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Homogeneous Games as Anti Step Functions

By P. Sudhölter[1]

Abstract: In this paper the class of homogeneous n-person games "without dummies and steps" is characterized by two algebraic axioms. Each of these games induces a natural vector of length n, called incidence vector of the game, and vice versa. A geometrical interpretation of incidence vectors allows to construct all of these games and to enumerate them recursively with respect to the number of persons.
In addition an algorithm is defined, which maps each directed game to a minimal representation of a homogeneous game. Moreover both games coincide, if the initial game is homogeneous.

1 Basic Notations and Definitions

A *simple n-person game* is a pair (Ω,v). Here $\Omega = \{1,\ldots,n\}$ is called the set of *players*, and $v : \mathscr{P}(\Omega) \to \{0,1\}$, $v(\emptyset) = 0$, is the "characteristic function" in the sense of Game Theory. An element S of $\mathscr{P}(\Omega)$, i.e. a subset of Ω, is a *coalition.* The coalition S is often identified with the indicator function 1_S.

A coalition S is *winning* if $v(S) = 1$ and *losing* otherwise. All considered simple games are *monotonous*, i.e. subcoalitions of the losing coalitions are losing. If each proper subcoalition of a winning coalition is losing, this winning coalition is called *minimal.* It should be noted that a monotone game is completely determined by its minimal winning coalitions.

The set of minimal winning coalitions is denoted by W_* or $W_*(\Omega,v)$, if the dependence of the game is to be stressed.

Sometimes the expression "n-person" is deleted.

Let $m = (m_1,\ldots,m_n) \in \mathbb{N}_0^n$ and let $\lambda \in \mathbb{N}$. In the context to be discussed we shall call m_i player i's weight and $m(S) := \sum_{i\in S} m_i$ the *weight* of coalition S. Finally λ is called the level. This terminology is justified by the following construction: Assume that $0 < \lambda \leq m(\Omega)$ and define a simple game (Ω,v) by

$$v(S) = \begin{cases} 1 \ , \ \text{if } m(S) \geq \lambda \\ 0 \ , \ \text{if } m(S) < \lambda, \end{cases}$$

[1] Peter Sudhölter, University of Bielefeld, IMW, Postfach 8640, 4800 Bielefeld, F.R.G.

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v is also written

$$v = 1_{[\lambda, m(\Omega)]} \circ m = v_\lambda^m$$

where 1_T is the indicator function of T. Intuitively we interpret v ("the characteristic function") as to represent the power structure of a parliament or committee where each member (or rather each party) is commanding a certain number of votes according to its weight. Obviously, various pairs (λ, m) may result in the same v.

Thus a pair (λ, m) resulting in a function v as specified above is called a representation of (Ω, v); we shall use notations like $(\lambda, (m_1, \dots, m_n))$ and $(\lambda; m_1, \dots, m_n)$ simultaneously.

A simple game having a representation is called *weighted majority game*.

If a weighted majority game has a representation (λ, m) such that all minimal winning coalitions are exactly of weight λ, then both the simple game and the representation are called *homogeneous*. For a special case the terms "simple", "weighted majority", and "homogeneous" were introduced by von Neumann and Morgenstern [13]. Shapley [12] considered homogeneous games in general. Isbell [1,2,3], Ostmann [5], Peleg [7] and Rosenmüller [8,9,10,11] should also be mentioned in this context.

Following Ostmann [5] each player i of Ω belongs to an equivalence class $\tilde{i}$, called *type of* i: $i \sim j$, if there is a permutation π of Ω such that $v = v \circ \pi$ and $\pi(i) = j$. Let $\tilde{i}$ be the set $\{j \in \Omega \mid i \sim j\}$. In the paper just mentioned it is shown that all representations of a given weighted majority game (Ω, v) are inducing the same order of the types of Ω, i.e. for all types $t_1 \neq t_2$ and all players $i_1 \in t_1$, $i_2 \in t_2$ either $m_{i_1} > m_{i_2}$ or $m_{i_1} < m_{i_2}$ for all representations (λ, m).

Let (Ω, v) be a simple game. The relation $\precsim \ \subseteq \Omega^2$, defined by $i \precsim j$, if $v(\{i\} \cup S) \leq v(\{j\} \cup S)$ for all coalitions S with $\{i, j\} \cap S = \emptyset$, is called *desirability* relation of (Ω, v). For this definition we refer to [4]. The simple game (Ω, v) is called *ordered*, if its desirability relation is complete, and is called *directed*, if additionally $1 \succsim 2 \succsim 3 \succsim \dots \succsim n$ is valid.

Two simple games (Ω, v) and (Ω, v') are *equivalent*, if there is a permutation π of Ω such that $v \circ \pi = v'$. Consequently the equivalence class of an ordered game can be identified with its unique directed representative. As our interest is restricted to these equivalence classes of ordered games only, it is assumed w.l.o.g. from now on that all considered games are directed.

Each weighted majority game is ordered and thus directed by the assumption, which implies that it has a *directed* representation, i.e. a representation (λ, m) with the property $m_1 \geq \dots \geq m_n$.

For these definitions and assertions we refer to [6].

The representations (λ, m) of a weighted majority game are ordered by *total mass* $m(\Omega)$. Ostmann [5] has shown that there is a unique minimal representation of a homogeneous game, which is automatically homogeneous. Two further proofs are contained in [10]. Therefore a homogeneous game is often identified with its minimal representation, which automatically is directed, since in [5] it is shown that players of the same type have the same weights in the minimal representation.

Let $D = D(\Omega,v) = \{i \in \Omega \mid v(S \cup \{i\}) = 0$ for all losing coalition $S\}$ be the set of *dummies*.

The fact that two coalitions S and T do *not differ earlier than* at i, formally meaning $S \cap [1,i\text{-}1] = T \cap [1,i\text{-}1]$, is abbreviated by $S \sim_i T$. If i and j are non-negative integers, we put $[i,j]_n := [i,j] \cap \mathbb{N}_0$. Sometimes, if a misunderstanding is excluded, the lower index "n" is deleted.

A player $i \in \Omega \setminus D$ is a *sum*, iff there are two coalitions $S, T \in W_*$ such that $i \in S$, $i \notin T$ and $S \sim_i T$, and is a *step* otherwise. If $D(\Omega,v)$ is nonvoid, the decreasing order of the weights m_i cause the existence of a first dummy player j, i.e. $D(\Omega,v) = \{j,\dots,n\}$.

If (λ,m) is the minimal representation of a homogeneous n-person game, the following assertion is obviously true:

$$m_i = \begin{cases} 0, & \text{iff } i \text{ is a dummy} \\ \lambda, & \text{iff } \{i\} \text{ is a one-person winning coalition} \end{cases}$$

Lemma 1.1: There are canonical bijections

$$\left\{ (\lambda,m) \,\middle|\, \begin{array}{l}(\lambda,m) \text{ is the minimal representation of} \\ \text{a homogeneous } n\text{-person game}\end{array} \right\}$$

$$\longrightarrow$$

$$\bigcup_{t=1}^{n} \left\{ (\lambda,m) \,\middle|\, \begin{array}{l}(\lambda,m) \text{ is the minimal representation of a} \\ \text{homogeneous } t\text{-person game without dummies}\end{array} \right\}$$

and

$$\left\{ (\lambda,m) \,\middle|\, \begin{array}{l}(\lambda,m) \text{ is the minimal representation of a} \\ \text{homogeneous } n\text{-person game without dummies}\end{array} \right\}$$

$$\longrightarrow$$

$$\{(1;\underbrace{1,\dots,1}_{n\text{-times}})\} \cup \bigcup_{t=2}^{n} \left\{ (\lambda,m) \,\middle|\, \begin{array}{l}(\lambda,m) \text{ is the minimal representation} \\ \text{of a homegeneous } t\text{-person game} \\ \text{without dummies and without} \\ \text{one-person winning coalitions}\end{array} \right\}$$

Proof: Definition of the first mapping:
If (λ,m) is a minimal representation of a homogeneous game, then there is a player $1 \le i_0 \le n$ such that $D(\Omega,v) = \{i_0+1,\dots,n\}$. The vector $(\lambda;m_1,\dots,m_{i_0})$ clearly is the minimal representation of a homogeneous game without dummies and the mapping $(\lambda,m) \mapsto (\lambda;m_1,\dots,m_{i_0})$ is bijective.

Definition of the second mapping:
Distinguish two cases:

1. $(\lambda,m) = (1; \underbrace{1,\ldots,1}_{n\text{-times}}) \mapsto (\lambda,m)$
2. If (λ,m) does not coincide with some $(1;1,\ldots,1)$, then there is an i_0, $1 \leq i_0 \leq n\text{-}1$, such that $m_1 = \ldots = m_{i_0-1} = \lambda > m_{i_0}$. The vector $(\lambda;m_{i_0},\ldots,m_n)$ is the minimal representation of a homogeneous game without one-person winning coalitions.

The inverse function is constructed as follows:

If (λ,m) is the minimal representation of a homogeneous t-person game without one-person winning coalitions, define the image of this function as $(\lambda; \underbrace{\lambda,\ldots,\lambda}_{(n-t)\text{-times}}, m_1,\ldots,m_t)$, otherwise $(1; \underbrace{1,\ldots,1}_{n\text{-times}})$, i.e. if $(\lambda,m) = (1; \underbrace{1,\ldots,1}_{n\text{-times}})$. q.e.d.

In view of the last lemma only homogeneous games without dummies and without one-person winning coalitions are being considered in the following presentation.

Let (Ω,v) be a directed n-person game. The matrix with n columns

$$X := X(\Omega,v) := \begin{bmatrix} \vdots \\ S \\ \vdots \end{bmatrix}_{S \in W_*}$$

with lexicographically ordered rows is called *incidence matrix of* (Ω,v).

Let (Ω,v) be a homogeneous n-person game without dummies and one-person winning coalitions. From Ostmann [5] we know the following algorithm, which generates the minimal representation of this game:

If i is a sum, let $S(i)$, $T(i)$ be the lexicographically first pair in W_* such that $S(i) \sim_i T(i)$, $i \in S(i)$, $i \notin T(i)$.

If i is a step, define $h_m(i) := \max\{m(H) \mid H \in \{[i,n] \setminus S \mid i \in S, S \in W_*\}\}$, $m \in \mathbb{N}^n$.

