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Star-shapedness of the kernel for homogeneous games

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Abstract

Homogeneous games and weighted majority games were introduced by von Neumann–Morgenstern (*Theory of Games and Economic Behavior*, 1944) in the constant-sum case. Peleg (*Illinois Journal of Mathematics*, 1966, 10, 39–48; *SIAM Journal of Applied Mathematics*, 1968, 16, 527–532) studied the kernel and nucleolus for these classes of games. The general theory of homogeneous, not necessarily constant-sum, games was developed by Ostmann (*International Journal of Game Theory*, 1987a, 16, 69–81), Rosenmüller (On homogeneous weights for simple games, Working Paper 115, 1982; *Zeitschrift für Operations Research*, 1984, 12, 123–141; *Mathematics of Operations Research*, 1987a, 12, 309–330), and Sudhölter (*International Journal of Game Theory*, 1989, 18, 433–469). Peleg–Rosenmüller (*Games and Economic Behaviour*, 1992, 4, 588–605) used it to discuss several solution concepts for homogeneous games without steps. A reduction theorem for the nucleolus and kernel of homogeneous games with steps was proved by Rosenmüller–Sudhölter (*Discrete Applied Mathematics*, 1994, 50, 53–76) and Peleg et al. (The kernel of homogeneous games with steps, *Essays in Game Theory in Honor of Michael Maschler*, 1994), respectively. On the basis of these results, this paper shows that the kernel of each homogeneous game without winning players is star-shaped. Moreover, each of these games possesses a truncated game that is uniquely determined and homogeneous itself. The normalized vector of weights of the minimal representation of the truncated game is a center of both of the kernels of the original game and the truncated game. Moreover, this preimputation is Lorenz maximal within the kernel and can be characterized as the unique minimizer of a social welfare ordering in the kernel. To be more precise, the center solution uniquely minimizes a weighted Gini coefficient. Every weighted majority game occurs as a reduced game of certain homogeneous games called homogeneous extensions. The kernel of a weighted majority game arises from that of each of its homogeneous extensions in a very simple way. Moreover, the kernels of partition games are shown to be singletons.

Keywords: Weighted majority game; Homogeneity; Incidence vector; Prekernel; Kernel; Partition game

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1. Introduction

In this section a short survey providing the necessary foundations from the theory of weighted majority and, particularly, homogeneous games, as well as some motivation, is presented. For a detailed definition, see Sections 3–7.

Two classes of simple games, the weighted majority games and the subset of homogeneous games, are considered in this paper. A simple game is a co-operative multi-person game in which each coalition either wins, i.e. obtains a fixed positive payoff, or loses, i.e. obtains no payoff. If it is possible to separate winning coalitions from losing ones by assigning non-negative weights to the players such that the aggregated weight of each winning coalition exceeds, or is equal to, a positive level, whereas the weight of each losing coalition is less than the level, then the game is a weighted majority game. The vector that consists of both, the level and the weights, is a representation of the game. If, in addition, there is a representation such that each winning coalition contains a 'smallest' winning coalition, i.e. a minimal winning one, with a weight exactly hitting the level, then the game is homogeneous. For the explicit definitions, see Section 2.

The terms 'simple', 'weighted majority', and 'homogeneous' were introduced by von Neumann and Morgenstern (1944). However, they were dealing with constant-sum games only. Both simple and weighted majority games appear in many applications of game theory (see, for example, Shapley, 1962). Concerning the structure of homogeneous games, Isbell (1956, 1958, 1959) and Peleg (1968) in the constant-sum case, Ostmann (1987a), Rosenmüller (1982, 1984, 1987a), and Sudhölter (1989) in the general case, should also be mentioned.

Weighted majority games are used, for example, to model decision problems in democratic parliaments. A coalition, i.e. the collection of all members of certain parties in the gremium, wins if it possesses the required majority (the 'simple' majority or the 'two-thirds' majority, for example). The Shapley and the Banzhaf value are frequently used to measure the (relative) power of the parties in the parliament (see, for example, Dubey and Shapley, 1978). The problem of finding a voting rule that generates a 'small' parliament, such that the game among the voters coincides with the game in the parliament (see, for example, Ostmann, 1983) directly leads to the question of finding a minimal integer representation of a given weighted majority game. For the homogeneous constant-sum case, von Neumann and Morgenstern (1944) had already noticed the existence of unique minimal representations. Isbell (1959) found an example of a non-homogeneous weighted majority constant-sum game that possesses two minimal representations. He asked whether there is a canonical representation (a representation that can intuitively be justified and that coincides with the unique minimal representation in the homogeneous case) for an arbitrary weighted majority constant-sum game. Peleg (1968) proved that the nucleolus possesses the desired properties up to normalization. Ostmann (1987a) showed that, also, non-constant-sum homogeneous games have a unique minimal integer representation. This representation, as in the constant-sum case, is automatically homogeneous itself. With the help of this result and more involved methods, Sudhölter (1989) was able to describe homogeneous games by 'incidence' vectors, which can be characterized algebraically. This 'theory of incidence vectors' allows us to give a formula for the number of homogeneous n-person games recursively with respect to the number of players. Moreover, an algorithm to create all homogeneous games can be deduced.

In the meantime, it turned out that every weighted majority game possesses a canonical representation, given by a certain solution concept applied to the game. This solution concept is defined on the family of all transferable utility (TU)-games and possesses axiomatizations (Sudhölter, 1996a,b).

Homogeneity occurs in many places in the literature. This property, together with some kind of non-degeneracy, can be viewed as a surrogate for non-atomicity in the finite case, as Rosenmüller (1987b) showed. To come back to von Neumann and Morgenstern, they used homogeneity to discuss their 'stable set'. Indeed, in the homogeneous constant-sum case, the main simple solution is obtained by restricting and normalizing the vector of minimal integer weights to each minimal winning coalition. This procedure yields a stable set. Unfortunately, this approach cannot be generalized to non-constant-sum games. Peleg (1966, 1968) was the first person who considered different solution concepts (nucleolus and kernel), but, again, for constant-sum games. Peleg and Rosenmüller (1992) used the structural results on the general homogeneous games mentioned above to show that the kernel always contains the normalized minimal representation for homogeneous games 'without' steps. However, they observed that the least core and, thus, the nucleolus, do not necessarily contain the minimal representation. The fact that steps play a very dominant role in the homogeneous game makes us believe that solution concepts like the nucleolus and the kernel should react according to the law, 'steps rule their followers', by giving a considerable advantage to players preceding the first step. Indeed, it turned out that the nucleolus vanishes after the first step and, hence, the computation of the nucleolus of a general homogeneous game can be reduced to the computation of the nucleolus of the reduced or truncated game, which is obtained by cutting off the players after the first step (see Rosenmüller–Sudhölter, 1994). Peleg et al. (1994) showed the same property for the kernel (see the Reduction Lemma of Section 3 for a summary).

