the game. For “large games” the modiclus behaves as indicated in Theorems 3.4, 4.1 and 4.5.

5. Large games, balancedness, and nondegeneracy

This section has the character of an interlude. We want to introduce the notion of “large games” in a suitable sense and show that the results of the previous sections indeed clarify the treatment of corners when “many players” (of the smallest type) are present. In fact it will turn out that the t-fold replication of a min-game, the determining measures of which are integervalued and assign weight 1 to at least one player, satisfies all assumptions employed in the theorems of the subsequent sections, if t is large enough.

In order to simplify the framework, we will tentatively change the notation and replace $(C^\rho, \boldsymbol{\lambda}^\rho)$ by $(I, \boldsymbol{\lambda})$. Thus, we consider a finite set I of cardinality n and a positive measure $\boldsymbol{\lambda} \geqslant 0$ on I with total weight $\boldsymbol{\lambda}(I) = m$. Moreover, we fix a total ordering $\prec$ on I satisfying $\lambda_i \geqslant \lambda_j$ whenever $i \prec j$ holds true. Throughout this section we shall assume that $\boldsymbol{\lambda}$ is integervalued. Also we write $\lambda^{\max}$ for the maximum of $\{\lambda_i \,|\, i \in I\}$.

Lemma 5.1. *Let $p \in \mathbb{N}$ satisfy $\lambda^{\max} \leqslant p \leqslant \boldsymbol{\lambda}(I)$. Then the system*

$$\underline{\underline{\mathbf{S}}}_{\boldsymbol{\lambda}, \prec, p} := \left\{ S \in \underline{\underline{\mathbf{P}}} \,\middle|\, \begin{array}{l} \boldsymbol{\lambda}(S) \leqslant p,\ \boldsymbol{\lambda}(S + \{i\}) > p (i \in I - S), \\ \boldsymbol{\lambda}((S + j) \cap \{k \in I \,|\, k \preceq j\}) \leqslant p \\ (j \in I - S, j \prec \max S) \end{array} \right\} \tag{72}$$

is balanced.

Proof. We proceed by induction. If $|I| = 1$, the requirements imply immediately that I is the unique member of $\underline{\underline{\mathbf{S}}} := \underline{\underline{\mathbf{S}}}_{\boldsymbol{\lambda}, \prec, p}$ and the lemma follows.

Assume now, that $|I|$ exceeds 1 and the lemma has been verified for all player sets of less cardinality. Moreover, w.l.o.g. assume that $I = \{1, \ldots, n\}$ and that $\prec$ is the natural ordering of integers. Let $\bar{S} \in \underline{\underline{\mathbf{S}}}$ be the lexicographically first coalition (i.e., collect the largest weights until reaching but not exceeding p). Fix player $i \in \bar{S}$ and consider the following two cases that may occur:

(1) $\boldsymbol{\lambda}(I - \{i\}) \leqslant p$. Then $I - \{i\}$ is an element of $\underline{\underline{\mathbf{S}}}$. Moreover, this coalition is the unique element which does not contain i.
(2) $\boldsymbol{\lambda}(I - \{i\}) > p$. Then, by induction hypothesis, the system $\underline{\underline{\mathbf{S}}}^i$ which is obtained on $I - \{i\}$ using p and the restrictions of $\boldsymbol{\lambda}$ and $\prec$, is balanced. It turns out that $\underline{\underline{\mathbf{S}}}^i = \{S \in \underline{\underline{\mathbf{S}}} | i \notin S\}$. For, the inclusion $\subseteq$ is straightforward. Moreover, $\supseteq$ follows from the fact that every subcoalition of $\{k \in I | k \preceq i\}$ has measure less than or equal to p.

Consequently, in both cases, the indicator $1_{I - \{i\}}$ is a positive linear combination of the indicators 1_S $(S \in \underline{\underline{\mathbf{S}}}, i \notin S)$. Finally, we can write

$$1_I = \frac{1}{|\bar{S}|} \left(1_{\bar{S}} + \sum_{i \in \bar{S}} 1_{I - \{i\}} \right),$$

which proves the lemma. □

Theorem 5.2. *Let $M^1 \in \mathbb{N}$ be such that $\lambda^{\max} \leqslant M^1 < m$ holds true. Suppose $J_1 \subseteq I$ consists of players of weight 1 only. If the conditions*

$$m_+ := \boldsymbol{\lambda}(I - J_1) \geqslant M^1, \tag{73}$$

$$\boldsymbol{\lambda}(J_1) = |J_1| > \frac{2m_+ \lambda^{\max}}{M^1 - \lambda^{\max} + 1} \tag{74}$$

and

$$|J_1|^2 > 2m_+\lambda^{\max} \tag{75}$$

are fulfilled, then the system

$$\underline{\underline{\mathbf{Q}}}_{M^1} := \{S \in \underline{\underline{\mathbf{P}}} | \boldsymbol{\lambda}(S) = M^1\} \tag{76}$$

is balanced and nondegenerate.

Proof. *Step* 1: Assume $I=\{1,\ldots,n\}$ and $\lambda_1 \geqslant \cdots \geqslant \lambda_n$. Thus λ_1 is the maximal weight. Define $m_1 := m - m_+ = |J_1| = \boldsymbol{\lambda}(J_1)$ and let $p \in \mathbb{N}$ satisfy

$$\lambda_1 \leqslant p \leqslant m_+. \tag{77}$$

We denote by $\underline{\underline{\mathbf{S}}}_p^+$ the system on $I - J_1 := I^+$ which is obtained via Lemma 5.1 applied to the restriction of $\boldsymbol{\lambda}$, the natural ordering, and p.

By Lemma 5.1 there are balancing coefficients $b_R^+(p) = b_R^+ > 0$ ($R \in \underline{\underline{\mathbf{S}}}_p^+$) satisfying

$$\sum_{R \in \underline{\underline{\mathbf{S}}}^+} b_R^+ 1_R = 1_{I^+}.$$

By definition of $\underline{\underline{\mathbf{S}}}_p^+$ the weight $\boldsymbol{\lambda}(R)$ of any coalition $R \in \underline{\underline{\mathbf{S}}}_p^+$ satisfies

$$\boldsymbol{\lambda}(R) \geqslant p - \lambda_1 + 1. \tag{78}$$

By integration with $\boldsymbol{\lambda}$ we conclude that

$$m_+ = \sum_{R \in \underline{\underline{\mathbf{S}}}^+} b_R^+ \boldsymbol{\lambda}(R) \geqslant (p - \lambda_1 + 1) \sum_{R \in \underline{\underline{\mathbf{S}}}^+} b_R^+ \tag{79}$$

holds true. Using (77) we obtain that $p > \lambda_1 - 1$ holds and, thus, we obtain an estimate

$$\sum_{R \in \underline{\underline{\mathbf{S}}}^+} b_R^+ \leqslant \frac{m_+}{p - \lambda_1 + 1}. \tag{80}$$

Let $q \in \mathbb{N}$ now satisfy

$$p \leqslant q \leqslant m - \lambda_1 + 1 \tag{81}$$

and define

$$\underline{\underline{\mathbf{S}}}_{p,q} := \{R + T | R \in \underline{\underline{\mathbf{S}}}_p^+,\ T \subseteq J_1,\ \boldsymbol{\lambda}(T) = q - \boldsymbol{\lambda}(R)\} \subseteq \underline{\underline{\mathbf{Q}}}_{M^1}. \tag{82}$$