Define

$$m_i := \begin{cases} m(T(i) \setminus S(i)), & \text{if } i \text{ is a sum} \\ 1+h_m(i) & , \text{ if } i \text{ is a step,} \end{cases}$$

$\lambda = m(S)$, where S is the lexicographically first minimal winning coalition.

It follows

Lemma 1.2: (λ,m) is the minimal representation of (Ω,v).

Remark 1.3:

1. (λ,m) is well defined, since it can be calculated successively by starting at n:

$$m_n = 1 + h_m(n) = 1 + m(\emptyset) = 1.$$

2. It is not necessary to use the lexicographically first pair $S(i)$, $T(i)$, in case i is a sum, to construct the minimal weights, only the fact $S(i) \sim_i T(i)$, $i \in S(i)$, $i \notin T(i)$ is needed.
3. In the following all games considered are assumed to have no one-person winning coalitions, unless otherwise specified.

In order to classify homogeneous games without dummies we can restrict our attention to those "without" steps, i.e. with just one step, since player n always is a step. Indeed, if $\mathbb{H}_n$ denotes the set of minimal representations of homogeneous n-person games without dummies, steps and one-person winning coalitions, the following assertion is true.

Lemma 1.4: There is a canonical bijection from

$$\mathbb{H}_{n+1} \text{ to } \left\{ (\lambda,m) \;\middle|\; \begin{array}{l}(\lambda,m) \text{ is the minimal representation of a} \\ \text{homogeneous } n\text{-person game without dummies}\end{array} \right\}$$

Proof: $(\lambda; m_1,\ldots,m_{n+1}) \rightarrow (\lambda; m_1,\ldots,m_n)$ has the desired properties: $(\lambda; m_1,\ldots,m_n)$ is a homogeneous representation of a simple game and the above algorithm shows the minimality – note: $m_n = 1$ if $(\lambda; m_1,\ldots,m_{n+1})$ has no steps. On the other hand the algorithm also shows that

$$(\lambda; m_1,\ldots,m_n) \mapsto (\lambda, m_1,\ldots,m_n,1)$$

maps homogeneous games to those without steps. This map is obviously inverse to the first. q.e.d.

By using the identification of homogeneous games with their minimal representations, we also identify $\mathbb{H}_n$ with

$$\{(\Omega,v) \mid (\Omega,v) \text{ has a representation } (\lambda,m) \in \mathbb{H}_n\}$$

and denote this set again by $\mathbb{H}_n$.

2 The Incidence Vector of a Homogeneous Game

The incidence matrix of a simple game is frequently of a respectable size. It is desirable to select an appropriate submatrix which allows for a unique identification of the game. Given the incidence matrix of a homogeneous game without dummies and steps we are going to show that there exists an $n \times n$ submatrix which completely determines the game. The n rows defining the submatrix are chosen in such a way that for each player $i \neq n$ there exist at least two rows S, T with $i \in S$, $i \notin T$ and $S \sim_i T$.

At first some notation is needed. Let (Ω,v) be a directed game, not necessarily homogeneous and without dummies and steps.

Definition 2.1: For a nonempty coalition let $l(S)$ be the *length* of S, meaning the one player of S who has the highest index, i.e.

$$l(S) := \max\{j \mid j \in S\}.$$

If $\overline{S} = \begin{bmatrix} S_1 \\ \vdots \\ S_t \end{bmatrix}$ is a matrix of coalitions, define $l(\overline{S}) = (l(S_1),\ldots,l(S_t))$.

Given $S \in W_*$, let j be such that $[j, l(S)] \subseteq S$.
If $S \setminus \{j\} \cup [l(S)+1, n]$ is winning, define

$$\rho_j(S) := S \setminus \{j\} \cup [l(S)+1,t],$$

where t is minimal such that $S \setminus \{j\} \cup [l(S)+1,t]$ is winning.

For a minimal winning coalition T, which is not the lexicographically maximal one, define

$$\varphi(T) := T \cup \{r\} \setminus [t,l(T)],$$

where $r = \max\{j \notin T \mid j < l(T)\}$ and

$$t = \min\{t' \mid T \cup \{r\} \setminus [t', l(T)] \text{ is winning}\}.$$

Player r exists, since T is not lexicographically maximal, and player t exists, since (Ω,v) is directed.

Remark 2.2:

(i) $\rho_j(S)$ is the lexicographically next minimal winning coalition to S, in which j is substituted by players of smaller or equal type.
(ii) With the above notations the following holds true:

$$\rho_r(\varphi(T)) = T, \; \varphi(\rho_j(S)) = S.$$

(iii) From the "Basic Lemma" of Rosenmüller ([8,9]) we know the following: If (λ,m) is homogeneous, then

$$m_j = m([l(S)+1,t]),$$

$$m_r = m([t,\, l(T)]).$$

The existence of n rows of the incidence matrix with the desired properties is a direct consequence of the following

Theorem 2.3: Let (Ω,v) be a game of $\mathbb{H}_n$ and S_1 be the lexicographically maximal minimal winning coalition.

If $S_2,\ldots,S_k$ are minimal winning coalitions and $k < n$, such that for all $j \in [2,k]$ there is an $i < j$ with

$$\rho_{j\text{-}1}(S_i) = S_j\,,$$

then there is $i_0 \in [1,k]$ such that

$$S_{k+1} := \rho_k(S_{i_0})$$

is defined.

Note that the property $\rho_{j\text{-}1}(S_i) = S_j$ can be replaced by $\varphi(S_j) = S_i$ (see Remark 2.2 (ii)).

Proof: Assume the contrary.

For each coalition S define

$$r(S) = \max\left(\{j \mid j \notin S, j < l(S)\} \cup \{0\}\right).$$

$$\text{Let } \bar{S} \text{ be a coalition in } \left\{ S \;\middle|\; \begin{array}{l} S \text{ is a minimal winning coalition and} \\ \rho_k(S) \text{ is defined} \end{array} \right\}$$

– which is indeed nonempty, since (Ω,v) has no steps except n –, such that $r := r(\bar{S})$ is maximal, thus $r \geq 1$, since $\bar{S}$ cannot be the lexicographically maximal minimal winning coalition S_1. We distinguish two cases.

1. $l(S_{r+1}) < l(\bar{S})$. Then $l(S_{r+1}) < k$, otherwise $\rho_k(S_{r+1})$ would be defined. Therefore

$$\rho_k(\bar{S} \cup \{r\} \setminus [l(\varphi(S_{r+1}))+1,\ l(S_{r+1})])$$

is defined, contradicting the maximality of r.

2. $l(S_{r+1}) \geq l(\bar{S})$. Then it is obvious that $l(S_{r+1}) > l(\bar{S})$ and $l(\varphi(S_{r+1})) < k$, otherwise $\rho_k(S_{r+1})$ resp. $\rho_k(\varphi(S_{r+1}))$ would be defined. As a direct consequence

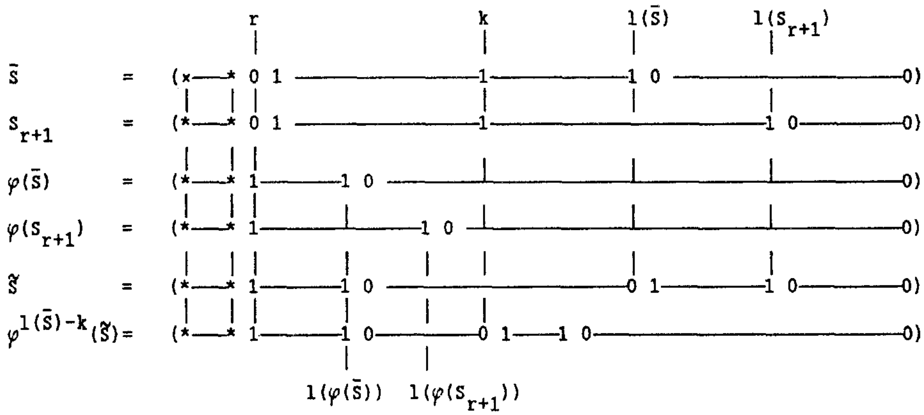
$$\tilde{S} := S_{r+1} \cup \{r\} \setminus [l(\varphi(\bar{S}))+1,\ l(\bar{S})]$$

is a minimal winning coalition with $[k\text{-}1,\ l(\bar{S})] \cap \tilde{S} = \emptyset$.

A simple computation shows that

$$\rho_k(\varphi^{l(\bar{S})+1-k}(\tilde{S})) = \varphi^{l(\bar{S})-k}(\tilde{S}),$$

but $r < k$-1, a contradiction to the maximality of r. Let us illustrate this situation by an example:



q.e.d.

With the help of the last important theorem we will define a unique sequence of minimal winning coalitions recursively.

Definition 2.4:

(1) For each vector $l = (l_1,...,l_n) \in \mathbb{N}^n$ we define $\Pi^l = (\Pi_1^l,...,\Pi_n^l)$ and $\omega^l = (\omega_1^l,...,\omega_n^l)$ by $\Pi_k^l = \min\{l_j \mid j \leq k \leq l_j\}$, $\omega_k^l = \min\{j \mid l_j = \Pi_k^l\}$, if the corresponding sets are nonvoid, and $\Pi_k^l = \omega_k^l = 0$, otherwise, for all $k \in [1,n]_n$.

(2) Let (Ω,v) be a game in $\mathbb{H}_n$ and S_1 be the lexicographically maximal minimal winning coalition.
If $S_2,...,S_k$ $(k < n)$ are already defined and $l = (l(S_1),...,l(S_k))$, then

$$S_{k+1} := \rho_k(S_{i_0}),$$

where

$$i_0 = \omega_k^l.$$

(3) $\overline{S}^v = \begin{bmatrix} S_1 \\ \vdots \\ S_n \end{bmatrix}$ is the *characterizing incidence submatrix* of (Ω,v).

In view of Theorem 2.3 the characterizing incidence submatrix of (Ω,v) is well defined and hence unique. Besides note that it may be useful to compare this procedure with the context of section 5, pp. 324-327, in [10].

Corollary 2.5: The function

$$\mathbb{H}_n \to (\mathscr{P}(\Omega))^n,$$

defined by

$$(\Omega,v) \mapsto \overline{S}^v$$

is injective.

Proof: Let (Ω,v) be a game with the desired properties and $\overline{S}^v = \begin{bmatrix} S_1 \\ \vdots \\ S_n \end{bmatrix}$.