The present paper starts at this point and asks how the minimal representation can be characterized within the kernel of a homogeneous game without steps. Moreover, the special shape of the kernel is discussed. Though the Reduction Lemma is used in several proofs, all results are stated in their general form.

Section 2 presents the notation, partially adopted from Peleg et al. (1994). Moreover, necessary foundations and results concerning weighted majority games and, in particular, homogeneous games, are summarized. The center solution, which is defined, roughly speaking, as the normalized vector of weights of the minimal representation of the truncated game, is introduced. The center solution coincides with the normalized vector of minimal weights in the case of a homogeneous game without steps.

In Section 3, the main result of Peleg et al. is recalled and some basic properties of the prekernel are formulated and proved. The center solution is a member of the prekernel with the largest carrier.

Section 4 presents the Technical Lemma, which is the most important tool for all new results. Basically, it is shown that the maximal surplus of one player over some other at any prekernel member is attained by some minimal winning coalition. Peleg (1966) proved the same assertion for a subclass of the homogeneous constant-sum games, namely the partition games, but his proof cannot be generalized to the class of homogeneous games or even to those that possess the constant-sum property in an obvious way. The approaches are totally different and the characterization of homogeneous games via 'incidence' vectors (see Sudhölter, 1989) plays the central role in the proof of the Technical Lemma. Similar to Peleg's approach, star-shapedness of the prekernel is a straightforward consequence of this lemma. Moreover, the center solution is a center of the prekernel, which means that every line segment connecting an arbitrary element of the kernel with the center solution is completely contained in the prekernel. Theorem 4.4 shows that the center solution is an extreme point of the convex hull of the prekernel, which means (see Section 6) that the prekernel is a cone over a union of polytopes all lying in a lower-dimensional hyperplane. For a sketch, see Figs. 1 and 2 below.

An application of these results is presented in Section 5. Peleg (1986) showed that the prekernel can be justified by a set of intuitive axioms. The prenucleolus is a single valued solution in the prekernel that also possesses an axiomatization (Sobolev, 1975). This section is meant to show that the center solution can be regarded as a justified proposal on how to share the worth of the grand coalition among the players in the homogeneous case. The notion of 'interpersonal comparison of utilities' is frequently used in the context of TU-games. With this notion, it seems feasible at a first view to propose an arbitrary preimputation of the prekernel, since surpluses are balanced at every element of the prekernel. The center solution assigns a Lorenz maximal preimputation within the prekernel to every homogeneous game. Moreover, it uniquely minimizes a certain weighted Gini inequality index that puts 'higher weights to poorer people' and, thus, maximizes the social welfare of 'small' players within the prekernel. For the theoretical results, see Lemma 4.3 and Theorem 5.1. The notation of this section is adopted from Moulin (1988).

Section 6 shows that the prekernel of an arbitrary weighted majority game is a 'canonical' image of the prekernels of certain homogeneous games, called homogeneous extensions. Indeed, the prekernel of the original game arises from restricting that of each of its homogeneous extensions to the preimputations of the original game. In this sense, the preceding result constitutes a new justification for considering homogeneous games as a subfamily of weighted majority games.

In Section 7 it is shown that Peleg's result on partition games can be sharpened. The kernel of a partition game is not only star-shaped but a singleton consisting of the center solution, which coincides in this case with the normalized minimal representation of the game. Finally, some illustrating examples are presented and the relation of the prekernel and the kernel is discussed.

2. Notation, homogeneous games, and the center solution

During this paper we let $\mathbb{N} = \{1, 2, 3, \ldots\}$ denote the universe of *players*. Finite subsets of $\mathbb{N}$ are called *coalitions*; *intervals* are subsets of $\mathbb{N} \cup \{0\}$ of the form:

$$[a, b] = \{i \in \mathbb{N} \cup \{0\} \mid a \leq i \leq b\},$$

where $a, b \in \mathbb{N} \cup \{0\}$.

The *grand coalition* is an interval $\Omega = \Omega_n = [1, n]$ for some $n \in \mathbb{N}$. If

$$v: \mathcal{P}(\Omega) \to \mathbb{R}, v(\emptyset) = 0,$$

where $\mathcal{P}(\Omega) = \{S \mid S \subseteq \Omega\}$, is a mapping (the *coalitional* or *characteristic function*), then $(\Omega, \mathcal{P}(\Omega), v)$ is a *(TU-) game (with n persons)*. Since the nature of Ω and $\mathcal{P}(\Omega)$ is determined by the characteristic function, v is called a game as well. A coalition $S \subseteq \Omega$ is often identified with the indicator function 1_S, considered as an n-vector.

Given a game v, the *desirability relation* (see Maschler and Peleg, 1966) *of* v is a binary relation on players. Player $j \in \Omega$ is more desirable than $i \in \Omega$ (written $i \precsim j$ or $i \precsim_v j$), if $v(S \cup \{i\}) \leq v(S \cup \{j\})$ for all $S \subseteq \Omega \setminus \{i, j\}$. Note that $\precsim$ is a relation with respect to players, which can be generalized to coalitions (see, for example, Einy, 1985). If $i \sim j$ (i.e. $i \precsim j$ and $j \precsim i$), then i and j are *interchangeable* or *equivalent in* v. The game v is *ordered* if its desirability relation is complete. An ordered game is a *directed* game if, additionally,

$$n \precsim n-1 \precsim \cdots \precsim 1$$

is valid. We always assume that ordered games are directed, since this can be enforced by just renaming the players. For every coalition S the *length of* S is defined to be

$$l(S) = \max S .$$

Note that in a directed game $l(S)$ is the 'weakest', 'smallest', or 'the last' player of coalition S.