We conclude from (74), (77), and (78) that $m_1 + \boldsymbol{\lambda}(R) \geqslant q$ holds true for any $R \in \underline{\underline{\mathbf{S}}}_p^+$, thus the coefficients

$$b_{R+T}(p,q) = b_{R+T} := \frac{b_R^+}{|\{T \subseteq J_1 | R + T \in \underline{\underline{\mathbf{S}}}_{p,q}\}|} \tag{83}$$

are well defined. We obtain

$$\sum_{R+T\in\underline{\underline{\mathbf{S}}}_{p,q}} b_{R+T}1_{R+T} = 1_{I^+} + K(p,q)1_{J_1} =: \boldsymbol{x}^{p,q} \tag{84}$$

with a suitable constant $K(p,q) \geqslant 0$. We want to show that this constant can be estimated. Indeed, for $R+T \in \underline{\underline{\mathbf{S}}}_{p,q}$, inequality (78) implies that

$$|T| = q - \boldsymbol{\lambda}(R) \leqslant q - p + \lambda^1 - 1$$

holds true. By (80) we obtain

$$K(p,q) \leqslant \frac{m_+(q-p+\lambda_1-1)}{m_1(p-\lambda_1+1)}. \tag{85}$$

Step 2: We are going to apply (85) in the case $p=q=M^1$. Indeed, the assumption (74) shows that (81) holds in this case. Moreover, $\underline{\underline{\mathbf{S}}} := \underline{\underline{\mathbf{S}}}_{M^1,M^1}$ is a subset of the system $\underline{\underline{\mathbf{Q}}}_{M^1}$, thus $\boldsymbol{x} := \boldsymbol{x}^{M^1,M^1}$ is a nonnegative linear combination of indicators of this system. The inequalities

$$\begin{aligned} 1 \quad &> \quad \frac{2m_+\lambda_1}{m_1(M^1-\lambda_1+1)} && \text{(by (74)} \\ &\geqslant \quad \frac{m_+(\lambda_1-1)}{m_1(M^1-\lambda_1+1)} \geqslant K(M^1,M^1) && \text{(by (80))} \end{aligned}$$

show that $K := K(M^1,M^1) < 1$ holds true.

Step 3: We are going to apply (85) in the case $p := m_+ - \max\{0, M^1 - m_1\}$ and $q := m - M^1$. First of all note that p satisfies (77) by (74). Secondly q satisfies (81) by (74) and the fact that $m = m_+ + m_1 \geqslant M^1 + m_1$ holds true. Next we shall show that $L := K(p,q)$ is strictly less than 1. Two cases may be distinguished:

(1) If $M^1 \leqslant m_1$ holds true, then $p = m_+$ is valid. In this case $\underline{\underline{\mathbf{S}}}_{p,q}$ consists of all coalitions of the form $I^+ + T$ where $T \subseteq J_1$ satisfies $|T| = m_1 - M^1$. Hence $L < 1$ is satisfied.

(2) If $M^1 > m_1$ holds true, then the inequalities

$$\begin{aligned} L &\leqslant \frac{m_+(m-M^1-m_+ + M^1 - m_1 + \lambda_1 - 1)}{m_1(m_+ - M^1 + m_1 - \lambda_1 + 1)} && \text{(by (85))} \\ &< \frac{m_+(\lambda_1-1)}{m_1(m_+ - M^1 + \frac{m_1}{2})} && \text{(because (74) implies } m_1 > 2\lambda_1) \\ &< \frac{2m_+\lambda_1}{(m_1)^2} < 1 && \text{(by (75))} \end{aligned}$$

show the assertion.

Let $b_{R+T}(R+T \in \underline{\underline{\mathbf{S}}}_{p,q})$ be the coefficients as defined in (83) and put $\beta := \sum_{R+T\in\underline{\underline{\mathbf{S}}}_{p,q}} b_{R+T} - L > 0$. Then the equation

$$\frac{1}{\beta}\sum_{R+T\in\underline{\underline{\mathbf{S}}}_{p,q}} b_{R+T}1_{I-(R+T)} = \frac{1}{\beta}((\beta+L)1_I - 1_{I^+} - L1_{J_1})$$

$$= \frac{\beta+L-1}{\beta}1_{I^+} + 1_{J_1} = \gamma 1_{I^+} + 1_{J_1} =: \boldsymbol{y} \tag{86}$$

shows that $\boldsymbol{y}$ is a positive linear combination of the indicators of the system

$$\underline{\underline{\mathbf{T}}} := \{S \in \underline{\underline{\mathbf{P}}} | I - S \in \underline{\underline{\mathbf{S}}}_{p,q}\}.$$

Moreover, $\gamma < 1$ holds, because $L < 1$ is valid. The definition of p and q implies that $\underline{\underline{\mathbf{T}}}$ is a subset of $\underline{\underline{\mathbf{Q}}}_{M^1}$.

Step 4: The third system of coalitions that will be used is the set $\underline{\underline{\mathbf{R}}}$ which is defined as follows. For any $i \in I^+$ define the system $\underline{\underline{\mathbf{R}}}_{(i)}$ and $\underline{\underline{\mathbf{R}}}$ by

$$\underline{\underline{\mathbf{R}}}_{(i)} := \{R - \{i\} + T | R \in \underline{\underline{\mathbf{S}}}^+, T \subseteq J_1, |T| = M^1 - \boldsymbol{\lambda}(R - \{i\})\} \tag{87}$$

and $\underline{\underline{\mathbf{R}}} = \bigcup_{i\in I^+} \underline{\underline{\mathbf{R}}}^{(i)}$. Here the natural notation $\underline{\underline{\mathbf{S}}}^+ = \{S \cap I^+ | S \in \underline{\underline{\mathbf{S}}}\}$ is used. Let b_R^+ $(R \in \underline{\underline{\mathbf{S}}}^+)$ be balancing coefficients of this system. Condition (74) implies that $m_1 \geqslant M^1 - \boldsymbol{\lambda}(R - \{i\})(R \in \underline{\underline{\mathbf{S}}}^+)$ holds true, thus the coefficients

$$b^{(i)}_{R-\{i\}+T} := \frac{b_R^+}{|\{T \subseteq J_1 | R - \{i\} + T \in \underline{\underline{\mathbf{R}}}_{(i)}\}|} \tag{88}$$

are well defined. Similarly to (84) it is seen that

$$\sum_{R-\{i\}+T\in\underline{\underline{\mathbf{R}}}_{(i)}} b^{(i)}_{R-\{i\}+T} 1_{R-\{i\}+T} = 1_{I^+-\{i\}} + K^{(i)}1_{J_1} =: \boldsymbol{x}^{(i)} \tag{89}$$

holds. Summing up the vectors $x^{(i)}$ and normalizing yields

$$\frac{1}{n-m_1-1}\sum_{i\in I^+} \boldsymbol{x}^{(i)} = 1_{I^+} + \tilde{K}1_{J_1} =: \boldsymbol{z}. \tag{90}$$

Hence, we have shown that $\boldsymbol{z}$ can be expressed as a positive linear combination of the indicators of the system $\underline{\underline{\mathbf{R}}}_{(i)} \subseteq \underline{\underline{\mathbf{Q}}}_{M^1}$.

Step 5: Put $\underline{\underline{\mathbf{Q}}} := \underline{\underline{\mathbf{R}}} \cup \underline{\underline{\mathbf{S}}} \cup \underline{\underline{\mathbf{T}}}$. The last three steps show that $\underline{\underline{\mathbf{Q}}}$ is, indeed, a subsystem of $\underline{\underline{\mathbf{Q}}}_{M^1}$. In view of Remark 2.3 it suffices to show that $\underline{\underline{\mathbf{Q}}}$ is balanced and nondegenerate.

In view of the fact that $K > 1$ holds true, we can find $1 > \varepsilon > 0$ such that $K - \varepsilon(K - \tilde{K}) > 1$ is true. Then $\tilde{\boldsymbol{x}} := (1-\varepsilon)\boldsymbol{x} + \varepsilon\boldsymbol{z}$ can be expressed as

$$\tilde{\boldsymbol{x}} = 1_{I^+} + \bar{K}1_{J_1}$$

with a suitable $0 \leqslant \bar{K} < 1$. Moreover, the equation

$$\frac{1-\bar{K}}{1-\bar{K}\gamma}\boldsymbol{y} + \frac{1-\gamma}{1-\bar{K}\gamma}\tilde{\boldsymbol{x}} = 1_I$$

shows that $\underline{\underline{\mathbf{Q}}}$ is balanced, because the coefficients are strictly positive.