Define successively

$$m_n = 1, m_i = m\,(S_{i+1} \setminus S_{i_0}),$$

where $\varphi(S_i) = S_{i_0}$,

and $\lambda = m(S_1)$.

With this notation it can be shown analogously to Lemma 1.2 that $(\lambda;m_1,...,m_n)$ is the minimal representation of (Ω,v). q.e.d.

Corollary 2.6: The mapping

$$\mathbb{H}_n \to \{(l_1,...,l_n) \in \mathbb{N}^n\},$$

$$(\Omega,v) \mapsto l(\bar{S}^v)$$

is injective.

Proof: From $l := l(\bar{S}^v)$ the matrix $\bar{S}^v$ can be reconstructed successively:

$$S_1 = \{i \in \Omega \mid 1 \le i \le l_1\},$$

if $S_1,...,S_k$ are already constructed, then

$$S_{k+1} = S_{i_0} \setminus \{k\} \cup [l_{i_0} + 1, l_{k+1}]_n ,$$

where

$$i_0 = \omega_k^l.$$

q.e.d.

In the following it will be shown that the image of the mapping given in Corollary 2.5, i.e. the vectors $(l(S_1),...,l(S_n))$, can be characterized by algebraic means.

Lemma 2.7: If (Ω,v) is a homogeneous n-person game without steps and dummies and $\bar{S}^v$ is as defined before, then

(i) $l(S_i) > \Pi_{i-1}^{l(\bar{S}^v)}$ for all $2 \le i \le n$

(ii) $l(S_{i+1}) \le l(S_i)$, if $\Pi_{i-1}^{l(\bar{S}^v)} \ge i$.

Proof: Assertion (i) is a direct consequence of the successive construction of the sequence $S_1,...,S_n$.

The order of the minimal weights, i.e. $m_1 \geq ... \geq m_n$ for $(\lambda;m_1,...,m_n)$ being the minimal representation of the game, directly implies assertion (ii).

q.e.d.

We show now, roughly speaking, that the converse is also true, i.e. every vector which fulfills (i) and (ii) of the last lemma is of the form $l(\bar{S}^v)$ for some homogeneous game (Ω,v). Thus, a new characterization of this class of homogeneous games is obtained as soon as a proof of the theorem, containing the above mentioned assertion, is provided.

The following notation simplifies the formulation of this important result. For technical reasons $n \geq 3$ is presumed.

Definition 2.8:

(1) A vector $l = (l_1,...,l_n) \in \{2,...,n\}^n$ is called an n-person *incidence vector,* iff – for all $i \in [2,n]$ –

(i) $l_i > \Pi^l_{i\text{-}1}$

(ii) $l_{i+1} \leq l_i$, if $\Pi^l_{i\text{-}1} \geq i$.

(2) The matrix $\bar{S}^l = \begin{bmatrix} S_1 \\ \vdots \\ S_n \end{bmatrix}$, defined by $S_1 = [1, l_1]_n$, $S_{k+1} =$

$S_{i_0} \setminus \{k\} \cup [l_{i_0} + 1, l_{k+1}]_n$, where $i_0 = \omega^l_k$, is called *associated* to l.

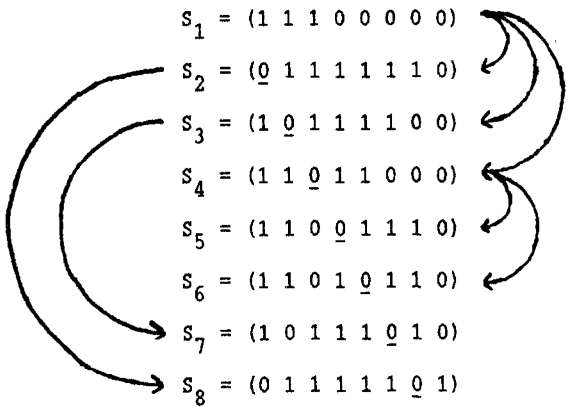
(3) l *generates* $M^l := (\lambda;m_1,...,m_n)$ via

$$m_n = 1,\ m_i = m_{i_0+1} + ... + m_{l_{i+1}}$$

where $i_0 = \Pi^l_i$, $\lambda = m(S^v_1)$.

(4) Let I_n denote the set of n-person incidence vectors.

In order to illustrate the last definition we give an explicit example:
The vector $l = (3,7,6,5,7,7,7,8)$ is an 8-person incidence vector.
The associated matrix of coalitions is



The coalition at the origin of each arrow is needed to construct the coalition at the top of the arrow.

The generated representation turns out to be

(15; 6,5,4,2,2,1,1,1).

Remark 2.9:

(1) A direct consequence of the definition of an incidence vector l is that both the matrix $\overline{S}^l$ and the generated vector M^l are well defined.

(2) If (Ω,v) is a homogeneous game in some $\mathbb{H}_n$, then by Lemma 2.7 the vector $l := l(\overline{S}^v)$ is an incidence vector, $\overline{S}^v$ is associated to l and l generates a tupel (λ,m), which is – this can be shown analogously to Lemma 1.2 by the way – the minimal representation of (Ω,v).

The main result of this chapter is stated in form of the following

Theorem 2.10: Each homogeneous n-person game without dummies and steps can be identified with some n-person incidence vector and vice versa, formally:

$$L_n : \mathbb{H}_n \to I_n$$

$$(\Omega,v) \mapsto l(\overline{S}^v)$$

is bijective.

Proof: The injectivity is already shown. So it is enough to prove that the vector $(\lambda;m_1,\dots,m_n)$, generated by an incidence vector l, is the minimal representation of some homogeneous game without dummies and steps. The associated matrix $\overline{S}^l$, the members S_i of which must then be minimal winning coalitions, guarantees that there cannot be dummies or steps and that (λ,m) is a minimal representation, if it is a representation of a homogeneous game at all.

First of all, the order of the weights, i.e. $m_1 \geq \dots \geq m_n$, is shown by induction on n-i:

$$m_n = 1 \leq m_i \text{ for all } 1 \leq i \leq n.$$

Assume $m_i \geq \dots \geq m_n$, then

$$m_{i-1} = m_{i_0+1} + \dots + m_{l_i},$$

where

$$i_0 = \Pi^l_{i-1}.$$

Two cases are distinguished:

1. $l_i > i_0 \geq i$: Then $m_i = m_{i_0+1} + \dots + m_{l_{i+1}}$ and $l_{i+1} \leq l_i$ (see Definition 2.8, (ii)), thus $m_{i-1} \geq m_i$.

2. $i_0 = i$-1 : Then $m_{i-1} = m_i + \dots + m_{l_i} \geq m_i$.

Referring to Definition 2.8 (i) the case $i_0 \geq l_i$ cannot occur, thus the induction is finished.

The first part of this proof implies that $(\lambda;m)$ is a representation of some simple game (Ω,v). It remains to show the homogeneity of (Ω,v):

Let $|\{l_1,\dots,l_n\}| =: r$ for some $2 \leq r \leq n$-1 and write

$$\{l_1,\dots,l_n\} = \{l^1,\dots,l^r\}$$

such that

$$l_1 = l^1 < \dots < l^r = l_n = n, \; l^0 := 0.$$

It is enough to show per induction on $0 \leq i \leq r$-1:
if $S \in W_*(\Omega,v)$ with $l^i < l(S) \leq l^{i+1}$, then $l(S) = l^{i+1}$ and $m(S) = \lambda$.

For $i = 0$ the assertion is immediately implied, since S^l_1 is the lexicographically first minimal winning coalition in (Ω,v).

Let S_0 be a coalition in $\{S \in W_* \mid l(S) = \min\{l(T) \mid T \in W_* \text{ and } l(T) > l^i\}\}$, such that $r := r(S_0)$ is maximal, where $r(S)$ is defined as in the proof of Theorem 2.3. Define $l_0 := l(S_0)$. Observe that $l_0 \leq l^{i+1}$.

It can be remarked, that $r > 0$ since S_0 cannot be the lexicographically first coalition in W_*.

The inductive hypothesis implies

$$l(\varphi(S_0)) \leq l^i,\ m(\varphi(S_0)) = \lambda\ .$$

Let l_j be minimal with $l_j \geq r$, thus $r \in S_j^l$ – otherwise $l_j > l(\varphi(S_j^l)) \geq r$. The minimality of l_j shows that

$$l_j \leq l(\varphi(S_0)),$$

thus

$$\max\{l(S) \mid S \in W_*,\ l(S) < l(S_j^l)\} < r.$$

If $l_j = l(\varphi(S_0))$ nothing remains to be shown, because of the definition of S_{r+1}^l. Therefore, assume $l_j < l(\varphi(S_0))$ and define

$$\tilde{S}_0 = S_j^l \setminus \{r\} \cup \{l(\varphi(S_0)) + 1, \ldots, l(S_0)\}.$$

With this notation

$$l(\tilde{S}_0) = l(S_0) \text{ and } m(\tilde{S}_0) = m(S_0).$$

Since additionally

$$r < l(\varphi(S_0)) \notin \tilde{S}_0\ ,$$

this assumption contradicts the maximality of r. q.e.d.

The proof of this theorem also implies the following

Corollary 2.11: Let $(\Omega,v) \in \mathbb{H}_n$, $S_1,\ldots,S_n$ the members of $\overline{S}^v$ and $S \in W_*(\Omega,v)$.
Then there is a $j \in [1,n]$, not necessarily unique, such that

$$l(S_j) = l(S).$$

Remark 2.12: The identification L_n of homogeneous games and incidence vectors permits us to provide an upper bound for the number of these n-person games: If $l \in I_n$, then l_1 is determined by the other components of the vector, since Definition 2.8 (i), (ii) guarantees that

$$l_2 \geq \ldots \geq l_{l_1+1} < l_{l_1+2},$$

showing that

$$l_1 = \max \{t \mid 2 \leq t \leq n, l_2 \geq \ldots \geq l_t\} -1.$$

Additionally the just mentioned definition implies

$$3 \leq l_2 \leq n, k \leq l_k \leq n \text{ for all } k \in [3,n].$$

Therefore l_2 can run through at most n-2 values and l_k can run through at most n-k+1 values. This implies

$$| \mathbb{H}_n | \leq (n\text{-}2)\,!\,(n\text{-}2) < (n\text{-}1)\,!$$

So far the number (n-1) ! was the smallest known upper bound for the cardinality of the set of homogeneous n-person zero-sum games (see [3]). Clearly this set is by comparison a very small subset of the considered class of simple games $\mathbb{H}_n$; hence it would seem that the preceding result is certainly an improvement. However, in the next chapter it will turn out, that we can achieve much more: We will construct an explicit recursive formula for the number of incidence vectors.