A game v is *simple*, if it is monotonic (i.e. $v(S) \leq v(T)$ for $S \subseteq T \subseteq \Omega$) and satisfies $v(S) \in \{0,1\}$ for $S \subseteq \Omega$. A coalition S is *winning* (in a simple game v), if $v(S)=1$, and *losing*, otherwise. In this case, the set of winning coalitions is abbreviated by W_v. Simple directed games are discussed in Ostmann (1987b, 1993), and Krohn and Sudhölter (1995). In a simple game, all subcoalitions of losing coalitions are losing as well. If each proper subcoalition of a winning coalition is a losing one, then this winning coalition is called a *minimal* winning (*min-win*) coalition. It should be noted that a simple game is completely determined by its set of min-win coalitions, denoted by W_v^m. To simplify matters we exclude the 'degenerate' simple games having no winning coalitions at all.

A *weighted majority game* v (with n players) is a simple n-person game having a *representation* $(\lambda; m)$, i.e. a *level* $\lambda \in \mathbb{R}_{>0}$, and a vector of *weights* – a measure – $m \in \mathbb{R}^n_{\geq 0}$, such that

$$v(S) = \begin{cases} 1\,, & \text{if } m(S) \geq \lambda\,, \\ 0\,, & \text{if } m(S) < \lambda\,. \end{cases}$$

Here, we use $m(S) = \sum_{i \in S} m_i$ $(S \subseteq \Omega)$ and call $m(S)$ the *weight* of coalition S. Since a weighted majority game is automatically ordered, it is directed by the assumption. Therefore, we can choose the components of m to be non-increasingly ordered. Moreover, the aggregated weight of the grand coalition exceeds the level, i.e. $m(\Omega) \geq \lambda$. Additionally, λ is assumed to be maximal, which means that there is at least one minimal winning coalition S of v satisfying $m(S) = \lambda$. Note that every weighted majority game possesses an *integer* representation (i.e. $m \in \mathbb{N}_0^n$, $\lambda \in \mathbb{N}$) with the preceding properties. A weighted majority game is *homogeneous* if it has a *homogeneous representation*, i.e. a representation $(\lambda; m)$ satisfying the property that every coalition $S \in W_v$ possesses a subcoalition T with $m(T) = \lambda$.

Let v be a directed simple game. There is a unique min-win coalition with minimal length. This coalition is an interval of the form $[1, t]$ and the *lexicographically maximal* (*lex-max*), min-win coalition of v. Player $i \in \Omega$ is a *nullplayer* if $v(S \cup \{i\}) = v(S)$ for all $S \subseteq \Omega$. Clearly, nullplayers are equivalent. A player is a *vetoer* if she is a member of every winning coalition. The equivalence classes or types of players (e.g. vetoers, winning and nullplayers) establish a decomposition of Ω.

There is another decomposition of Ω, in the case of a homogeneous n-person game v, into sets of players of equal *character*. Let $(\lambda; m)$ be a homogeneous representation of v. There are three characters, called 'sum', 'step', and 'nullplayer'. The definition of a nullplayer was given above and remains unchanged. So the two others have to be defined. We fix a non-nullplayer $i \in \Omega$ and consider the minimal length of min-win coalitions containing i, say,

$$l^{(i)} = \min\{l(S) \mid i \in S \in W_v^m\}\,.$$

The *domain* of i is

$$C^{(i)} = [l^{(i)} + 1, n] .$$

Player i is a *sum* if

$$m_i \leq m(C^{(i)}) ,$$

and otherwise i is a *step*.

A sum can be replaced in at least one min-win coalition by a coalition of smaller players, the weight being exactly the sum of the weights of these smaller players by the homogeneity of $(\lambda; m)$. However, 'steps rule their followers', i.e. whenever a smaller player – a player with a larger index – is a member of a min-win coalition, any preceding step also is a member.

Therefore, the characters (nullplayer, step, sum) form a partition of Ω. A homogeneous game may have no nullplayers or sums (e.g. the unanimity game of the grand coalition) but steps are always present. The smallest non-nullplayer is always a step.

Remark 2.1. (1) A homogeneous game v has a unique minimal representation – i.e. an integer valued $(\lambda; M)$ representing v such that $M(\Omega)$ is minimal among all integer representations of v – which is automatically homogeneous itself (see Osmann, 1987a, and Rosenmüller, 1982). Moreover, $M_i = M_j$, iff i and j are equivalent $(i, j \in \Omega)$. The normalized vector of weights of this minimal representation is abbreviated by $m(v) = M/M(\Omega)$.

(2) Let $(\lambda; m)$ be a homogeneous representation of the homogeneous game v and $S \in W_v^m$. The set

$$\{i \in \Omega \mid l(S) > i \notin S\}$$

is the set of *dropouts* of S. If S is not the lex-max, min-win coalition, then S possesses dropouts. In this case, the *last* dropout is denoted by

$$r(S) = \max\{i \in \Omega \mid i \text{ is a dropout of } S\} .$$

Clearly, there exists a unique $t \in [r(S) + 1, l(S)]$ such that

$$\varphi(S) = S \cup \{r(S)\} \backslash [t, l(S)]$$

is a min-win coalition. That means that φ inserts the last dropout and cuts off a tail of S to generate a min-win coalition. The aggregated weight of this tail coincides with the weight of player $r(S)$ by homogeneity. If α is the number of dropouts of S, then $\varphi^{\alpha}(S)$ – i.e. the α iterate of φ applied to S – coincides with the lex-max, min-win coalition.

To define the 'inverse' map, suppose $j \in S$ is such that

$[j, l(S)] \subseteq S$ and $S \backslash \{j\} \cup [l(S) + 1, n] \in W_v$ .Then we say that j is *expend-*

able in S, i.e. *replaceable* by a 'tail' $[l(S)+1, t]$. To be more precise, t is defined to be the player such that

$$\rho_j(S) = S \setminus \{j\} \cup [l(S)+l, t]$$

is a min-win coalition. Again, the aggregated weight of the tail coincides with the weight of j. Note that there is a unique min-win coalition that does not possess expendable players at all. This is the *lexicographically minimal*, min-win coalition. Clearly,

$$\rho_{r(S)}(\varphi(S)) = S \,, \quad \text{if } S \text{ is not lex-max}\,,$$

and

$$\varphi(\rho_j(S)) = S \,, \quad \text{if } j \text{ is expendable in } S\,.$$

(3) Let k be a player of the homogeneous game v such that all persons $1, \ldots, k$ are sums. Then there exists a sequence of min-win coalitions, $S_1, \ldots, S_{k+1} \in W_v^m$, such that the following conditions are satisfied:

(i) S_1 is the lex-max, min-win coalition.