Now we turn to nondegeneracy. The vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ can be used to show that 1_{I^+} and 1_{J_1} are spanned by the indicators of $\underline{\underline{\mathbf{Q}}}$. Additionally using the $\boldsymbol{x}^{(i)}$ $(i \in I^+)$ defined in (89) shows that every indicator $1_{\{i\}}(i \in I^+)$ as well belongs to the span. Then pick any $i \in I^+$ and any coalition $R \in \underline{\underline{\mathbf{S}}}^+$ which contains i. All indicators 1_T satisfying $R - \{i\} + T \in \underline{\underline{\mathbf{R}}}_{(i)}$ are spanned. The corresponding coalitions are exactly those subsets of J_1 that possess the cardinality $M^1 - \boldsymbol{\lambda}(R) + \lambda_i$. This cardinality is, by (74), strictly less than m_1 and, by definition of $\underline{\underline{\mathbf{S}}}^+$, it is strictly positive. Therefore $1_{\{i\}}(i \in J_1)$ is spanned. □

Now we draw the conclusions of our results. To this end, we return to the original setup within which we deal with a min-game. Recall that the shape of the modiclus (with respect to the coefficients determining the share of the cartels) was clarified in Sections 3 and 4. We want to show that the conditions employed are satisfied if there are sufficiently many small players present.

For $\tau \in \mathbb{N}$, the t-fold replication of any measure $\boldsymbol{\lambda}$ is denoted by $\boldsymbol{\lambda}^{(t)}$. Likewise, $I^{(t)}$ is used for the t-fold replication of I. Thus, we assume that the t-fold replication of the game $(I, \underline{\underline{\mathbf{P}}}, \boldsymbol{v})$, denoted by $(I^{(t)}, \underline{\underline{\mathbf{P}}}^{(t)}, \boldsymbol{v}^{(t)})$ is a concept well known to the reader.

Corollary 5.3. *Let $\boldsymbol{v} = \bigwedge\{\boldsymbol{\lambda}^1, \ldots, \boldsymbol{\lambda}^r\}$ be an integer valued min-game. Assume that, for some $\rho > \sigma$, there is at least one player with weight 1 in corner C^ρ. Then there is $t_0 \in \mathbb{N}$ such that for any $t \geqslant t_0$ with respect to the replicated game $\boldsymbol{v}^{(t)}$ the system of partners of maximal diagonal coalitions, i.e., the system*

$$\underline{\underline{\mathbf{D}}}^{m\rho(t)} = \{S \in \underline{\underline{\mathbf{P}}}^{(t)} | \boldsymbol{\lambda}^{\rho(t)}(S) = tM^1\} \tag{91}$$

is balanced and nondegenerate.

Proof. Given ρ, let k be a player with weight 1 in corner ρ. We appeal to Theorem 5.2 which will be applied to $C^{\rho(t)}, \boldsymbol{\lambda}^{\rho(t)}$ and tM^1. To be more precise, we have $\boldsymbol{\lambda}^\rho(C^\rho - \{j\}) \geqslant M^1$ and hence, for any natural t, we have $\boldsymbol{\lambda}^{\rho(t)}(C^{\rho(t)} - J_1) \geqslant tM^1$ where J_1 is the coalition of all t copies of player k. Thus, using $\boldsymbol{\lambda} = \boldsymbol{\lambda}^{\rho(t)}$ for the moment, condition (73) is satisfied for all $t \in \mathbb{N}$.

Now, the right-hand term in (74) is clearly bounded in t. For, tm_+ as well as tM^1 increase linearly and $\max_{j \in C^{\rho(t)}} \lambda_j$ does not change with t. therefore, if $|J_1| = t$ is large enough, Eq. (74) will be satisfied.

Similarly, the left-hand side in (75) equals t^2 while the right-hand side again increases linearly. It is now obvious how to choose the desired bound t_0 in order to ensure the statement of Theorem 5.2. Thereafter, it satisfies to realize that $\underline{\underline{\mathbf{Q}}}_{tM^1}$ as defined in (76) equals the system of partners we are concerned with, that is (91). □

Remark 5.4. (1) Note that, under the assumptions of Corollary 5.3, t_0 can be chosen in such a way that the vector $(\boldsymbol{\lambda}^{(t)1}, \ldots, \boldsymbol{\lambda}^{(t)r})$ of replicated measures allows matches (cf. (62) of Section 3) for $t \geqslant t_0$.

(2) The index of relative powers, i.e., the quantity $\imath(\boldsymbol{v})$ (cf. formula (42) of Section 4) is preserved under replication. This means that a min-game possesses a strong long

side, a strong short side, or balanced sides, respectively, if and only if this property holds for any replicated game.

(3) It is not hard to see that another procedure can be implemented which also preserves the index of relative powers and ensures that Theorem 3.4 holds true eventually. One can add players of weight 1 in large numbers to each corner. This way the mass relations can be kept constant and again it is possible to show that the balancedness as well as the nondegeneracy condition (see Theorem 5.2) is ensured after finitely many steps. The proof is actually much easier and we will not dwell on this subject excessively. We refer to this procedure by *adding small players*.

(4) We shall say that an integervalued min-game $\bigwedge\{\boldsymbol{\lambda}^1,\ldots,\boldsymbol{\lambda}^r\}$ is *large*, if $\underline{\underline{\mathbf{D}}}^{m\rho}$ is balanced and nondegenerate, C^ρ contains a player of weight 1 ($\rho=\sigma+1,\ldots,r$), and $(\boldsymbol{\lambda}^1,\ldots,\boldsymbol{\lambda}^r)$ allows matches.

Corollary 5.5. *Let $\boldsymbol{v}=\bigwedge\{\boldsymbol{\lambda}^1,\ldots,\boldsymbol{\lambda}^r\}$ be an integer valued min-game. Assume that, for all $\rho>\sigma$, there is at least one player with weight 1 in corner C^ρ. Then both, replication and adding small players, generate large games after finitely many steps. Hence, the assertions of all theorems of Sections 3 and 4 are valid.*

6. The VIP formula and a bankruptcy problem

Within this section we restrict ourselves to min-games $\boldsymbol{v}=\bigwedge\{\boldsymbol{\lambda}^1,\ldots,\boldsymbol{\lambda}^r\}$ satisfying the following conditions:

(1) The measures $\boldsymbol{\lambda}^1,\ldots,\boldsymbol{\lambda}^r$ are integervalued.
(2) The measures $\boldsymbol{\lambda}^2,\ldots,\boldsymbol{\lambda}^\sigma$ are *uniformly distributed*, i.e.,

$$\lambda_i^\rho=1 \quad (i\in C^\rho,\rho=2,\ldots,\sigma). \tag{92}$$

(3) $\boldsymbol{v}$ has either a strong long side or balanced sides.
(4) The corners $\underline{\underline{\mathbf{D}}}^{m\rho}$ are balanced and nondegenerate w.r.t. C^ρ for any $\rho=\sigma+1,\ldots,r$.
(5) Any weight in every nonminimal corner exceeds the sum of the smaller weights by at most one, i.e., that

$$\lambda_i^\rho \leqslant 1+\boldsymbol{\lambda}^\rho(\{j\in I|\lambda_j^\rho<\lambda_i^\rho\}) \quad (i\in C^\rho,\rho=\sigma+1,\ldots,r) \tag{93}$$

holds true.

Given integervalued measures, (93) is equivalent to the condition that every natural number smaller than or equal to M^ρ is the weight of some coalition with respect to $\boldsymbol{\lambda}^\rho$. Also, (93) yields $\lambda^{\min}=1$ and ensures that the long side shows small players (cf. Definition 4.2).