3 Geometrical Description of Incidence Vectors as Anti Step Functions, Providing a Recursive Formula for the Number of Homogeneous Games

It is the aim of this chapter to enumerate the homogeneous games without dummies and steps recursively w.r.t. the number of players. This will be done by partitioning the corresponding class of n-person incidence vectors into certain subsets which will be defined later on.

We assume $n \geq 3$, unless otherwise specified.

Definition 3.1: Let $l = (l_1,\ldots,l_n)$ be an n-person incidence vector. We identify Π^l with the *quadratic n-person step function*

$$Q^l : [0, n\text{-}1] \to [0, n\text{-}1],$$

defined by

$$Q^l(0) = 0,\ Q^l(x) = \Pi^l_j \text{ for all } x \in (j\text{-}1, j] \text{ and } j \in [1, n\text{-}1]_n.$$

If $k_1 < \ldots < k_r$ are the values of a quadratic n-person step function Q^l and $k_0 := 0$, it can easily be seen by Definition 2.8 that $k_1 \geq 2$, $k_r = n\text{-}1$ and that Q^l can be redefined:

$$Q^l(0) = 0,\ Q^l(x) = k_i \text{ for all } x \in (k_{i\text{-}1}, k_i], \text{ if } i \in [1,r]_n.$$

Let Q_n denote the set of vectors Π^l, $l \in I_n$, i.e.

$$Q_n = \{\Pi^l \mid l \in I_n\}.$$

The incidence vector l is identified with the *n-person anti step function*

$$A^l : [0, n\text{-}1] \to (0, n\text{-}1],$$

defined by

$$A^l(0) = \Pi^l_1,\ A^l(x) = l_{i+1} \text{ for all } x \in (i\text{-}1, i], \text{ if } i \in [1, n\text{-}1]_n.$$

For $\Pi \in Q_n$ define $I(\Pi) = \{l \in I_n \mid \Pi^l = \Pi\}$.

Remark 3.2:

(1) The denotation "quadratic" reflects the obvious fact that each step of a quadratic step function is as high as long.
(2) The step function A^l is called "anti" step function, since Definition 2.8 directly implies that A^l is – not necessarily strictly – decreasing on sections where Q^l is constant, i.e.

$$A^l|_{(k_{i\text{-}1}, k_i]} \text{ is not increasing for all } i \in [1, r]_n.$$

(3) Two further properties of Q^l, A^l should be noted here:

(a) $A^l([0, n\text{-}1]) = (Q^l([0, n\text{-}1]) \setminus \{0\}) \cup \{n\}$,

(b) $A^l(x) > Q^l(x)$ for all $x \in [0, n\text{-}1]$.

Example 3.3: The following sketches illustrate the graphs of the quadratic step function $f = Q^l$ and the anti step function $h = A^l$, where

$$l = (3,14,14,7,13,12,10,10,14,14,13,13,13,14)$$

is a 14-person incidence vector:

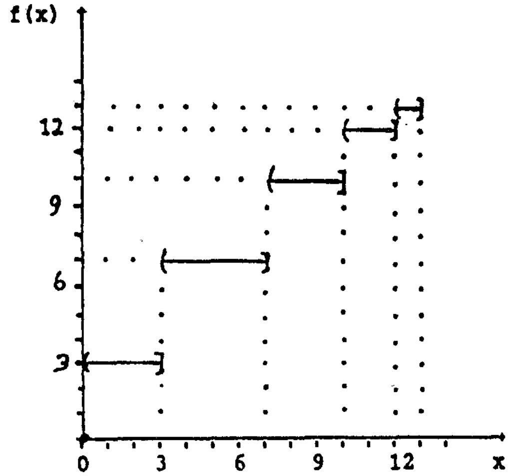


Fig. 1.

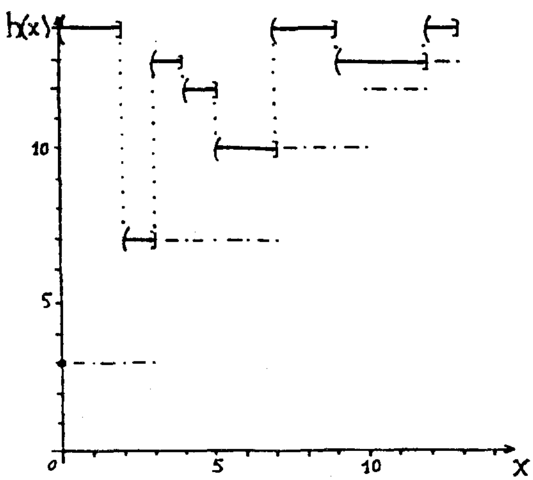


Fig. 2. The lines "-·-·-·-" represent the graph of f.

In the following we often use the identifications $l \to A^l$, $l \to Q^l$, since the corresponding graphs can be nicely illustrated as shown above.

Lemma 3.4:

(1) $$Q_n = \left\{ \Pi \in \mathbb{N}^n \,\middle|\, \begin{array}{l} \text{There is a natural } r \text{ and } \Pi_i \in \mathbb{N},\ i \in [1,r]_n \text{ such that } 2 \leq k_1 \\ < \ldots < k_r = n\text{-}1 \text{ and } \Pi = (\underbrace{k_1,\ldots,k_1}_{k_1 \text{times}}, \underbrace{k_2,\ldots,k_2}_{(k_2-k_1)\text{times}}, \ldots, \underbrace{k_r,\ldots,k_r}_{(k_r-k_{r-1})\text{times}}, n) \end{array} \right\}$$

and $\mid Q_n \mid = 2^{n-3}$.

(2) If $\Pi \in Q_n$, then

$$I(\Pi) = \left\{ l \in \mathbb{N}^n \,\middle|\, \begin{array}{l} \{l_i \mid i \in [1,n]\} = \{\Pi_i \mid i \in [1,n]\}, (l_i > \Pi_{i-1}, \text{ if } i \in [2,n]), \\ l_1 = \Pi_1, (l_i \leq l_{i-1}, \text{ if } \Pi_{i-1} = \Pi_{i-2} \text{ for all } i \in [3,n]) \end{array} \right\}.$$

Proof: ad (1): One inclusion of the first part of assertion (1) is trivially satisfied. For the other inclusion take a vector Π with the desired properties and observe that

$$l = (\underbrace{\mathrm{k}_1, k_2,\ldots,k_2}_{k_1 \text{ times}}, \ldots, \underbrace{k_r,\ldots,k_r}_{(k_{r-1}-k_{r-2}) \text{ times}}, \underbrace{n,\ldots,n}_{(k_r-k_{r-1})\text{times}})$$

is an n-person incidence vector with $\Pi^l = \Pi$.

The assertion concerning the cardinality of Q_n is verified by induction on n : It is clear that $Q_3 = \{(2,2,3)\}$.

From a vector $\Pi \in Q_n$ two vectors $\Pi^1, \Pi^2 \in Q_{n+1}$ are constructed, namely

$$\Pi^1 = (k_1,\ldots,k_r, n, n+1)$$

and

$$\Pi^2 = (k_1,\ldots,k_{r-1}, \underbrace{n,\ldots,n}_{(n-k_{r-1}) \text{ times}}, n+1).$$

The maps $\Pi \to \Pi^1$ and $\Pi \to \Pi^2$ are injective and have disjoint images. It is abvious that the union of these images contains Q_{n+1}, thus this part of the proof is finished.

Assertion (2) is a trivial consequence of Definition 2.8 (1). q.e.d.

In order to enumerate the homogeneous games the set of incidence vectors will be decomposed into subsets, the members of which having a common property called type. To begin with some notation will be needed.

Definition 3.5: Let $l \in I_n$. Then $l \in I(\Pi)$ for some $\Pi \in Q_n$ with values $k_0 = 0 < k_1 < \ldots < k_r = n\text{-}1$ (see Definition 3.1).
Define

$$\alpha_i := |\{j \in \mathbb{N} \mid j \in (k_{i-1}, k_i] \text{ and } l_{j+1} = n\}|.$$

Then there is a chain

$$i_1 < \ldots < i_t$$

for some $t \in \mathbb{N}$ such that

$$\{i_1, \ldots, i_t\} = \{i \in \mathbb{N} \mid \alpha_i \neq 0 \text{ and } i \in [1,r]\}.$$

Since $\alpha_r \neq 0$, this last set must be nonvoid. The vector $(\alpha_{i_1}, \ldots, \alpha_{i_t})$ is called *ceiling* of l.

With this notation (2,2,1) is the ceiling of the incidence vector 1, given in Example 3.3, since $(2,0,2,0,1) = (\alpha_1, \alpha_2, \ldots, \alpha_5)$.