(ii) $S_{j+1} = \rho_j(S_{i_0})$ for each $j \in [1, k]$, where i_0 is minimal such that $S_{i_0} \in S^j$ and $l(S_{i_0}) = \min\{l(S) \mid S \in S^j\}$, where $S^j = \{S_i \mid i \in [1, j], j \in S_i\}$.

This assertion follows directly from Theorem 2.3 and Definition 2.4 in Sudhölter (1989). Moreover, let $j \in [1, k+1]$, $r_0 = l(S_j)$, and $r_1 > \cdots > r_\alpha = 0$ be defined by:

$$\{r_i \mid i \in [1, \alpha - 1]\}$$

is the set of dropouts of coalition S_j. Then

$$\begin{aligned} l(\varphi^\beta(S_j)) &= \min\{l(S) \mid r_{\beta+1} \notin S \in W_v^m, l(S) \geq r_\beta\} \\ &= \min\{l(S) \mid r_\beta \in S \in W_v^m\} \end{aligned}$$

for each $\beta \in [1, \alpha - 1]$. For a proof, see the same paper.

In Section 4 it will be shown that the prekernel of a homogeneous game is star-shaped. Here, a subset A of some real vectorspace is called a *star* if it contains an element c such that for every $a \in A$ the line segment with endpoints a and c is contained in A. A vector c satisfying this property is a *center of* A. We are going to define the center solution that is a center, and frequently the unique center, of the prekernel (see Section 4).

Definition 2.2. Let v be a homogeneous n-person game with representation $(\lambda; m)$.

(1) Let $\tau(v)$ denote the last player that is equivalent to the first step of v, i.e.

$$\tau(v) = \max\{\tau \in \Omega \mid \tau \sim_v \min\{i \in \Omega \mid i \text{ is a step}\}\}\,.$$

Note that $\tau = \tau(v)$ is a step.

(2) The *truncated game of* v is the τ-person weighted majority game v^t represented by $(\lambda - m([\tau+1, n]; m_{[1,\tau]})$, i.e.

$$v^t(S) = v(S \cup [\tau+1, n]) , \quad \text{for } S \subseteq \Omega_\tau .$$

Note that v^t is the reduced game of v with respect to Ω_τ and any vector $x \in \mathbb{R}^\Omega$ satisfying $x_i = 0$ for $i > \tau$. For the definition of reduced games, see Davis and Maschler (1965).

Without proof we state a result of Rosenmüller and Sudhölter (1994, Lemma 3.7 and Corollary 3.9).

Lemma 2.3. *Let v be a homogeneous game and $\tau = \tau(v)$. Then v^t is a homogeneous game without nullplayers and without non-equivalent steps.*

A homogeneous game without nullplayers and without non-equivalent steps is called a *homogeneous standard* game. Now the *center solution* $\mathfrak{c}(v)$ can be defined, For every homogeneous game v let

$$\mathfrak{c}(v) = (m(v^\tau), \underbrace{0, \ldots, 0}_{n-\tau}) ,$$

i.e. $\mathfrak{c}(v)$ is the vector that arises from the normalized vector of weights of the minimal representation of the truncated game by adding sufficiently many zero components. In the standard homogeneous case, $\mathfrak{c}(v)$ is the normalized minimal vector of homogeneous weights

Example 2.4. For simpler reading, the parentheses in an integer representation $(\lambda; M)$ are omitted. The characters are indicated by σ for sum and τ for step. For $n = 7$:

$$\begin{array}{llllllll} 19; & 7 & 7 & 5 & 2 & 2 & 1 & 1 \\ & \tau & \tau & \sigma & \tau & \tau & \sigma & \tau \end{array}$$

is a minimal representation of a homogeneous game v with $\tau(v) = 2$, and thus v^t is the two-person game represented by 8; 7 7 or by

$$\begin{array}{lll} 2; & 1 & 1 \\ & \tau & \tau \end{array}.$$

Therefore, $\mathfrak{c}(v) = (1, 1, 0, 0, 0, 0, 0)/2$.

Let u be represented by

$$\begin{array}{llllllll} 13; & 7 & 6 & 3 & 3 & 1 & 1 & 1 \\ & \sigma & \sigma & \sigma & \sigma & \sigma & \sigma & \tau \end{array}$$

Then the center solution of u coincides with $(7,6,3,3,1,1,1)/22$. If w is the game that arises from u by dropping player 7, i.e. w is represented by

$$\begin{matrix} 13; & 7 & 6 & 3 & 3 & 1 & 1 \\ & \sigma & \sigma & \tau & \tau & \sigma & \tau \end{matrix},$$

then w^t possesses the minimal representation 4; 2 2 1 1, and thus $c(w) = (2,2,1,1,0,0)/6$.

In a homogeneous game, equivalent steps occur in every min-win coalition either simultaneously or not at all (by 'steps rule their followers', see the games v and w of the preceding example). Therefore, the truncated game either possesses a unique step, namely the last player (see $u = u^t$), or the set of steps consists of all players equivalent to the last player (see, for example, v^t, w^t of Example 2.4).

Ostmann (1987a) describes an easy method to compute the minimal representation of a homogeneous game. For homogeneous standard games, this algorithm assigns weight 1 to all players equivalent to the last player. All other players are non-nullplayers and, thus, sums. Sums can be replaced by a tail of smaller players. Therefore, the minimal vector of weights can be determined recursively. For a non-standard homogeneous game, the vector of minimal integer weights assigns zero to all nullplayers and the weight of the domain is increased by one to every step. All sums are treated analogously to the standard case.

3. Preliminary results about the prekernel

This section serves to recall some definitions, some results, and to prove some elementary properties of the prekernel, which are needed in what follows.

Definition 3.1. Let v be an n-person game. (1) The set of *preimputations of (Pareto-optimal payoffs of)* v is denoted by

$$X^*(v) = X^* = \{x \in \mathbb{R}^n \mid x(\Omega) = 1\}.$$

Note that a preimputation may have negative entries.