Given these assumptions, we are going to classify the behavior of $\hat{\boldsymbol{x}}:=\boldsymbol{\psi}(\boldsymbol{v})$ by a formula involving the shape of the initial assignments represented by $\boldsymbol{\lambda}^1$. First of all Theorems 3.4, 4.3 and 4.5 completely determine the shape of the modiclus restricted to the union of nonminimal corners $I-\tilde{S}$. Here $\tilde{S}=\sum_{\rho=1}^{\sigma}C^\rho$ is the short side of the market as in Section 4. Moreover, these theorems determine the vector $\boldsymbol{c}$ of convexifying coefficients given by $c_\rho M^1=\hat{\boldsymbol{x}}(C^\rho)$ for any $\rho=1,\ldots,r$. As a consequence, for any player in $C^2, \ldots, C^\sigma$ the modiclus is completely determined by the equal treatment property (see [14]).

Imagine a situation in which the modiclus $\hat{\boldsymbol{x}}$ is agreed upon by the bargaining process of the representatives of the various cartels (corners), and hence is externally fixed. As in Section 5, for the sake of the internal discussion, we will tentatively replace corner C^1 by I—this will now be the player set. The initial assignment $\boldsymbol{\lambda}^1$ will be replaced by $\boldsymbol{\lambda}$ and because of the external influence the players will have to agree on the distribution of $M^1 - \hat{\boldsymbol{x}}(\sum_{\rho=2}^r C^\rho)$. This quantity is now replaced by a positive real E. Which kind of "internal game" should we have in mind in order to discuss the bargaining process inside the cartel C^1?

Of course players will internally argue with their strength in the global game $\boldsymbol{v}$ given the modiclus (which is fixed on the corners outside). These arguments may formally be based on the quantity

$$\max \left\{ \boldsymbol{v}(\{i\} + T) - \hat{\boldsymbol{x}}(T) \,\middle|\, T \subseteq \sum_{\rho=2}^r C^\rho \right\} \tag{94}$$

for $i \in C^1$. That is, player i points to coalitions he could form with partners (who are already assigned a definite share by the modiclus based on the uniform distribution in their corner). Player i could try to join these partners at the same conditions and then he would get the surplus. In view of Lemma 2.1 and Corollary 2.2 we expect this quantity to be maximal, when player i attempts to form *diagonal* coalitions (the excess appears more or less in Eq. (94)).

Now, based on $\hat{\boldsymbol{x}}$ and the coefficient c_ρ of corner C^ρ, we compute for player $k \in C^\rho$ the payoff

$$\hat{x}_k = \frac{M^1}{M^\rho} c_\rho \lambda_k^\rho = \frac{M^1}{M^\rho} c_\rho.$$

Hence, the quantity specified in (94) when $\{i\} + T$ is diagonal turns out to be

$$\boldsymbol{v}(\{i\} + T) - \sum_{\rho=2}^r \frac{M^1}{M^\rho} c_\rho \lambda_i^1 = \lambda_i^1 \left(1 - \sum_{\rho=2}^r \frac{M^1}{M^\rho} c_\rho \right).$$

This quantity is now abbreviated by $\lambda_i^1 \beta$. Similarly, a coalitions $S \subseteq C^1$ would have an aspiration of $\boldsymbol{\lambda}^1(S)(1 - (M^1/M^\rho)c_\rho)$ or $\boldsymbol{\lambda}^1(S)\beta$. Note that $E \leqslant \beta\boldsymbol{\lambda}^1(C^1)$ can be verified.

Let us focus on a player set I, a measure $\boldsymbol{\lambda}$ and positive real numbers E and β satisfying $E \leqslant \beta\boldsymbol{\lambda}(I)$. Each player enters the discussion with a "claim" based on his external possibilities. This claim is given by $\lambda_i\beta$. However, the total of claims, i.e. $\beta\boldsymbol{\lambda}(I)$ (weakly) exceeds the "estate" E that can be allotted at all inside the cartel. This kind of problem is well known in the literature and was first discussed by Aumann–Maschler [1] who treat a bankruptcy problem that appears already in the Talmud. In this context, the data $\beta\lambda_i$ appear as "debts" of the estate towards the contestants. The game $\boldsymbol{w}$ derived from this problem is given by

$$\boldsymbol{w}(S) := (E - \beta\boldsymbol{\lambda}(I - S))^+ \quad (S \in \underline{\mathbf{P}}) \tag{95}$$

and reflects a pessimistic attitude: If the opposing coalition $I - S$ successfully leaves booking its claims, the remainder towards E is what is left for coalition S to distribute. The solution concept mentioned in the Talmud according to Aumann–Maschler is the "contested garment consistent solution" (the *CG-solution*). It coincides with the nucleolus of the corresponding game $\boldsymbol{w}$ (the *CG-game*).

The solution concept one might adopt is, therefore, suggested by the procedure developed in [1]. In the present context, we are going to introduce this concept as follows.

Imagine that a quantity of $\beta\lambda_i/2$ is guaranteed to each of the players. This is the average of his individually rational payoff (which is 0) in the global game $\boldsymbol{v}$ and the aspiration in the endogenous game of the cartel.

Now the rich players have to pay a constant fee ε and the poor ones are allotted $\beta\lambda_i/2$. Who is considered to be rich and who is poor depends on the size of the fee which is determined by the requirement

$$\sum_{i \in I} \max\left(\beta\lambda_i - \varepsilon, \frac{\beta\lambda_i}{2}\right) = E. \tag{96}$$

Thereafter, if $\varepsilon(E, \beta)$ is the (unique) solution of (96), the labels "rich" and "poor" can immediately be allotted. The smallest rich player is the one, say k_0, such that $\lambda_{k_0} - \varepsilon(E, \beta)$ just exceeds or equals $(\beta\lambda_{k_0})/2$ and $\lambda_{k_0+1} - \varepsilon(E, \beta)$ is below $(\beta\lambda_{k_0+1})/2$.

To have a nice term, we call the rich players in this context the *VIP*s. The final formula arising eventually for the modiclus of the corner with big chunks of initial assignments will be called the *VIP formula*.

Remark 6.1. Recall that the total mass is $\boldsymbol{\lambda}(I) =: m$. Now, for $\beta m > E \geqslant \beta m/2$, it is not too hard to see that (96) indeed admits of a unique solution $\varepsilon(E, \beta) \geqslant 0$.

Now we are going to present the endogenous solution in a precise manner. The result will be called the *E-β-CG measure*.

Definition 6.2. Let E, β be real numbers. Assume that (E, β) satisfies

$$0 < \beta \quad \text{and} \quad \frac{\beta}{2} m < E \leqslant \beta m. \tag{97}$$

Define the real number $\varepsilon(E, \beta)$ by the requirement

$$\sum_{i \in I} \max\left\{\lambda_i \beta - \varepsilon(E, \beta), \frac{\lambda_i}{2}\beta\right\} = E \tag{98}$$

and the *E-β-CG measure* $\boldsymbol{x}^{(E,\beta)}$ by

$$\boldsymbol{x}_i^{(E,\beta)} := \max\left\{\lambda_i \beta - \varepsilon(E, \beta), \frac{\lambda_i}{2}\beta\right\}. \tag{99}$$

Remark 6.3. (1) Assumption (97) implies that $\varepsilon(E,\beta)$ and, thus, $\boldsymbol{x}^{(E,\beta)}$ are well defined. Moreover, by definition, we have

$$\boldsymbol{x}^{(E,\beta)}(I) = E. \tag{100}$$

Again (97) implies that $\varepsilon(E,\beta)$ is nonnegative. Note that $\varepsilon(E,\beta)=0$ holds true if and only if E coincides with βm.