Lemma 3.6: It $\beta = (\beta_1, \ldots, \beta_t)$ is the ceiling of some n-person incidence vector and $\tilde{\beta} = (\tilde{\beta}_1, \ldots, \tilde{\beta}_t) \in \mathbb{N}^t$ is a vector such that $\sum_{i=1}^{t} \beta_i = \sum_{i=1}^{t} \tilde{\beta}_i$, then there is a canonical bijection from

$$\{l \in I_n \mid \text{the ceiling of } l \text{ is } \beta\}$$

to

$$\{l \in I_n \mid \text{the ceiling of } l \text{ is } \tilde{\beta}\}.$$

Proof: Let l be an incidence vector of ceiling β, let us say $l \in I(\Pi)$ for some Π with values $k_0 = 0 < k_1 < \ldots < k_r \doteq n\text{-}1$ and let $\alpha = (\alpha_1, \ldots, \alpha_r)$ and $(\alpha_{i_1}, \ldots, \alpha_{i_t})$ be defined according to Definition 3.5, implying

$$\beta = (\alpha_{i_1}, \ldots, \alpha_{i_t}).$$

We put

$$\tilde{\alpha}_k := \begin{cases} \tilde{\beta}_j \text{, if } k = i_j \text{ for some } 1 \leq j \leq t \\ 0, \text{ otherwise} \end{cases} ; 1 \leq k \leq r$$

and

$$\tilde{k}_j := k_j - \alpha_j - \alpha_{j-1} - \ldots - \alpha_1 + \tilde{\alpha}_j + \ldots + \tilde{\alpha}_1,\ 1 \leq j \leq r.$$

Conclude that

$$\tilde{k}_i - \tilde{k}_{i-1} = k_i - k_{i-1} + \tilde{\alpha}_i - \alpha_i \geq \tilde{\alpha}_i$$

and thus

$$\tilde{k}_i - \tilde{k}_{i-1} - \tilde{\alpha}_i = k_i - k_{i-1} - \alpha_i .$$

Observe that $l_{i+1} = n$ is equivalent to $i \in [k_j+1, k_j+\alpha_{j+1}]_n$ for some $j \in [0,r\text{-}1]_n$ and define analogously

$$\tilde{l}_{i+1} := \begin{cases} n, \text{ if } i \in [\tilde{k}_j+1, \tilde{k}_j+\tilde{\alpha}_{j+1}]_n \text{ for some } 0 \leq j \leq r\text{-}1 \\ \tilde{k}_r, \text{ if } i = \tilde{k}_j+\tilde{\alpha}_{j+1}+s \leq \tilde{k}_{j+1} \text{ for some } j \text{ and } s, \text{ such that} \\ \qquad l_{k_j+\alpha_{j+1}+s+1} = k_r . \end{cases}$$

A simple computation shows that $\tilde{l}$ is an incidence vector of ceiling $\tilde{\beta}$ and $\tilde{l} \in I(\tilde{\Pi})$, where $\tilde{\Pi}$ is the vector of Q_n with values $0 < \tilde{k}_1 < \ldots < \tilde{k}_r = n\text{-}1$.

The inverse mapping can be defined analogously by interchanging the rôles of $\tilde{\beta}$ and β. q.e.d.

The following example graphically represents the preceding canonical bijection: Let

$$\beta = (2,2,1),\ \tilde{\beta} = (1,1,3)$$

and l be the incidence vector given in Example 3.3.

With the notation used in the above proof we get

$$(\alpha_1,\ldots,\alpha_r) = (2,0,2,0,1),\ (\tilde{\alpha}_1,\ldots,\alpha_r) = (1,0,1,0,3)$$

and

$$(\tilde{k}_1,\ldots,\tilde{k}_r) = (2,6,8,10,13).$$

Figure 3 illustrates the anti step functions $h = A^l$ and $\tilde{h} = A^{\tilde{l}}$.

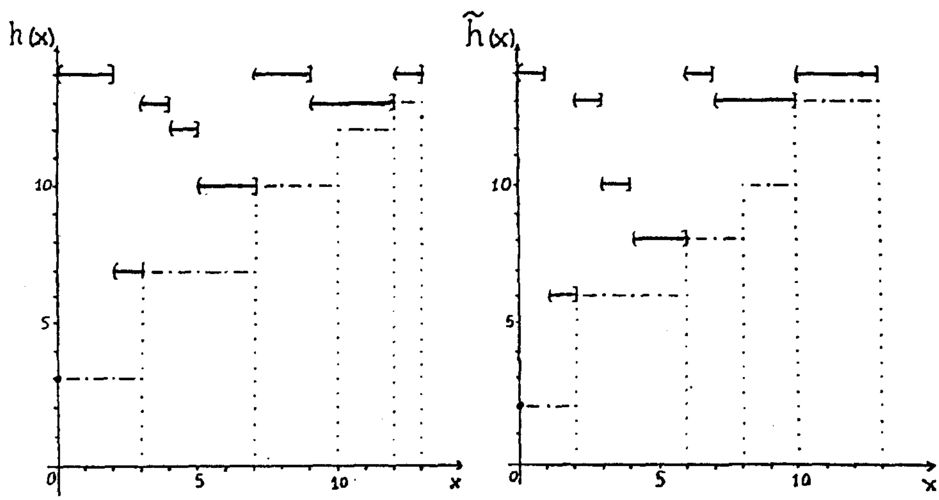


Fig. 3. The lines "-·-·-" represent the underlying quadratic step functions.

Definition 3.7: Let l be a member of I_n with ceiling $(\beta_1,\ldots,\beta_t)$.
Then (t,p) is called *type* of l, if

$$p = \sum_{i=1}^{t} \beta_i - t.$$

The subset of I_n, whose elements are of type (k,p), is denoted by $I_n^{k,p}$ and, in addition, the cardinality of this set is abbreviated by $a_n^{k,p}$, formally:

$$I_n^{k,p} = \{l \in I_n \mid h \text{ is of type } (k,p)\},\ a_n^{k,p} = |I_n^{k,p}|.$$

Note that if $a_n^{k,p} \neq 0$, then $k \in \mathbb{N}$, $p \in \mathbb{N}_0$.

The rest of this chapter will be used to give a recursive description of the cardinalities $a_n^{k,p}$ and starts with the following important

Theorem 3.8: Let $n \geq 3$. Then the following assertions are valid.

(i) $a_{n+1}^{1,0} = \sum_{k \geq 1} \sum_{p \geq 0} a_n^{k,p}$;

(ii) $a_{n+1}^{k,0} = \sum_{\tilde{k} \geq k-1} \binom{\tilde{k}}{k-1} \sum_{p \geq 0} a_n^{\tilde{k},p} - a_n^{k-1,0}$, if $k \geq 2$;

(iii) $a_{n+1}^{k,p} = a_n^{k,p-1} \cdot \frac{p+k-1}{p}$, if $p \geq 1$.

Proof: The canonical identification of incidence vectors with anti step functions may help to illustrate the formal arguments.

ad (i): If l is a member of $\bigcup_{k \geq 1} \bigcup_{p \geq 0} I_n^{k,p}$, then

$$(l_1, \ldots, l_n, n+1) \in I_{n+1}^{1,0}.$$

Conversely, if $\bar{l} \in I_{n+1}^{1,0}$, then

$$(l_1, \ldots, l_n) \in \bigcup_{k \geq 1} \bigcup_{p \geq 0} I_n^{k,p}.$$

These considerations induce a function and its inverse and thus verify assertion (i).

ad (ii): If $l \in I_{n+1}^{k,0}$, then l has the ceiling $\underbrace{(1, \ldots, 1)}_{k \text{ times}}$.

If $k_1 < \ldots < k_r = n$ are the values of this anti step function A^l (exept $n+1$), then $k_{r-1} = n-1$ and the vector $(0, k_1, \ldots, k_r)$ defines a quadratic step function Q^Π, such that $l \in I(\Pi)$. Define

$$\bar{l}_{i+1} := \min \{l_{i+1}, n\} \text{ for all } i \in [0, n-1]_n.$$

Let $Q^{\tilde{\Pi}}$ be the quadratic step function defined by the vector of values $(0, k_1, \ldots, k_{r-1})$ and observe that

$$\tilde{l} \in I(\tilde{\Pi}).$$

If $(\beta_1,...,\beta_t)$ is the ceiling of $\tilde{l}$, then it is clear that

$$\sum_{i=1}^{t} \beta_i \geq k\text{-}1 \text{ and } t \geq k\text{-}1,$$

since $\{i \leq n \mid l_{i+1} = n+1\} \subseteq \{i \leq n \mid \tilde{l}_{i+1} = n\}$.

As there is at least one player i such that $l_i = n = \tilde{l}_i$, the case $\sum_{i=1}^{t} \beta_i = k\text{-}1 = t$ cannot occur.

For the converse let $l = (l_1,...,l_n)$ define a member of $I_n^{\tilde{k},p}$ of some ceiling $(\beta_1,...,\beta_{\tilde{k}})$ such that

$$p + \tilde{k} \geq k, \tilde{k} \geq k\text{-}1.$$

Let $0 < k_1 <...< k_r = n\text{-}1$ define the quadratic step function Q^{Π} such that $l \in I(\Pi)$ and put

$$\alpha_i = |\{j \in \mathbb{N} \mid l_j = n \text{ and } k_{i\text{-}1} < j\text{-}1 \leq k_i\}|$$

(the vector $(\alpha_1,...,\alpha_r)$ has already been constructed in Definition 3.5).

Following this definition there is an increasing subsequence $(i_1,...,i_{\tilde{k}})$ of $(1,...,r)$ such that

$$(\beta_1,...,\beta_{\tilde{k}}) = (\alpha_{i_1},...,\alpha_{i_{\tilde{k}}}).$$

Let T be an arbitrary subset of $\{1,...,\tilde{k}\}$ of cardinality k-1.
Define

$$l_i^T := \begin{cases} n+1, \text{ if } i = k_{i_s\text{-}1}+1 \text{ for some } s \in T \text{ or } i=n+1 \\ l_i, \text{ otherwise} \end{cases}$$

To verify that $l^T = (l_1^T,...,l_{n+1}^T)$ is an incidence vector in $I_{n+1}^{k,0}$ is straightforward and therefore skipped here.

Note that the case $\tilde{k} = k\text{-}1$ and $p = 0$ must be excluded, since then T automatically coincides with $\{1,...,k\text{-}1\}$ and l^T cannot be an incidence vector ($l_{n+1}^T = n+1 = \min \{l_j^T \mid 1 \leq j \leq n, l_j^T \geq n\}$, thus condition (i) of Definition 2.8 is violated).

With the above notation the following assertions are valid:

(i) $l^T = l$,

(ii) For each $l \in I_{n+1}^{k,0}$ there is a unique $T \subseteq \{1,...,\tilde{k}\}$ of cardinality k-1 such that $\tilde{l}^T = l$ and $\tilde{l} \in I_n^{\tilde{k},p}$.

From combinatorics it is known that the binomial coefficient $\binom{\tilde{k}}{k-1}$ describes the cardinality of the set of subsets of $\{1,...,\tilde{k}\}$ containing k-1 elements, thus assertion (ii) of the theorem is shown.