(2) For different players $i, j \in \Omega$ we write

$$T_{ij} = \{S \subseteq \Omega \mid j \notin S \ni i\},$$

and for every coalition $S \subseteq N$:

$$e(S, x, v) = e(S, x) = v(S) - x(S),$$

denotes the *excess of* S *at* $x \in \mathbb{R}^n$ *(with respect to* v*)*. The *maximal* excess of $x \in \mathbb{R}^n$ is

$$\mu(x) = \mu(x, v) = \max_{S \subseteq \Omega} e(S, x) ,$$

and

$$s_{ij}(x) = s_{ij}(x, v) = \max_{S \in T_{ij}} e(S, x)$$

is the *maximal surplus of i over j*.

(3) The corresponding systems of coalitions reaching maximal excess or maximal surplus are given by

$$\mathcal{D}(x) = \mathcal{D}(x, v) = \{S \subseteq \Omega \mid e(S, x) = \mu(x)\}$$

and

$$\mathcal{D}_{ij}(x) = \mathcal{D}_{ij}(x, v) = \{S \in T_{ij} \mid e(S, x) = s_{ij}(x)\} .$$

(4) The *prekernel of v* is given by

$$\mathcal{PK}(v) = \{x \in X^* \mid s_{ij}(x) = s_{ji}(x)(i, j \in \Omega, i \neq k)\} .$$

A vector $x \in \mathbb{R}^n$ is *balanced* if $s_{ij}(x) = s_{ij}(x)$ for $i, j \in \Omega$ satisfying $i \neq j$.

(5) The *kernel* is the set:

$$\mathcal{K}(v) = \{x \in X^* \mid x_j \geq v(\{j\}) \text{ and } (s_{ij}(x) \leq s_{ji}(x) \text{ or } x_j = v(\{j\})), i, j \in \Omega, i \neq j\}.$$

A vector $x \in \mathbb{R}^n$ satisfying $x_j \geq v(\{j\})$ for $j \in \Omega$ is *individually rational*.

The kernel was introduced by Davis and Maschler (1965), see also Maschler et al. (1979), Maschler and Peleg (1966, 1967), and Peleg (1966). Both the kernel and the prekernel *respect the desirability relation*, i.e. $i \lesssim_v j$ implies $x_i \geq x_j$ for every element of the (pre)kernel.

It is obvious that a game that arises by dropping some or all nullplayers inherits the directedness, the weighted majority property, and the homogeneity, respectively. Moreover, the (pre)kernel of the new game arises from the original one by dropping the corresponding zero components of each element. A solution concept with this property is said to satisfy the *strong nullplayer property*.

For a *weakly superadditive* game v (i.e. a game satisfying $v(S \cup \{i\}) \geq v(S) + v(\{i\})$ for all $i \in \Omega$ and $S \subseteq \Omega \setminus \{i\}$) prekernel and kernel coincide. This means that the kernel and prekernel of a directed simple game coincide unless there is a winning player. If exactly one winning player is present, then the kernel consists of the unique vector that distributes one to the winning player and assigns zero to any other player. In the case where there are multiple winning players, the kernel is empty because there is no individually rational Pareto-optimal payoff at all. Therefore, we restrict our attention to the shape of the prekernel and recall some 'reduction properties' (see Peleg et al., 1994, Corollary 2.7, Theorems 3.1, 3.2, 5.2, and 5.4) without proofs.

Reduction Lemma. *Let v be a directed simple n-person game.*

(i) *If v processes vetoers, then the (pre)kernel consists of the unique vector that distributes $v(\Omega)$ equally among the vetoers, i.e.*

$$\mathcal{PK}(v) = \mathcal{K}(v) = \{(\underbrace{1, \ldots, 1}_{p}, \underbrace{0, \ldots, 0}_{n-p})/p\},$$

where p is the number of vetoers.

(ii) *If v possesses $p > 0$ winning players and other non-nullplayers, then*

$$\mathcal{PK}(v) = \{(\underbrace{\sigma^x, \ldots, \alpha^x}_{p}, x_1, \ldots, x_{n-p})/(1 + p\alpha^x) \mid x \in \mathcal{PK}(w)\},$$

where $\sigma^x = \min\{x(S) \mid S \in W_w\}$ for $x \in \mathbb{R}^{n-p}$ and w is the $(n-p)$-person game that arises from v by dropping all winning players, i.e. w is the game defined by $w(S) = v(\{p + s \mid s \in S\})$, for $S \subseteq \Omega_{n-p}$.

(iii) *If v is homogeneous and $\tau = \tau(v)$ denotes the last player equivalent to the first step, then the prekernel of v arises from the prekernel of the truncated game v' of v by adding zero components for players not belonging to v', i.e.*

$$\mathcal{PK}(v) = \{(x_1, \ldots, x_\tau, \underbrace{0, \ldots, 0}_{n-\tau}) \mid x \in \mathcal{PK}(v')\}.$$

The Reduction Lemma, together with Theorem 5.2 of Peleg and Rosenmüller (1992), yields the following result:

Theorem 3.2. *If v is a homogeneous game, then the center solution $\mathfrak{c}(v)$ is a member of its prekernel.*

The main reason for the fact that $\mathfrak{c}(v) \in \mathcal{PK}(v)$ for any homogeneous game is that the maximal surplus of one player over a non-equivalent other player is attained by a minimal winning coalition and coincides with the maximal excess as long as both players belong to the carrier of $\mathfrak{c}(v)$. Indeed, all minimal winning coalitions possess the same weight according to the center solution, by construction. Peleg (1966) shows that for every element of the kernel of a partition game, the maximal surplus is attained by a min-win coalition. He then uses this fact to prove that the kernel must be star-shaped and that the center solution is a center of the kernel. The idea of the present approach is similar. The next section is devoted to the proof that the maximal surplus is attained by a min-win coalition in a more general context. In order to verify the assertion of the Technical Lemma of Section 4, the following results will be useful.

Lemma 3.3. *If v is a directed simple game without vetoers and $x \in \mathcal{PK}(v)$*, then $\mathcal{D}(x) \subseteq W_v$.