(2) The following procedure shows how to compute $\varepsilon(E,\beta)$ recursively. For any $\lambda \in \{\lambda_i \mid i\in I\}$ let $S_\lambda := \{i\in I \mid \lambda_i \geqslant \lambda\}$ be the set of players of a weight weakly exceeding λ and define ε_λ by the requirement

$$\sum_{i\in S_\lambda} (\beta\lambda_i - \varepsilon_\lambda) + \sum_{j\in I-S_\lambda} \beta\frac{\lambda_j}{2} = E, \tag{101}$$

i.e., by

$$\varepsilon_\lambda := \frac{1}{|S_\lambda|}\left(\frac{\beta m}{2} + \frac{\beta\boldsymbol{\lambda}(S_\lambda)}{2} - E\right). \tag{102}$$

Let $\lambda^{\max}$ and $\lambda^{\min}$ denote the maximum and minimum of $\{\lambda_i \mid i\in I\}$ and observe that

$$\varepsilon_{\lambda^{\min}} = \frac{1}{n}(\beta m - E) \geqslant 0 \quad \text{(by (97))}$$

holds true as well as

$$\begin{aligned} 2\varepsilon_{\lambda^{\max}} &= \frac{1}{|S_{\lambda^{\max}}|}(\beta m + \boldsymbol{\lambda}(S_{\lambda^{\max}}) - 2E) \\ &< \frac{1}{|S_{\lambda^{\max}}|}\beta\boldsymbol{\lambda}(S_{\lambda^{\max}}) = \beta\lambda_i (i\in S_{\lambda^{\max}}). \end{aligned}$$

Thus $\bar{\lambda} := \min\{\lambda_i \mid i\in I, 2\varepsilon_{\lambda_i} \leqslant \beta\lambda_i\}$ is a member of $\{\lambda_i \mid i\in I\}$. A comparison of (98) and (101) shows that $\varepsilon(E,\beta)$ coincides with $\varepsilon_{\bar{\lambda}}$.

(3) The measure $\boldsymbol{x}^{(E,\beta)}$ is indeed the nucleolus of the game $\boldsymbol{w}$ given by (95), hence it is the *contested garment consistent* solution of the underlying bankruptcy problem [1].

In order to describe the modiclus of the game $\boldsymbol{v}$ recall the quantity γ (Formula (63) of Definition 4.4). Define the quantities

$$\beta := \frac{1+(r-\sigma)(\sigma-1)\gamma}{\sigma} - \gamma\sum_{\rho=\sigma+1}^{r}\frac{M^1}{M^\rho}, \qquad E := M^1\left(\frac{1-(r-\sigma)\gamma}{\sigma}\right). \tag{103}$$

Remark 6.4. If $\boldsymbol{v}$ has a strong long side, then β can be written as

$$\beta = 1 - \frac{1}{r}\sum_{\rho=2}^{r}\frac{M^1}{M^\rho} = \frac{1}{r}(r - \imath(\boldsymbol{v}) + 2) \tag{104}$$

and E can be written as

$$E = \frac{M^1}{r}, \tag{105}$$

thus

$$\frac{1}{r} \leqslant \beta < \frac{2}{r}$$

is valid. Therefore,

$$M^1 \frac{\beta}{2} \leqslant E < M^1 \beta \tag{106}$$

holds true. Moreover (106) is also valid in the case that $\boldsymbol{v}$ has balanced sides. Indeed, in this case β and E are given by

$$\beta = \frac{1 + 2M^r}{\sigma + rM^r} \quad \text{and} \quad E = M^1 \left(\frac{1 + M^r}{\sigma + rM^r} \right), \tag{107}$$

thus (106) is valid even with strict inequalities in this case.

Hence, the pair (E, β) satisfies condition (97) and the quantity $\varepsilon(E, \beta)$ and the E-β-CG measure $\boldsymbol{x}^{(E,\beta)}$ are well defined. Of course we apply the corresponding definitions to the finite set C^1 and to the restriction of $\boldsymbol{\lambda}^1$ to C^1. In what follows the measure $\boldsymbol{x}^{(E,\beta)}$ on C^1 is as well considered as a measure on I with carrier C^1 whenever this is needed.

Theorem 6.5. *The modiclus of $\boldsymbol{v}$ is the imputation given by*

$$\boldsymbol{\psi}(\boldsymbol{v}) = \hat{\boldsymbol{x}}^{(E,\beta)} + \gamma \sum_{\rho=2}^{r} \frac{\boldsymbol{\lambda}^\rho}{M^\rho}. \tag{108}$$

Proof. By Theorems 3.4, 4.3, 4.5 and Corollary 2.6 of [14] the modiclus $\boldsymbol{\psi}(\boldsymbol{v}) =: \hat{\boldsymbol{x}}$ has the desired form, when restricted to $I - C^1$.

Let $\boldsymbol{w}$ be the bankruptcy game with player set C^1 defined by

$$\boldsymbol{w}(S) = (E - \beta\lambda^1(C^1 - S))^+.$$

By Remark 6.4 $\boldsymbol{x} := \boldsymbol{x}^{(E,\beta)}$ is the nucleolus of $\boldsymbol{w}$ (see [1]). In view of Lemma 2.7 it suffices to show that $\boldsymbol{w}$ coincides with the derived game $\boldsymbol{v}_{C^1,\hat{\boldsymbol{x}}}$. For the trivial coalitions, coincidence is certainly true. Let $R \subseteq C^1, \emptyset \neq R \neq C^1$ be a nontrivial coalition and let $\boldsymbol{u}_1 := \boldsymbol{v}^{C^1,\hat{\boldsymbol{x}}}$ and $\boldsymbol{u}_2 := (\boldsymbol{v}^\star)^{C^1,\hat{\boldsymbol{x}}}$ be the corresponding reduced games. In view of (27) and (28) of Section 3 we obtain

$$\mu := \mu(\hat{\boldsymbol{x}}, \boldsymbol{v}) = M^1 \left((r - \sigma)\gamma - \gamma \sum_{\rho=\sigma+1}^{r} \frac{M^1}{M^\rho} \right) = \beta M^1 - E \tag{109}$$

and

$$\mu^\star := \mu(\hat{\boldsymbol{x}}, \boldsymbol{v}^\star) = M^1(1-\gamma). \tag{110}$$

In order to show that

$$\boldsymbol{u}_1(R) - \mu = E - \beta\boldsymbol{\lambda}^1(C^1 - R) \tag{111}$$

is satisfied let $Q \subseteq I - C^1$. An application of (15) of Section in 2 yields

$$\boldsymbol{v}(R+Q) - \hat{\boldsymbol{x}}(Q) \leqslant \boldsymbol{\lambda}^1(R)\left(1 - (1-(r-\sigma)\gamma)\frac{\sigma-1}{\sigma} - \gamma\sum_{\rho=\sigma+1}^{r}\frac{M^1}{M^r}\right),$$

thus $\boldsymbol{v}(R+Q) - \hat{\boldsymbol{x}}(Q) \leqslant \boldsymbol{\lambda}^1(R)\beta$ holds true as well as

$$\boldsymbol{v}(R+Q) - \hat{\boldsymbol{x}}(Q) - \mu \leqslant (\boldsymbol{\lambda}^1(R) - M^1)\beta + E = E - \beta\boldsymbol{\lambda}^1(C^1 - R). \tag{112}$$

On the other hand the measures allow matches. Take coalitions $Q^\rho \subseteq C^\rho$ $(\rho = 2, \ldots, r)$ satisfying $\boldsymbol{\lambda}^\rho(Q^\rho) = \boldsymbol{\lambda}^1(R)$, define $Q := \sum_{\rho=\sigma+1}^{r} Q^\rho$ and note that (112) is now, in fact, an equation. We conclude that (111) is satisfied.