Figure 4 illustrates per example how the map $l \to l^T$ works.

ad (iii): The mapping

$$(\beta_1,...,\beta_k) \to \{\beta_1, \beta_1+\beta_2,...,\beta_1+...+\beta_{k-1}\}$$

yields a bijection from

$$\{(\beta_1,...,\beta_k) \mid \sum_{i=1}^{k} \beta_i - k = p, \beta_i \in \mathbb{N}\} =: T^{k,p}$$

to

$$\{M \subseteq \{1,...,k+p-1\} \mid |M| = k-1\}$$

for each $k \in \mathbb{N}$, $p \in \mathbb{N}$,
thus

$$| T^{k,p} | = \binom{k+p-1}{k-1}.$$

Analogously it can be shown:

$$| \{(\beta_1,...,\beta_k) \in T^{k,p} \mid \beta_k \geq 2\} | = \binom{k+p-2}{k-1}$$

Let l be a member of $I^{k,p}_{n+1}$ with a ceiling $(\beta_1,...,\beta_k)$ such that $\beta_k \geq 2$. Then define for each $1 \leq i \leq n$

$$\tilde{l}_i = \begin{cases} l_i, & \text{if } l_i \leq n\text{-}2 \\ n\text{-}1, & \text{if } l_i = n \\ n, & \text{if } l_i = n+1 \end{cases}$$

The case $l_i = n\text{-}1$ cannot occur since $\beta_k \geq 2$ directly implies $\{i \leq n+1 \mid l_i = n\text{-}1\} = \emptyset$. It is clear (see Definition 2.8) that $\tilde{l} = (\tilde{l}_1,...,l_n)$ is an incidence vector, thus $\tilde{l} \in I^{k,p\text{-}1}_n$.

Furthermore the function

$$\{l \in I^{k,r}_{n+1} \mid l \text{ has the ceiling } (\beta_1,...,\beta_k) \text{ with } \beta_k \geq 2\} \to I^{k,p\text{-}1}_n$$
$$l \mapsto \tilde{l}$$

is bijective, because it is obvious how to define the inverse mapping.

Combining the above observations and definitions, and using Lemma 3.6, we get: Fix a ceiling $\beta = (\beta_1,...,\beta_k)$ such that $\sum_{i=1}^{k} \beta_i\text{-}k = p \geq 1$, then

$$|\{l \in I^{k,p}_{n+1} \mid l \text{ has the ceiling } \beta\}| \cdot \binom{k+p\text{-}1}{k\text{-}1} = a^{k,p}_{n+1}$$

$$|\{l \in I^{k,p}_{n+1} \mid l \text{ has the ceiling } \beta\}| \cdot \binom{k+p\text{-}2}{k\text{-}1} =$$

$$|\{l \in I^{k,p}_{n+1} \mid l \text{ has the ceiling } (\tilde{\beta}_1,...,\tilde{\beta}_k) \text{ with } \tilde{\beta}_k \geq 2\}| = a^{k,p\text{-}1}_n.$$

Consequently

$$a^{k,p}_{n+1} = a^{k,p\text{-}1}_{\mathrm{n}} \cdot \frac{\binom{k+p\text{-}1}{k\text{-}1}}{\binom{k+p\text{-}2}{k\text{-}1}} = a^{k,p\text{-}1}_{\mathrm{n}} \cdot \frac{p+k\text{ -}1}{p}. \qquad \text{q.e.d.}$$

The following example illustrates the maps $l \to l^T$ constructed in part (ii) of the preceding proof.

Example 3.9: Let $n = 7$, $k = 3$, and $l = (3,7,7,5,7,6,7)$. Then $l \in I_7^{3,1}$, since (2,1,1) is the ceiling of l. There are exactly 3 subsets of {1,2,3} with two elements, namely $T_1 = \{1,2\}$, $T_2 = \{1,3\}$, $T_3 = \{2,3\}$. So we have to construct l^{T_i}, $i = 1,2,3$.

Since $l \in I(\Pi)$, where Q^{Π} is the quadratic step function, defined by the values (3,5,6), we get:

$$l^{T_1} = (3,8,7,5,8,6,7,8),$$

$$l^{T_2} = (3,8,7,5,7,6,8,8)$$

$$l^{T_3} = (3,7,7,5,8,6,8,8).$$

These incidence vectors are sketched in the following diagrams (as graphs of the corresponding anti step functions).

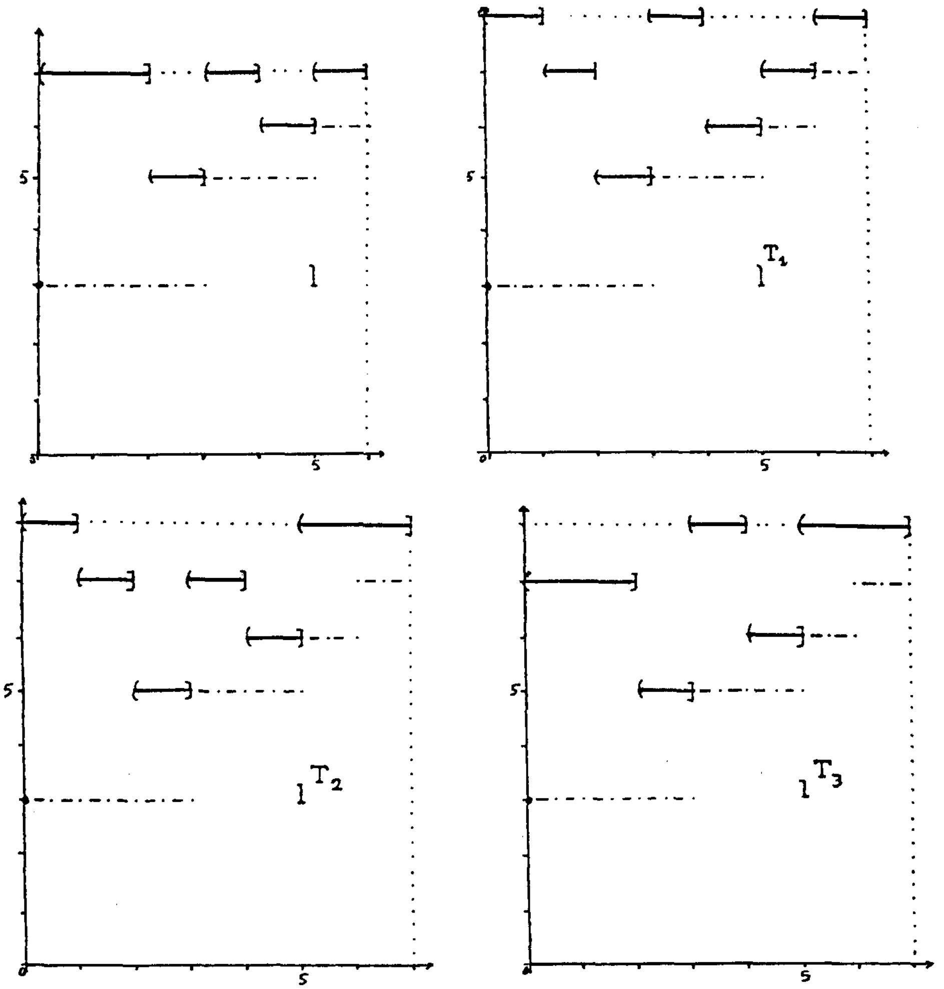


Fig. 4. The lines "·-·-·-·" represent the underlying quadratic step functions.

Corollary 3.10:

(i) $a_n^{1,n-3} = 0$

(ii) Let $(k,p) \neq (1,n\text{-}3)$. Then $a_n^{k,p} \neq 0$ if and only if the following holds true:

(a) $k \leq [\frac{n}{2}]$

(b) $p + 2k \leq n$.

(c) $(k,p) \in \mathbb{N} \times \mathbb{N} \cup \{0\}$.

These assertions can be verified by induction on n:

Since (2,3,3) is the only 3-person incidence vector, the corollary is valid in the case $n = 3$. The last theorem directly completes the proof.

Let us introduce the following notation: $D_n = \{(k,p) \mid a_n^{k,p} \neq 0\}$.
The next assertion is a direct consequence of the last corollary.

Corollary 3.11:

$$|D_n| = \begin{cases} \frac{n^2-5}{4}, & \text{if } n \text{ is odd} \\ \frac{n^2-4}{2}, & \text{if } n \text{ is even.} \end{cases}$$

Since the number of homgeneous n-person games without dummies and steps equals $a_{n+1}^{1,0}$ by Theorem 3.8 (i), it is very useful to eliminate the $a_n^{k,p}$, $p > 0$. Define

$$a_n^k := a_n^{k,0}$$

and for technical reasons

$$a_2^1 := 1.$$

With these notations the following recursive formulae are valid.

Theorem 3.12:

$$\text{(i)} \quad a_{n+1}^1 = \sum_{k=1}^{[\frac{n}{2}]} \sum_{p=0}^{n-2k} \binom{k+p-1}{p} a_{n-p}^k = \sum_{p=0}^{n-2} \sum_{k=1}^{[\frac{n-p}{2}]} \binom{k+p-1}{p} a_{n-p}^k$$

(ii) $$a^k_{n+1} = \sum_{\tilde{k}=k}^{[\frac{n}{2}]} \binom{\tilde{k}}{k-1} a^{\tilde{k}}_n + \sum_{\tilde{k}=k-1}^{[\frac{n-1}{2}]} \binom{\tilde{k}}{k-1} \sum_{p=1}^{n-2\tilde{k}} a^{\tilde{k}}_{n-p} \binom{\tilde{k}+p-1}{p}$$

$$= \sum_{p=0}^{n-2k+2} \left[\sum_{\tilde{k}=k-1}^{[\frac{n-p}{2}]} \binom{\tilde{k}}{k-1} \binom{\tilde{k}+p-1}{p} a^{\tilde{k}}_{n-p} \right] - a^{k-1}_n ,$$

if $2 \leq k \leq [\frac{n+1}{2}]$.

The proof of this theorem will be postponed, as we shall have to apply the following formula which describes $a^{k,p}_n$ in terms of some $a^k_{\tilde{n}}$.

Lemma 3.13:

$$a^{k,p}_n = a^k_{n-p} \binom{k+p-1}{p} \text{ for all } k \leq [\tfrac{n}{2}], p \leq n-2.$$

Proof (by induction on n):

In the case $p = 0$ nothing remains to be shown. Since (2,3,3) is the only 3-person incidence vector, we have

$$a^{1,1}_3 = 1 = a^1_2.$$

If $p \geq 1$, then by Theorem 3.9 (iii):

$$a^{k,p}_{n+1} = a^{k,p-1}_n \left(\frac{p+k-1}{p}\right)$$

$$= a^{k,0}_{n-p+1} \binom{k+p-2}{p-1} \left(\frac{p+k-1}{p}\right) \text{ (by inductive hypothesis)}$$

$$= a^{k,0}_{n+1-p} \binom{k+p-1}{p}. \qquad \text{q.e.d.}$$

We now proceed by proving the theorem:
It is straightforward – by interchanging the summation indices – that the second equality of (i) resp. (ii) holds.