Proof. Since the prekernel satisfies the strong nullplayer property, assume v does not possess nullplayers. By Peleg et al. (1994, Lemmas 2.4 and 2.5), see also Peleg (1989), we know that $x \geq 0$ and $x_1 \geq \cdots \geq x_n$ holds true. Assume, on the contrary, that $\mathscr{D}(x)$ contains a losing coalition, say T. For each winning coalition S we have

$$e(S, x) \geq 1 - x(\Omega) = 0 ,$$

and

$$e(T, x) = -x(T) \leq 0$$

Thus,

$$e(T, x) = 0 ,$$

and $x_i = 0$ for all $i \in T$ by the assumption. Moreover, the excess of S must vanish. Thus, $x(S) = 1$ for each $S \in W_v$. Consequently, each player i with $x_i > 0$ must be a member of each winning coalition; thus, i is a vetoer that is excluded. Q.E.D.

In the case where vetoers are present, Lemma 3.3 is false (see the Reduction Lemma, part (i)). For the next lemma the absence of vetoers is also needed.

Lemma 3.4. *Let i and j be different players of the directed simple game v without vetoers and $x \in \mathscr{PK}(v)$. Then*

(a) $s_{ij}(x) \geq \mu(x) - x_i$ *and*

(b) $s_{ij}(x)$ *is attained by a min-win coalition or by a coalition of the form* $S \cup \{i\}$, *where* $S \in W_v^m$ *is a coalition with maximal excess and* $l(S) < i$; *formally written*:

$$\mathscr{D}_{ij}(x) \cap (W_v^m \cup \{S \cup \{i\} \mid S \in \mathscr{D}(x) \cap W_v^m, l(S) < i\}) \neq \emptyset .$$

Proof. (a) Take any minimal winning coalition T with maximal excess. By Lemma 3.3 such a T exists. If $j \notin T$, then

$$e(T \cup \{i\}, x) \geq \mu(x) - x_i ,$$

thus assertion (a) is true. If $j \in T$, then take any $k \in \Omega \setminus T$, $\Omega \setminus T \neq \emptyset$, since v is assumed to have no vetoers, and observe that there is a coalition $S \in \mathscr{D}(x)$ with $k \in S$ and $j \notin S$ by the balancedness of x. Lemma 3.3 guarantees that S is winning. We can assume without loss of generality that S is a min-win coalition (otherwise take any min-win subcoalition of S). Then, again.

$$e(S \cup \{i\}, x) \geq \mu(x) - x_i$$

holds true.

(b) Take any winning coalition $T \in \mathscr{D}_{ij}(x)$ and define

$$\bar{T} = T \cap [1, i-1] .$$

If $\bar{T} \in W_v$, then choose any minimal winning subcoalition S of $\bar{T}$ and observe that

$$e(S,x) - x_i = e(S \cup \{i\}, x) \geq e(\bar{T} \cup \{i\}, x) \geq e(T,x) \geq \mu(x) - x_i ,$$

Thus $S \in \mathcal{D}(x) \cap W_v^m$, $l(S) < i$, $j \not\in S$ and the proof is completed.

If $\bar{T} \not\in W_v$, then there is a min-win coalition $\bar{T} \cup \{i\} \subseteq S \subseteq T$ and $e(S,x) \geq e(T,x)$; thus, the proof is finished. Q.E.D.

4. Star-shapedness

It is the aim of this section to show that the prekernel of a homogeneous game is star-shaped. This assertion will be a consequence of the Technical Lemma, in which it is shown that the maximum surplus of i over j ($i, j \in \Omega, i \neq j$) is attained by at least one minimal winning coalition for any homogeneous standard game unless both players are equivalent steps. This is true for the center solution, which assigns equal weight to every min-win coalition. Recall that a homogeneous standard game is a homogeneous game without non-equivalent steps and without non-nullplayers. Recall that if i and j are equivalent steps, then they appear in every min-win coalition either simultaneously or not at all ('steps rule their followers'). In the case of a homogeneous standard game, all steps are equivalent to the last player. That means that i or j has to be a sum in this case.

Peleg (1966) showed the same assertion for certain pairs (i, j) in the constant-sum case. However, his approach cannot be generalized to arbitrary homogeneous games and the 'theory of incidence vectors' (see Sudhölter, 1989) is strongly used in this paper. The center solution assigns equal weight to all min-win coalitions of a homogeneous standard game, since it coincides, up to normalization, with the vector of weights of the minimal homogeneous representation.

Technical Lemma. *Let v be a homogeneous standard n-person game, $x \in \mathcal{PK}(v)$, and let i and j be different players such that at least one of them is a sum.*

(i) *Then $s_{ij}(x)$ is attained by a min-win coalition.*

(ii) *If the last component of x is positive, then $s_{ij}(x) > \mu(x) - x_i$.*

Proof. If vetoers are present, then every player is a step and is equivalent to the last player. Therefore, there is no pair of players that satisfies the desired properties in this case. Thus, both assertions are trivially valid. We assume from now on that vetoers are absent and (i, j) is a pair of players with $i \neq j$ and $\min\{i, j\}$ is a sum.

(i) Assume, on the contrary, that $s_{ij}(x)$ is not attained by a min-win coalition, i.e.

$$\mathcal{D}_{ij}(x) \cap W_v^m = \emptyset . \tag{1}$$

Therefore,

$$s_{ij}(x) = \mu(x) - x_i , \tag{2}$$

by Lemma 3.4. Let t be the index of the last non-vanishing component of x, i.e.

$$t = \max\{i \in \Omega \mid x_i > 0\} . \tag{3}$$

For each coalition $S \in W_v$ there is a unique $t(S) \in S$ such that

$$S \cap [1, t(S)] \text{ is a min-win coalition} , \tag{4}$$

i.e. this min-win coalition arises from the winning coalition by dropping 'superfluous' small players.