Now let $Q \subseteq I - C^1$ be a coalition. Lemma 2.1 ((16) and (17) applied to $\tau = 1$) implies that

$$\boldsymbol{v}^\star(R+Q) - \hat{\boldsymbol{x}}(Q) \leqslant \max\{M^1(1-\gamma), \boldsymbol{\lambda}^1(R)\}$$

and, thus,

$$\boldsymbol{u}_2(R) \leqslant (\boldsymbol{\lambda}^1(R) - M^1(1-\gamma))^+ \tag{113}$$

hold true. On the other hand we obtain

$$\boldsymbol{v}^\star(R + C^r) - \hat{\boldsymbol{x}}(C^r) = M^1(1-\gamma),$$

thus $\boldsymbol{u}_2(R) \geqslant 0$ is valid. Hence, it suffices to show that

$$\boldsymbol{u}_1(R) - \mu \geqslant \boldsymbol{\lambda}^1(R) - M^1(1-\gamma) \tag{114}$$

holds true. By (63) of Section 4 (with $\lambda^{\min} = 1$) we obtain $r\gamma \leqslant 1$, thus inequality (114) implies that

$$E + M^1(1-\gamma) = M^1\left(1 + \frac{1-r\gamma}{\sigma}\right) \geqslant M^1 \tag{115}$$

holds true. Eq. (111) together with (115) show that

$$\begin{aligned}(\boldsymbol{u}_1(R) - \mu) - (\boldsymbol{\lambda}^1(R) - \mu^\star) &= E - \beta\boldsymbol{\lambda}^1(C^1 - R) - \boldsymbol{\lambda}^1(R) + M^1(1-\gamma)\\ &\geqslant M^1 - \beta M^1 - (1-\beta)\boldsymbol{\lambda}^1(R) \geqslant 0\end{aligned}$$

holds true. □

7. A strong short side

In this section, we discuss the modiclus of a min-game with a strong short side. Under some conditions it coincides with the barycenter of the measures on the short side. This means that the modiclus equals the nucleolus of the exact game generated by the measures on the short side. The preliminary result, therefore, deals with the nucleolus of exact min-games. Next, we show that the nucleolus and the modiclus of an exact min-game coincide, if and only if the nucleolus treats all corners equally. Recall that a min-game $\boldsymbol{v} = \bigwedge\{\boldsymbol{\lambda}^1, \dots, \boldsymbol{\lambda}^r\}$ is exact, iff $\sigma = r$ holds true.

Theorem 7.1. *Let $\boldsymbol{v} = \bigwedge\{\boldsymbol{\lambda}^1, \dots, \boldsymbol{\lambda}^r\}$ be an exact min-game and let $\boldsymbol{\lambda}^\rho$ $(\rho = 1, \dots, r)$ be integervalued. Denote by $C_1^\rho := \{i \in C^\rho \mid \lambda_i^\rho = 1\}$ and assume that, for all $\rho = 1, \dots, r$, the condition*

$$|C_1^\rho| \geqslant \max\left\{ \lambda_i^\tau - 1 \,\middle|\, i \in \sum_{\tau \neq \rho} C^\tau \right\} \tag{116}$$

is satisfied. Then the nucleolus is the barycenter of the measures involved, i.e.,

$$\boldsymbol{\nu}(\boldsymbol{v}) = \bar{\boldsymbol{x}} = \frac{1}{r} \sum_{\rho=1}^{r} \boldsymbol{\lambda}^\rho. \tag{117}$$

Proof. *Step* 1: We are going to show that the coalitions of maximal excess form a balanced system. Moreover, we show the same fact for the coalitions of second largest excess and prove that this system is nondegenerate. This suffices in view of Remarks 2.5 and 2.3.

First of all we discuss the maximal excess with respect to $\bar{\boldsymbol{x}}$. Since the game is exact and $\bar{\boldsymbol{x}}$ is in the core, this excess is 0 and it is attained exactly on diagonal sets. Note that the system $\underline{\underline{\mathbf{D}}}$ of diagonal sets is easily recognized to be balanced, as the complement of a diagonal set is diagonal as well.

Step 2: We turn to the second largest excess. Note that, in view of Eq. (116), there is at most one corner C^ρ with $C_1^\rho = \emptyset$. If so, we assume without loss of generality that this is the first corner.

Now, for every $j \in C_1^\rho$ $(\rho = 2, \dots, r)$ the excess of $\{j\}$ turns out to be $-1/r$.

Next, let S be an arbitrary coalition which is *not* diagonal. Then there are corners π and τ such that $\boldsymbol{\lambda}^\pi(S) > \boldsymbol{v}(S) = \boldsymbol{\lambda}^\tau(S)$ holds true. Then the excess is

$$\begin{aligned} \boldsymbol{v}(S) - \bar{\boldsymbol{x}}(S) &= \boldsymbol{\lambda}^\tau(S) - \frac{1}{r} \sum_{\rho=1}^{r} \boldsymbol{\lambda}^\rho(S) \\ &= -\frac{1}{r} \sum_{\rho=1}^{r} (\boldsymbol{\lambda}^\rho(S) - \boldsymbol{\lambda}^\tau(S)) \leqslant -\frac{1}{r}(\boldsymbol{\lambda}^\pi(S) - \boldsymbol{\lambda}^\tau(S)) \leqslant -\frac{1}{r}. \end{aligned}$$

Consequently, the second largest excess is $-1/r$.

Step 3: We define, for $\rho = 1, \ldots, r$ and $i \in C^\rho$ a system of coalitions

$$\underline{\underline{\mathbf{S}}}^{\rho,i} := \{S \in \underline{\underline{\mathbf{P}}} | S^\rho = \{i\}, S^\tau \subseteq C_1^\tau, |S^\tau| = \lambda_i^\rho - 1(\tau \neq \rho)\}. \tag{118}$$

Observe that these systems are contained in $\underline{\underline{\mathbf{S}}}(-1/r, \bar{\boldsymbol{x}}, \boldsymbol{v})$. Now by summing up we obtain for each ρ

$$\sum_{i \in C^\rho} \frac{1}{|\underline{\underline{\mathbf{S}}}^{\rho,i}|} \sum_{S \in \underline{\underline{\mathbf{S}}}^{\rho,i}} 1_S = 1_{C^\rho} + \boldsymbol{y}^\rho. \tag{119}$$

Here, $\boldsymbol{y}^\rho$ is a nonnegative vector which has positive coordinates exactly in $\sum_{\tau \neq \rho} C_1^\tau$. This we write

$$\sum_{S \in \underline{\underline{\mathbf{S}}}^\rho} c_S 1_S = 1_{C^\rho} + \boldsymbol{y}^\rho \tag{120}$$

with $\underline{\underline{\mathbf{S}}}^\rho := \bigcup_{i \in C^\rho} \underline{\underline{\mathbf{S}}}^{\rho,i}$ and nonnegative coefficients $c_\bullet$. From (120) we obtain by again summing up

$$\sum_{S \in \underline{\underline{\mathbf{S}}}} \hat{c}_S 1_S = 1_I + \hat{\boldsymbol{y}} \tag{121}$$

with $\underline{\underline{\mathbf{S}}} := \bigcup_{\rho=1,\ldots,r} \underline{\underline{\mathbf{S}}}^\rho$ and an obvious choice of $\hat{c}_\bullet$. Moreover, $\hat{\boldsymbol{y}}$ is nonnegative and positive exactly on $\sum_{\rho=1}^r C_1^\rho$. This coalition (the one of players with weight 1) we now abbreviate by $C_1 := \sum_{\rho=1}^r C_1^\rho$.

Next, for $\tau = 1, \ldots, r$, we introduce a further system

$$\underline{\underline{\mathbf{T}}}^\tau := \{T \in \underline{\underline{\mathbf{P}}} | \boldsymbol{\lambda}^\rho(T) = M^1 - 1(\rho \neq \tau),\ \boldsymbol{\lambda}^\tau(T) = M^1\} \tag{122}$$

the elements of which have second largest excess as well. Take $\underline{\underline{\mathbf{T}}} = \bigcup_{\tau=1}^r \underline{\underline{\mathbf{T}}}^\tau$ and observe that

$$\sum_{T \in \underline{\underline{\mathbf{T}}}} 1_T = |\underline{\underline{\mathbf{T}}}| 1_I - \hat{\boldsymbol{z}}, \tag{123}$$

where $\hat{\boldsymbol{z}}$ is a nonnegative vector with positive coordinates exactly on C_1.