The first equalities are shown by induction on n:
by Theorem 3.8 (i) resp. (ii) we get

$$a_4^1 = 1 \text{ resp. } a_4^2 = 1$$

which coincides with

$$\sum_{k=1}^{1} \sum_{p=0}^{1} \binom{k+p-1}{p} a_{3-p}^k \text{ resp. } \sum_{k=2}^{1} \binom{k}{1} a_3^k + \sum_{k=1}^{1} \binom{k}{1} \sum_{p=1}^{1} \binom{k+p-1}{p} a_{3-p}^k .$$

Assume the validity of the assertions for some n.
Then this assumption, Lemma 3.13 and Theorem 3.8 (i) resp. (ii) imply

$$a_{n+2}^1 = \sum_{k=1}^{[\frac{n+1}{2}]} \sum_{p=0}^{n+1-2k} a_{n+1}^{k,p} = \sum_{k=1}^{[\frac{n+1}{2}]} \sum_{p=0}^{n-2k} a_{n+1-p}^k \binom{k+p-1}{p}$$

resp.

$$a_{n+2}^1 = \sum_{\tilde{k}=k-1}^{[\frac{n+1}{2}]} \binom{\tilde{k}}{k-1} \sum_{p=0}^{n+1-2\tilde{k}} a_{n+1}^{\tilde{k},p} - a_{n+1}^{k-1}$$

$$= \sum_{\tilde{k}=k}^{[\frac{n+1}{2}]} \binom{\tilde{k}}{k-1} a_{n+1}^{\tilde{k}} + \sum_{\tilde{k}=k-1}^{[\frac{n}{2}]} \binom{\tilde{k}}{k-1} \sum_{p=1}^{n+1-2\tilde{k}} a_{n+1-p}^{\tilde{k}} \binom{\tilde{k}+p-1}{p},$$

if $k \geq 2$. q.e.d.

A sketch of the recursive development will be given in the following figures. We restrict our attention to the case that n is even and k is at least two. The other three cases can be treated analogously. We presume that all a_t^j, $t \leq n$-1, are already known.

The element in the j-th row and l-th column of Figure 5, which is often deleted for clearness reasons, shall represent a_j^l .

Figure 6 is Pascal's triangle, rotated to the left by $1\frac{1}{2}$ right angles. Thus the element in the j-th row and l-th column shall represent $\binom{j+1}{j}$.

The marked areas of Figures 5 and 6 cover each other and the distances of the vertical axes are equal. The binomial coefficients of the first column of the marked area of Fig. 6 have to be multiplied with the first element in the weak marked (k-1)-st row, the second column with the second element and so on.

The number a_n^k will be obtained by summing up all the products of the coefficients in the marked areas of Fig. 5 and the modified coefficients of the marked area of Fig. 6 (elementwise).

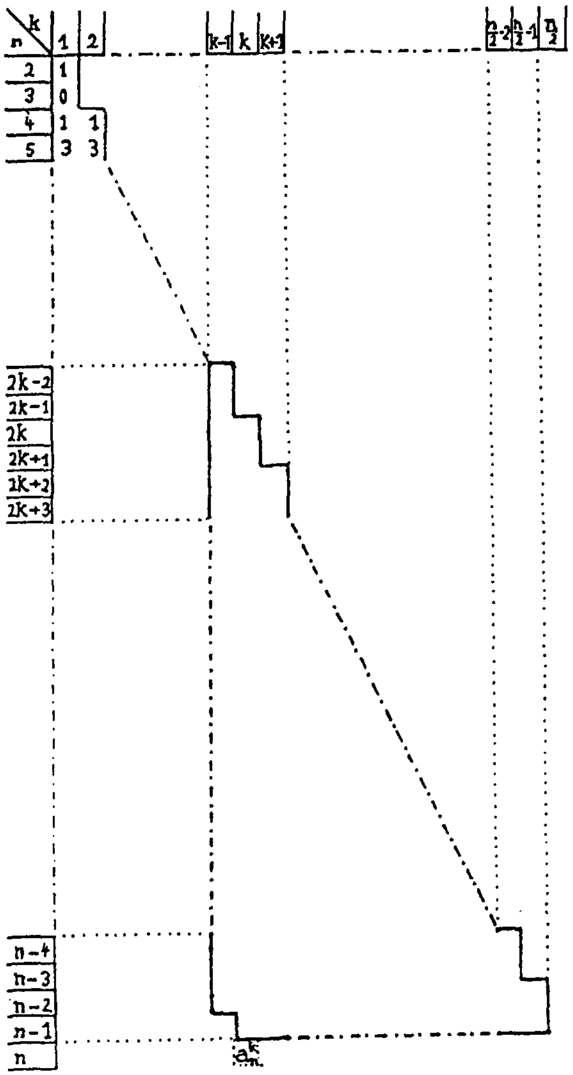


Fig. 5.

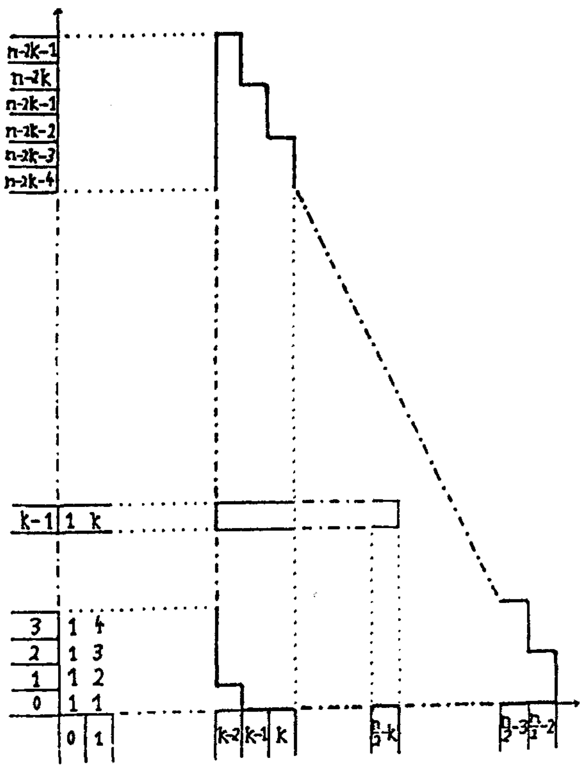


Fig. 6.

4 Test of Homogeneity

In this chapter an algorithm is constructed which enables us to decide whether a directed game is homogeneous or not.

In the following the homogeneity of a directed game (Ω, v) is tested.

Definition 4.1: Let (Ω, v) be a directed n-person game without one-person winning coalitions and let S_1 be the lexicographically maximal minimal winning coalition. If $S_1,\ldots,S_k$ are already constructed and $l = (l(S_1),\ldots,l(S_n))$, define

$$S_{k+1} = \begin{cases} \rho_k(S_{i_0}), \text{ where } i_0 = \omega_k^l, \text{ if } \omega_k^l \neq 0, \\ \emptyset, \text{ otherwise} \end{cases}$$

$\Omega \setminus \bigcup_{i=1}^{n} S_i$ is called set of *pseudo-dummies*. If $S_{i+1} = \emptyset$ and i is not a pseudo-dummy then it is called *pseudo-step*. Let $\{k_0+1,\ldots,n\}$ be the set of pseudo-dummies. Then $(l_1,\ldots,l_{k_0+1})$ is an incidence vector, where

$$l_j = \begin{cases} l(S_j), & \text{if } S_j \neq \emptyset \\ k_0+1, & \text{otherwise} \end{cases}$$

If (λ,m) is the minimal representation, generated by $(l_1,\ldots,l_{k_0+1})$ (for the expression "generated representation" Definition 2.8 is referred to), define

$$\iota(\Omega, v) = (\lambda; m_1,\ldots,m_{k_0}, \underbrace{0,\ldots,0}_{(n\text{-}k_0)\text{-times}}).$$

A straigthforward consequence of this definition, Remark 2.9 and Theorem 2.3 is the following

Theorem 4.2: Let (Ω, v) be a directed n-person game without one-person winning coalitions.

Then $\iota(\Omega, v)$ is the minimal representation of a homogeneous game, where exactly the pseudo-steps resp. pseudo-dummies of (Ω, v) are the steps resp. dummies of $\iota(\Omega, v)$.

Additionally, if (Ω, v) is homogeneous, then $\iota(\Omega, v)$ is the minimal representation of the same game.

Proof: The first part is obvious from Corollary 2.11 and Lemma 1.4. As the chain $S_1,\ldots,S_n$ constructed in the last definition, does not depend on the representation but only on the game it can be started with a minimal representation, which is itself automatically homogeneous, if the game is. Thus, the second part again is implied using the just mentioned assertions. q.e.d.

Remark: It is obvious how to generalize the preceding definitions and assertions to directed games containing one-person winning coalitions.

Corollary 4.3: A directed game (Ω, v) is homogeneous, iff the incidence matrices of (Ω, v) and $\iota(\Omega, v)$ coincide.

A practicabel, slightly modified test of homogeneity, which already has been implemented on a computer, is presented in what follows.

Let (Ω, v) be a directed game. A minimal winning coalition S is called *shiftminimal,* if $S \cup \{i+1\} \setminus \{i\}$ is losing for all players i such that $i \in S$, $i+1 \notin S$. (For this notation we refer to [6]). The matrix

$$X^* := X^*(\Omega, v) = \begin{bmatrix} \vdots \\ S \\ \vdots \end{bmatrix}_{S \text{ shiftminimal}}$$

with lexicographically ordered rows is called shiftminimal matrix of (Ω, v). Ostmann [6] has shown $X^*(\Omega, v)$ uniquely determines (Ω, v) and that neighboring players i, $i+1$ are of different type (for the definition of the term "type" we refer to section 1), iff there is a shiftminimal coalition S with $i \in S$, $i+1 \notin S$. The term "type" can easily be generalized to coalitions: S and S' have the same type – $S \sim S'$ –, iff there is a permutation π of Ω such that $\pi(S) = S'$ and $\pi(i) \sim i$ for all $i \in \Omega$. With this definition it is obvious that all coalitions of one type are winning resp. minimal winning if only one does. In the homogeneous case this notation trivially implies: each minimal winning coalition S corresponds to a unique shiftminimal coalition $\tilde{S} =: SH(S)$ satisfying $\tilde{S} \sim S$. In the general case $SH(S)$ is to be defined as the lexicographically last coalition such that $SH(S) \sim S$.