For each $T \in W_v^m$ let $\alpha(T) \in \mathbb{N} \cup \{0\}$ be minimal such that $\varphi^{\alpha(T)}(T)$ has no dropout $k > \max\{j, t\}$. Define

$$\tilde{S} = \varphi^{\alpha(S \cap [1, t(S)])}(S \cap [1, t(S)])$$

for each $S \in W_v$. We conclude that

$$\begin{aligned} e(S, x) = 1 - x(S) &\leq 1 - x(S \cap [1, t(S)]) \quad \text{(by (3))} \\ &= 1 - x(\tilde{S}) \quad \text{(by (3))} \\ &= e(\tilde{S}, x) , \end{aligned} \tag{5}$$

and thus,

$$\mathcal{M} = \{S \in \mathcal{D}(x) \cap W_v^m \mid \alpha(S) = 0\} \neq \emptyset \quad \text{(by (5) and Lemma 3.1)}. \tag{6}$$

Two subsets of $\mathcal{M}$ are defined as follows:

$$\mathcal{M}^- = \{S \in \mathcal{M} \mid [\min\{i, j\}, n] \cap S = \emptyset\} ,$$
$$\mathcal{M}^+ = \{S \in \mathcal{M} \mid [\min\{i, j), \max\{j, t\}] \subseteq S\} .$$

Step 1. $\mathcal{M} = \mathcal{M}^- \cup \mathcal{M}^+$ and $\mathcal{M}^- \neq \emptyset \neq \mathcal{M}^+$. Indeed, as soon as the equality is shown, it is easy to deduce the second part of the assertion. Take $S \in \mathcal{M}$. If $S \in \mathcal{M}^+$, then there is a dropout $k \not\in S$ – since S cannot be the grand coalition by the absence of vetoers. In this case, the balancedness of x applied to $(k, \min\{i, j\})$, i.e. $s_{k \min\{i,j\}} = s_{\min\{i,j\}k}$, guarantees the existence of $T \in \mathcal{D}(x)$ with $k \in T$, $\min\{i, j\} \not\in T$. By definition, $\tilde{T} \in \mathcal{M}$; thus, $\tilde{T} \in \mathcal{M}^-$ holds true. In each case $\mathcal{M}^-$ is non-empty. But $\mathcal{M}^+ \neq \emptyset$ is valid as well, which can be seen by distinguishing two cases: if $t < \min\{i, j\}$, then take $S \in \mathcal{M}$ with maximal length $l(S)$. Clearly, $l(S) \geq t$ holds true. The case $S \in \mathcal{M}^-$ cannot occur since then $T = \rho_{l(S)}(S)$ exists and $e(T, x) \geq e(S, x) = \mu(x)$; thus, $l(T) > l(S)$, $T \in \mathcal{M}$, a contradiction.

If $t \geq \min\{i, j\}$, then take $S \in \mathcal{M}^-$ and observe that there is $T \in \mathcal{D}(x)$, $\min\{i, j\} \in T, k \not\in T$ for each $k \in S$. Clearly, $\tilde{T} \in \mathcal{M}^+$.

To show that $\mathcal{M} = \mathcal{M}^+ \cup \mathcal{M}^-$ if suffices to verify that there is no coalition $S \in \mathcal{M}$ such that

$$S \cap [\min\{i, j\}, n] \neq \emptyset \neq [\min\{i, j\}, \max\{j, t\}] \backslash S \tag{7}$$

is satisfied. Assume, on the contrary, there is a coalition $S \in \mathcal{M}$ with property (7). Four cases are distinguished:

(a) $i<j, t<j$. If $i \in S$, then $j \in S$ by (1). Since $t<j$ there is $k \in [i, j]$ with $k \notin S$ by (6). Then $S \cup \{k\} \setminus \{j\}$ is winning (since $j \preceq k$) and contains a min-win coalition T containing i (see (4)).

Now

$$x(T) \leqslant x(S) + x_k \leqslant x(S) + x_i \quad \text{(by (3))}$$

holds true, a contradiction to (1).

If $i \notin S$ and $j \in S$, then the observation that $S \cup \{i\} \setminus \{j\}$ is winning yields a contradiction in the same way as before without using $t<j$.

If $i \notin S$ and $j \notin S$, then there is $k \in S \cap [i+1, n]$ (by (7)). Again, a contradiction is obtained by considering $S \cup \{i\} \setminus \{k\}$ without using $t<j$.

(b) $i<j \leqslant t$. We can assume, without loss of generality, that $i \in S$, thus $j \in S$, since all other cases can be treated in the same way as in (a). Again, there is $k \in [i+1, t] \setminus S$ by (7).

Since $s_{jk}(x) = s_{kj}(x)$, it follows that $\mathcal{D}_{kj}(x) \subseteq \mathcal{D}(x)$, thus $\mathcal{D}_{kj}(x) \cap \mathcal{M} \neq \emptyset$ (by (5), (6) and $x_k > 0$).

Take $T \in \mathcal{D}_{kj}(x) \cap \mathcal{M}$ and observe that in the case where $i \notin T$ a contradiction is obtained analogously to the last subcase of (a). The case $i \in T$ cannot occur since then

$$e(T, x) = \mu(x) > \mu(x) - x_i = s_{ij}(x) \quad \text{(by (2) and } x_i > 0) .$$

(c) $t<j<i$. Property (7) directly implies $j \notin S$, $[j+1, l(S)] \subseteq S$, $l(S) \geqslant j+1$; thus, $l(S) < i$ (by (1)). Take $T \in \mathcal{D}(x) \cap W_v^m$ with $l(T) < i$ such that $l(T)$ is maximal with these properties. Since all steps – they are interchangeable – either occur as a block or do not occur at all in a fixed min-win coalition (by 'steps rule their followers'), the last player $l(T)$ of T must be a sum. Clearly, $l(T) < n$, by definition. Therefore, $l(T)$ is expendable in T and a min-win coalition:

$$R = \rho_{l(T)}(T)$$

is obtained. It is obvious that

$$\mu(x) = e(R, x) \quad \text{(by } l(T) > t) ;$$

thus either $i \in R$ – a contradiction to (1) – or $l(R) < i$ – a contradiction to the maximality of $l(T)$.