Choose $\varepsilon > 0$ sufficiently small such that

$$(1-\varepsilon)(1_I + \hat{\boldsymbol{y}}) + \varepsilon(1_I - \hat{\boldsymbol{z}}) =: 1_I - \boldsymbol{z} \tag{124}$$

satisfies $\boldsymbol{z} \geqslant \mathbf{0}$. Again, $\boldsymbol{z}$ has positive coordinates at most on C_1. Now the system $\underline{\underline{\mathbf{R}}} := \{\{j\} | j \in C_1\}$ consists of coalitions of second largest excess (Step 2) and yields

$$\boldsymbol{z} = \sum_{\{j\} \in \underline{\underline{\mathbf{R}}}} z_j 1_{\{j\}}. \tag{125}$$

Note that $\underline{\underline{\mathbf{R}}} \subseteq \underline{\underline{\mathbf{S}}}$ holds true. Hence, $\underline{\underline{\mathbf{S}}} \cup \underline{\underline{\mathbf{T}}}$ is a balanced system. Moreover, this system (actually $\underline{\underline{\mathbf{S}}}$) is nondegenerate. $\square$

Theorem 7.2. *Let* $\boldsymbol{v} = \bigwedge\{\boldsymbol{\lambda}^1, \ldots, \boldsymbol{\lambda}^r\}$ *be an exact min-game. Then the following two assertions are equivalent*:
(1) *The nucleolus* $\boldsymbol{\nu}(\boldsymbol{v})$ *treats all corners equally*, *i.e.*, *it satisfies*

$$\boldsymbol{\nu}(\boldsymbol{v})(C^\rho) = \frac{M^1}{r} \quad (\rho = 1, \ldots, r).$$

(2) *The modiclus* $\boldsymbol{\psi}(\boldsymbol{v})$ *coincides with the nucleolus* $\boldsymbol{\nu}(\boldsymbol{v})$.

Proof. One direction ((2) $\Rightarrow$ (1)) is implied by Theorem 3.4, because condition (33) is automatically satisfied and the assumption is empty in the exact case ($\sigma = r$). It remains to prove the opposite direction.

Note that the inequalities

$$0 \leqslant x_i \leqslant \lambda_i^\rho \quad (i \in C^\rho, \rho = 1, \ldots, r) \tag{126}$$

and

$$\frac{M^1(r-1)}{r} - \boldsymbol{\lambda}^\tau(T) + \boldsymbol{x}(T^\tau) = -e\left(T^\tau + \sum_{\rho \neq \tau} C^\rho, \boldsymbol{x}, \boldsymbol{v}\right) \geqslant 0 \quad (T \in \underline{\underline{\mathbf{P}}}) \tag{127}$$

are immediate consequences of the fact that the nucleolus of the game must be a member of its core. Therefore, the maximal excesses $\mu := \mu(\boldsymbol{x}, \boldsymbol{v})$ and $\mu^\star := \mu(\boldsymbol{x}, \boldsymbol{v}^\star)$ satisfy the equations

$$\mu = 0 \quad \text{and} \quad \mu^\star = \frac{M^1(r-1)}{r} \tag{128}$$

and are attained by $\emptyset, I$ and by any corner C^ρ $(\rho = 1, \ldots, r)$, respectively. Let $\alpha \leqslant \mu^\star$. In view of Theorem 2.4 it remains to show that $\tilde{\underline{\underline{\mathbf{S}}}}(\alpha) : = \tilde{\underline{\underline{\mathbf{S}}}}(\alpha, \boldsymbol{x}, \boldsymbol{v})$ is balanced. Note that $(S, T) \in \tilde{\underline{\underline{\mathbf{S}}}}(\alpha)$ implies that

$$(S, C^\rho) \in \tilde{\underline{\underline{\mathbf{S}}}}(\alpha) \quad \text{and} \quad e(S, \boldsymbol{x}, \boldsymbol{v}) \geqslant \alpha - \mu^\star = : \beta \tag{129}$$

and

$$(\emptyset, T) \in \tilde{\underline{\underline{\mathbf{S}}}}(\alpha) \quad \text{and} \quad e(T, \boldsymbol{x}, \boldsymbol{v}^\star) \geqslant \alpha - \mu = \alpha \tag{130}$$

hold true. Moreover, all pairs $(\emptyset, C^\rho)$ $(\rho = 1, \ldots, r)$ belong to $\tilde{\underline{\underline{\mathbf{S}}}}(\alpha)$ as well. In view of the fact that balancedness of a system $\underline{\underline{\mathbf{S}}}$ implies balancedness of the system $\underline{\underline{\mathbf{S}}} \cup \{C^\rho | \rho = 1, \ldots, r\}$ it suffices to show that

$$\underline{\underline{\mathbf{S}}}(\beta, \boldsymbol{x}, \boldsymbol{v}) \cup \underline{\underline{\mathbf{S}}}(\alpha, \boldsymbol{x}, \boldsymbol{v}^\star)$$

is balanced. By Remark 2.3 and the characterization of the nucleolus (see Remark 2.5) it suffices to show that

$$\mathscr{S} := \{1_S | S \in \underline{\underline{\mathbf{S}}}(b, \boldsymbol{x}, \boldsymbol{v}) \cup \{C^\rho | \rho = 1, \ldots, r\}\}$$

spans $\{1_T | T \in \underline{\underline{\mathbf{S}}}(\alpha, \boldsymbol{x}, \boldsymbol{v}^\star)\}$. Let $T \in \underline{\underline{\mathbf{S}}}(\alpha, \boldsymbol{x}, \boldsymbol{v}^\star)$ and C^τ be some carrier satisfying $\boldsymbol{\lambda}^\tau(T) = \max_{\rho=1,\ldots,r} \boldsymbol{\lambda}^\tau(T) = \boldsymbol{v}^\star(T)$. By (127) and the fact that $\sum_{\rho \neq \tau} \boldsymbol{x}(C^\rho) = \mu^\star$ holds, we obtain the equation

$$-e\left(T^{\tau}+\sum_{\rho\neq\tau}C^{\rho},\boldsymbol{x},\boldsymbol{v}\right)=\mu^{\star}-e(T^{\tau},\boldsymbol{x},\boldsymbol{v}^{\star})$$

and the inequality

$$\boldsymbol{x}(T-T^{\tau})\leqslant\mu^{\star}-e(T^{\tau},\boldsymbol{x},\boldsymbol{v}^{\star})+\boldsymbol{x}(T-T^{\tau})=\mu^{\star}-e(T^{\tau},\boldsymbol{x},\boldsymbol{v}^{\star}).$$

Hence the coalitions $T^{\tau}+\sum_{\rho\neq\tau}C^{\rho}$ and $T-T^{\tau}$ both belong to the system $\underline{\underline{\mathbf{S}}}(\beta,\boldsymbol{x},\boldsymbol{v})$. The proof is completed by the observation that

$$1_T=\left(1_{T^{\tau}}+\sum_{\rho\neq\tau}1_{C^{\rho}}\right)+1_{T-T^{\tau}}-\sum_{\rho\neq\tau}1_{C^{\rho}}$$

holds true. □

Theorems 4.1, 7.1 and 7.2 yield the following result.

Corollary 7.3. *Suppose* $\boldsymbol{v}=\bigwedge\{\boldsymbol{\lambda}^1,\ldots,\boldsymbol{\lambda}^r\}$ *is a min-game which possesses a strong short side. Assume that* $\underline{\underline{\mathbf{D}}}^{m\rho}$ *is weakly balanced for every* $\rho=\sigma+1,\ldots,r$ *and that, for all* $\rho=1,\ldots,\sigma$, *condition* (116) *is satisfied and* $\boldsymbol{\lambda}^{\rho}$ *is integer valued. Then the modiclus is given by the equation*

$$\boldsymbol{\psi}(\boldsymbol{v})=\frac{1}{\sigma}\sum_{\rho=1}^{\sigma}\boldsymbol{\lambda}^{\rho}.$$

Remark 7.4. If we assume uniform distribution, then Corollary 7.3 and Theorem 6.5 (which rests on Theorem 4.5) imply the result presented in [8, Theorem 3.1].