Lemma 4.4: If (Ω, v) is a homogeneous n-person game, not necessarily without dummies and steps, and $S_1,\ldots,S_n$ are constructed according to Definition 4.1, then the following assertions are equivalent.

(i) $i \not\sim i+1$

(ii) There is a j, $1 \leq j \leq n$, such that $\{i, i+1\} \cap SH(S_j) = \{i\}$.

Proof: We only have to show that (i) implies (ii).

Assume $i \not\sim i+1$. If $S_{i+1} = \rho_i(S_j)$ for some $j \leq i$, then $i \in SH(S_j)$ by definition. If $i+1 \notin SH(S_j)$, nothing remains to be shown. In the other case $i+1 \in S_j$, thus $S_{i+2} = \rho_{i+1}(S_j)$ (see Definition 2.8). Consequently $SH(S_{i+2}) \cap \{i, i+1\} = \{i\}$.

If $S_{i+1} = \emptyset$, two cases may occur:

1. None of the coalition $S_1,\ldots,S_i$ contains player i. Then i and thus $i+1$ are pseudo-dummies of (Ω, v). Since $\iota(\Omega, v)$ represents (Ω, v), both players are dummies and consequently of the same type, which contradicts assumption (i).
2. Player i is a pseudo-step of (Ω, v), thus a step. The fact that $\iota(\Omega, v) = (\lambda; m_1,\ldots,m_{k_0}, 0,\ldots,0)$ is the minimal representation of (Ω, v) implies $m_i > m_{i+1}$ (see section 1) and thus – by Definition 2.8 – the existence of t such that $i \in S_t$, $i+1 \notin S_t$. Then $SH(S_t)$ satisfies property (ii) by definition. q.e.d.

Lemma 4.5: Let X^* be the shiftminimal $m \times n$ matrix of a homogeneous game. Then the following assertions are valid:

(*) (i) If $t < l(X_{1}^{*}.)$, $t \notin X_{1}^{*}.$, then $t \sim l(X_{1}^{*}.)$

(ii) If $t_0 := \min\{j \mid j \in X_{i}^{*}., j \notin X_{i+1}^{*}.\} < t < l(X_{i+1}^{*}.)$, $t \notin X_{i+1}^{*}.$, then $t \sim l(X_{i+1}^{*}.)$, for all $1 \leq i < m$.

Proof:

ad (i): The fact that $X_{1}^{*}. = SH(S)$, where S is the lexicographically maximal minimal winning coalition, directly implies (i).

ad (ii): Again $X_{i+1}^{*}. = SH(\rho_{t_0}(X_{i}^{*}.))$ directly implies (ii). q.e.d.

Definition 4.6: Let X^* be a shiftminimal matrix of a directed n-person game (Ω, v), satisfying condition (*). Let $i_1 < \ldots < i_r = n$ be the last players of the different types of the game and $i_0 = 0$. Let S_{1}^{Tv} be the first row of X^*. If $S_{1}^{Tv},\ldots,S_{k}^{Tv}$ are already constructed and $k < r$, define

$$S_{k+1}^{Tv} = \begin{cases} SH(\rho_{i_k}(S_{j}^{Tv})), & \text{if } j = \min\{j \leq i_k \mid j \in \{t \leq k \mid l(S_{k}^{Tv}) \geq i_k\} \mid |[i_k+1, n] \cap S_{t}^{Tv}| \text{ minimal}\}\} \\ & \text{the preceding set is nonvoid and } \rho_k(S_{j}^{Tv}) \text{ exists} \\ \emptyset, & \text{otherwise} \end{cases}$$

Analogously to Definition 2.8 (3) the matrix

$$\overline{S}^{Tv} = \begin{bmatrix} S_{1}^{Tv} \\ \vdots \\ S_{r}^{Tv} \end{bmatrix}$$

generates a vector (λ, m) via

$m_{k_0+1} = \ldots = m_n = 0$, where $k_0 = \max\{l(S_{i}^{Tv}) \mid 1 \leq i \leq r\}$, let us say $k_0 = i_{t_0}$; $m_{i_{t_0-1}} = \ldots = m_{i_{t_0}} = 1$. If $m_{i_t+1},\ldots,m_n$ are already constructed for some $t \leq t_0$ and $t \geq 1$, define

$$m_{i_{t-1}+1} = \ldots = m_{i_t} = \begin{cases} m([i_t+1,n] \cap S^{Tv}_{t+1} \setminus S^{Tv}_j), \text{ if } S^{Tv}_{t+1} \neq 0, \text{ where } SH\,\rho_{i_t}(S^{Tv}_j) = S^{Tv}_{t+1} \\ 1+m([i_t+1,n] \setminus S^{Tv}_j), \text{ otherwise where } i_t \in S^{Tv}_j \text{ and } |S^{Tv}_j \cap [\Sigma i_t+1,n]| \text{ is minimal with this property.} \end{cases}$$

$$\lambda := m(S^{Tv}_1).$$

Lemma 4.7: Let (Ω, v) be homogeneous. Then the representation $\iota(\Omega, v)$ coincides with the vector (λ,m) generated by $\overline{S}^{Tv}$.

Proof: Let $S_1,\ldots,S_n$ be defined according to Definition 4.1 Then it is obvious that $S_1 \sim S^{Tv}_1$. Using an inductive argument we easily see that $SH\, S_{i_{t-1}+1} \setminus [1,i_{t-1}] = S^{Tv}_t \setminus [1,i_{t-1}]$ for all t, $1 \le t \le r$. It is straightforward to finish the proof by comparing Definitions 2.8 (3) and 4.6. q.e.d.

Summarizing the preceding notation and assertions we get

Theorem 4.8: Let (Ω, v) be a directed game, whose shiftminimal $(m \times n)$ matrix X^* satisfies condition (*).

(Ω, v) is homogeneous, iff the following conditions are valid:

(i) $i \not\sim i+1$, iff there is a t such that $i \in S^{Tv}_t$, $i+1 \notin S^{Tv}_t$.

(ii) $m(S) = \lambda$ for each row S of X^*, where (λ,m) is the vector generated by $\overline{S}^{Tv}$.

(iii) Let $j \in [1,m]_n$. If $i \in X^*_{j\cdot}$, $i \quad t_0 = \min\{t \mid t \in X^*_{j\cdot}, t \notin X^*_{j+1\cdot}\}$, $i+1 \not\sim i \not\sim t_0$, where $X^*_{m+1\cdot}$ is the empty coalition, then $m([i+1,n] \setminus X^*_{j\cdot}) < m_i$ and, if $j = m$, $m([t_0+1,n] \setminus X^*_{m\cdot}) < m_{t_0}$.

Proof: Assume (Ω, v) is homogeneous. Assertion (i) is a direct consequence of Lemma 4.4 and the proof of Lemma 4.7 and again Lemma 4.7 implies (ii). Condition (iii) follows from the fact that $SH(S)$ is shiftminimal for each minimal winning coalition S.

Conversely if X^* satisfies (*), the matrix $\overline{S}^{Tv}$ is well defined and generates the minimal representation (λ,m) of a game $(\Omega, \tilde{v})$. Each winning coalition w.r.t. (Ω, v), does win w.r.t. $(\Omega, \tilde{v})$, since for each shiftminimal coalition S the equality $m(S) = \lambda$ holds true. Assume there is a losing coalition S w.r.t. (Ω, v) which wins w.r.t. $(\Omega, \tilde{v})$. W.l.o.g. let S be shiftminimal w.r.t. (Ω, v). Then there is a unique $j \in [1,m]$ such that $X^*_{j\cdot}$ is lexicographically greater than S is greater than $X^*_{j+1\cdot}$. Define $\tilde{i} = \min[t \mid t \in X^*_{j\cdot}, t \notin S\}$ and $i = \max\{t \mid t \sim \tilde{i}\}$. Then it is obvious that $i \quad t_0$, thus $i \sim t_0$ by condition (iii). Consequently, $X^*_{j+1\cdot}$ has a proper subcoalition $\tilde{S}$ with $m(\tilde{S}) = \lambda$, thus $l(X^*_{j+1\cdot})$ is a dummy w.r.t. (λ,m). By (i) we have $i := l(X^*_{j+1\cdot}) \le t_0$, $i \not\sim t_0$, which contradicts (iii). q.e.d.

The following 3 examples show that none of the conditions (i), (ii), (iii) can be deleted in Theorem 4.8. The shiftminimal matrices of the considered games, which all are weighted majority games, given in terms of representations, have property (*). Since it is straightforward to verify the relevant properties of these games, the concerning proofs are skipped.

The game (Ω, v) represented by $\begin{cases}(7;5,3,3,1,1,1)\\(10;5,5,2,2,2,2,2)\\(29;19,12,12,5,5,5,2,1,1)\end{cases}$

satisfies conditions $\begin{cases}\text{(i), (ii)}\\\text{(i), (iii)}\\\text{(ii), (iii)}\end{cases}$ but not $\begin{cases}\text{(iii)}\\\text{(ii)}\\\text{(i)}\end{cases}$ respectively

The following diagram illustrates some properties of these games.

representation of (Ω, v)	$X^*(\Omega, v)$	$\bar{S}^{Tv}$	representation, generated by $\bar{S}^{Tv}$	satisfied conditions
(7;5,3,3,1,1,1)	$\begin{bmatrix}101000\\100011\\011001\end{bmatrix}$	$\begin{bmatrix}101000\\011001\\100011\end{bmatrix}$	(5;3,2,2,1,1,1)	(i), (ii)
(10;5,5,2,2,2,2,2)	$\begin{bmatrix}1100000\\0100111\\0011111\end{bmatrix}$	$\begin{bmatrix}1100000\\0100111\end{bmatrix}$	(6;3,3,1,1,1,1,1)	(i), (iii)
(29;19,12,12,5,5,5,2,1,1)	$\begin{bmatrix}101000000\\100011000\\011001000\\001111100\\001111011\end{bmatrix}$	$\begin{bmatrix}101000000\\011001000\\100011000\\000000000\\000000000\end{bmatrix}$	(5;3,2,2,1,1,1,0,0,0)	(ii), (iii)

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