8. Examples and remarks

Within this section we present a few examples. In particular, these examples show that some conditions used in the theorems are crucial. We start out with an exact game. In the following example the nucleolus is not the barycenter of the measures involved and neither does it coincide with the modiclus. Clearly this is at variance with Theorem 7.1, the conditions of which are not satisfied.

Example 8.1. Let $r=3, C^{\rho}=\{\rho\}$ $(\rho=1,2)$ and $C^3=\{3,4\}$. The measures are defined by

$$\boldsymbol{\lambda}^1=(3,0,0,0),$$

$$\boldsymbol{\lambda}^2=(0,3,0,0),$$

$$\boldsymbol{\lambda}^3=(0,0,2,1).$$

Then the arising min-game $\boldsymbol{v}$ is exact. It can be shown that the nucleolus and the modiclus are given by

$$\boldsymbol{v}(\boldsymbol{v}) = \tfrac{1}{6}(5,5,5,3) =: \boldsymbol{x} \quad \text{and} \quad \boldsymbol{\psi}(\boldsymbol{v}) = \tfrac{1}{2}(2,2,1,1) =: \hat{\boldsymbol{x}}. \tag{131}$$

Let us add (at least) one small player of weight 1 to each corner. In view of $3 > l+1$ we cannot employ Theorem 7.1 directly. However, a stronger version of Theorem 7.1 exists as $\boldsymbol{\lambda}^1$ and $\boldsymbol{\lambda}^2$ (apart from a different carrier) look equal. Hence, the modiclus and the nucleolus coincide and are given by the barycenter. That is, the measures

$$\boldsymbol{\lambda}^1 = (3,1,0,0,0,0,0),$$
$$\boldsymbol{\lambda}^2 = (0,0,3,1,0,0,0),$$
$$\boldsymbol{\lambda}^3 = (0,0,0,0,2,1,1)$$

generate a min-game with modiclus and nucleolus equal to

$$\tfrac{1}{3}(3,1,3,1,2,1,1).$$

Example 8.2. Let $r=5$, let the measures $\boldsymbol{\lambda}^\rho$ on their carriers C^ρ $(\rho=1,2,3)$ be defined as in Example 8.1, and let $\boldsymbol{\lambda}^4, \boldsymbol{\lambda}^5$ be the uniform measures with carriers C^4, C^5 which are assumed to be disjoint, not to intersect $C^1 + C^2 + C^3$, and to satisfy

$$|C^5| \geqslant |C^4| > M^1 \quad \text{and} \quad M^1(|C^4| + |C^5|) < |C^4|\,|C^5|. \tag{132}$$

The arising min-game is denoted by $\boldsymbol{u}$. Then $\boldsymbol{u}$ has a strong short side, because

$$\frac{M^1}{M^4} + \frac{M^1}{M^5} < \frac{M^5}{M^4+M^5} + \frac{M^4}{M^4+M^5} = 1$$

holds true. Theorem 4.1 explains that the nucleolus of the derived game on the short side determines the modiclus. In view of Example 8.1 we, therefore, obtain

$$\boldsymbol{\psi}(\boldsymbol{u}) = \frac{1}{6}\left(5,5,5,3,\underbrace{0,\ldots,0}_{M^4+M^5}\right).$$

Of course, if we add (at least) one player in the first three corners and make sure that (132) is satisfied, then the derived game of the short side yields a modiclus which coincides with a nucleolus (cf. Example 8.1), hence an application of Corollary 7.3 results in a modiclus represented by

$$\tfrac{1}{3}(3,1,3,1,2,1,1,0,\ldots,0).$$

Remark 8.3. Note that the nucleolus of any replicated game of $\boldsymbol{v}$ or $\boldsymbol{u}$ of Examples 8.1 and 8.2 assigns the largest amount to the third corner $C^{3(t)}$. Namely, if $t \geqslant 2$, then the players with weight 2 receive the payoff 1, the players with weight 1 receive $\frac{1}{2}$, whereas all players in the other minimal corners receive $\frac{3}{4}$.

The following example shows that the second assertion of Theorem 3.4 does not hold without the weak balancedness of restrictions of maximal diagonal coalitions to the nonminimal corners.

Example 8.4. *Let* $10 \leqslant n \leqslant 29$, $r=3$, $C^1=\{1,2,3\}$, $C^2=\{4,5,6\}$, $C^3=\{7,\dots,n\}$, and $\boldsymbol{\lambda}^1, \boldsymbol{\lambda}^3$ be uniform measures, and let $\boldsymbol{\lambda}^2$ be given by

$$\boldsymbol{\lambda}^2 = \left(0,0,0,2,1,1,\underbrace{0,\dots,0}_{n-6}\right).$$

Finally, let $\boldsymbol{v}$ be the corresponding min-game. In what follows we shall use the abbreviation $k := |C^3| = M^3$ (i.e., $4 \leqslant k \leqslant 23$). Then one can see that

$$\boldsymbol{\psi}(\boldsymbol{v}) = \frac{1}{9k}\left(3k,3k,3k,7k-6,k+3,k+3,\underbrace{9,\dots,9}_{k\text{ times}}\right) =: \hat{\boldsymbol{x}} \tag{133}$$

holds true.

Hence, the modiclus treats all corners equally for $k=4,\dots,23$. For $k=12$ the game has balanced sides and for $k \geqslant 13$ it possesses a strong short side. Hence, Theorem 3.4 (2) is not true without the weak balancedness assumption. Moreover Theorem 4.3 is no longer valid when the assumption concerning the $\underline{\underline{\mathbf{D}}}^{m\rho}$ is not satisfied.

Example 8.5. Let $k=3, C^1=\{1\}, C^2=\{2,3\}, C^3=\{4,5,6\}$, and $\boldsymbol{\lambda}^\rho$ be given by

$$\boldsymbol{\lambda}^1 = (4,0,0,0,0,0),$$

$$\boldsymbol{\lambda}^2 = (0,3,3,0,0,0),$$

$$\boldsymbol{\lambda}^3 = (0,0,0,3,3,3).$$

The arising min-game $\boldsymbol{v}$ has a strong long side. However, in contrast to Theorem 3.4, the modiclus does not yield equal treatment of the corners. Indeed, we claim that

$$\boldsymbol{\psi}(\boldsymbol{v}) = \frac{1}{5}(8,3,3,2,2,2)$$

holds true. Indeed, the corners C^2 and C^3 are the only coalitions attaining maximal dual excess, whereas the maximal primal excess is attained by all coalitions containing 1 member of each corner and by all coalitions containing 1 member of the minimal and 2 members of each of the other corners. It can be checked that the pairs of coalitions of maximal biexcess form a nondegenerate and balanced system.

Remark 8.6. (1) In case $k \geqslant 25$ the modiclus of the game defined in Example 8.4 is concentrated to the first corner. Hence the "region" in which the modiclus guarantees equal treatment of the corners, is just much larger than in the case of the presence of weakly balanced $\underline{\underline{\mathbf{D}}}^{m\rho}$ ($\rho=\sigma+1,\dots,r$). We conjecture that the corresponding assertion (1) of Theorem 3.4 remains true, if "weak balancedness" is replaced by "nonemptiness."

(2) The t-fold replication of the game in Example 8.4 satisfies the balancedness and nondegeneracy property of $\underline{\underline{\mathbf{D}}}^{m2(t)}$ whenever $t \geqslant 2$, thus Theorems 3.4 and 4.3 can be applied in the replicated case.

(3) It should be noted that the modiclus of the t-fold replication of the game defined in Example 8.5 coincides with the barycenter of the measures involved, if t is sufficiently large. However, balancedness and nondegeneracy of $\underline{\underline{\mathbf{D}}}^{m\rho(t)}$ $(\rho = 2, 3)$ are only satisfied in the case that t is a multiple of 3.

(4) Finally it should be remarked that the modiclus treats the corners equally in the case that only two corners are present. In this case, no further conditions have to be satisfied in order to guarantee this kind of "equal treatment property" among corners. For a proof see [15].

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