(d) $j<i$ and $j \leqslant t$. If $j \notin S$, then there is $k>j$ with $k \in S$ (by (7)) and without loss of generality $k \leqslant t+1$, since $[t+1, l(S)] \subseteq S$ whenever $l(S) > t$ by (6). If $l(S) > i$ (implying $t \geqslant i$), then the consideration of a min-win coalition contained in $S \cup \{i\} \setminus \{l(S)\}$ again yields a contradiction. If $t \leqslant l(S) < i$, then we can proceed in the same way as in the last subcase of (c) by choosing any $T \in \mathcal{D}(x) \cap W_v^m$ with $l(T) < i$, $j \notin T$ such that $l(T)$ is maximal. If $l(S) < \min\{i, t\}$, then assume that $l(S)$

is maximal with these properties and take any $T \in \mathcal{M}$ with $l(S) \not\in T$, $l(S)+1 \in T$. There exists $k \in T$, $k > l(S)$, such that

$$R = T \cup \{l(S)\} \setminus [k, n] \in W_v^m \quad \text{(see (4))}.$$

Thus,

$$x_{l(S)} \geq x(T \cap [k, n]) \quad (\text{since } T \in \mathcal{M}). \tag{8}$$

But, by homogeneity,

$$Q := S \setminus \{l(S)\} \cup (T \cap [k, n]) \in W_v^m$$

holds true. Consequently, (8) is, indeed, an equality and $Q \in \mathcal{M}, l(Q) > l(S)$. Clearly, Q satisfies (7) and thus $l(Q) \geq \min\{i, t\}$ is impossible, as shown above. But $l(Q) < \min\{i, t\}$ contradicts the maximality of $l(S)$.

If $j \in S$, then there is $k \in [j+1, t] \setminus S$; thus, there is $\bar{S} \in \mathcal{M}$, $k \in \bar{S}$, $j \not\in \bar{S}$, which is impossible by the first part of (d).

From now on, let i and j be chosen in such a way that $i + j$ is minimal with the desired properties. Moreover, write $k = \min\{i, j\}$.

Step 2. Let $S \in \mathcal{M}^+$ with $r = r(S)$ maximal. Then

$$l(\varphi(S)) = r \quad \text{and} \quad \varphi(S) \in \mathcal{M}^-. \tag{9}$$

Recall that $r(S)$ denotes the last dropout of S, which exists because S cannot be the lex-max, min-win coalition. Since $r < k$ is valid, i.e. $r + j < i + j$, there is a min-win coalition $T \in \mathcal{D}_{rj}(x)$ by the minimality of $i + j$. By the balancedness property of x, namely $\mu(x) = s_{jr}(x) = s_{rj}(x)$, coalition T has maximal excess; thus $T \in \mathcal{M}^-$ by Step 1.

If $l(T) > r$, then there is $R \in \mathcal{D}(x)$ with $l(T) \not\in R$, $k \in R$, since $T \in \mathcal{D}(x) \cap \mathcal{D}_{l(T)k}$. Again, by the minimality of $i + j$ and $l(T) + k < i + j$, we can assume w.l.o.g. that $R \in W_v^m$ and $R \in \mathcal{M}^+$ is valid. Now, the existence of R contradicts the maximality of r. Therefore, $l(T) = r$. We conclude

$$x_r \geq x([l(\varphi(S)) + 1, l(S)]) \quad (\text{by } S \in \mathcal{M})$$

and

$$x_r \leq x([l(\varphi(S)) + 1, l(S)]) \quad (\text{by } T \in \mathcal{M} \text{ and homogeneity}),$$

thus the assertion (9).

Step 3. Now the proof can be completed. Let S be the coalition of Step 2 and again $r = r(S)$. Moreover, let $r_0 = l(S_{k+1})$ and

$$k = r_1 > \cdots > r_\alpha = 0$$

be defined via $\{r_1, \ldots, r_{\alpha-1}\}$ is the set of dropouts of S_{k+1} – for the definition of S_{k+1}, see the third part of Remark 2.1. By construction and this remark we have

$$r_{\beta+1} \notin \varphi^{\beta}(S_{k+1}) \ni r_{\beta} , \tag{10}$$

and

$$\begin{aligned} l(\varphi^{\beta}(S_{k+1})) &= \min\{l(S) \mid r_{\beta+1} \notin S \in W_v^m, l(S) > r_{\beta+1}\} \\ &= \min\{l(S) \mid r_{\beta} \in S \in W_v^m\} , \end{aligned} \tag{11}$$

for all $\beta \in [0, \alpha - 1]$.

Let f be defined by $r_{f+1} < r \leq r_f$ and let

$$T = \varphi^{f-1}(S_{k+1}) .$$

Three cases are distinguished:

(a) $f + 1 = \alpha$. Then

$$r \leq l(\varphi(T)) = l(S_1) \tag{12}$$

holds true, where S_1 is the lex-max, min-win coalition.

But $l(S_1) = \min\{l(S) \mid S \in W_v^m\} \leq l(\varphi(S)) = r$ (by (9)); thus (12) is an equality. Consequently, $l(S)$ is minimal such that $k \in S$, and thus k is expendable in S by Sudhölter (1989), a contradiction to the Step 1. Therefore, we assume $f + 1 < \alpha$ from now on.

(b) $r_{f+1} \notin S$. Then $r_f \in \varphi(S)$ and $l(\varphi(T)) \geq r_f \geq r$.

By the minimality of $l(\varphi(T))$ – see (11) – and (9) we obtain $r_f = r$. Now – by homogeneity – $l(T)$ has to coincide with $l(S)$, thus $k \in T$. By (11) k must be expendable in T, thus in S, a contradiction.

(c) $r_{f+1} \in S$. If $r_f = r$, then we obtain a contradiction analogously to (b). Therefore, $r_f > r$ is assumed. Choose $R \in \mathcal{M}$ with $r_{f+1} \notin R$, $r \in R$. The existence of R is guaranteed by the minimality of $i + j$ and the balancedness

$$\mu(x) = s_{r_{f+1}r}(x) = s_{rr_{f+1}}(x) .$$

R cannot be a member of $\mathcal{M}^-$, since otherwise – by $l(R) \geq l(T) > r$ (see (11)) – there is a coalition containing k and not $l(R)$ with maximal excess, which can be chosen to be min-win by $k + l(R) < i + j$. This contradicts the maximality of r. Therefore, $R \in \mathcal{M}^+$ holds true. Let $(\lambda; M)$ be the minimal representation of v (see Remark 2.1).

Now we have

$$M([r + 1, l(S)]) = M_r \leq M_b ; b \leq r \quad \text{(by (9) and Remark 2.1)}$$

and

$$M_{r+1} \geq M_k \quad \text{(by Remark 2.1)} .$$

Since k is not expendable in S, we conclude that

$$M([l(S) + 1, n]) < M_{r+1} ;$$

thus $l(\varphi(R)) \leqslant r$ is valid. Therefore, r_{f+1} cannot be a dropout of $\varphi(R)$ (by (11)). Since r_f is expendable in $\varphi(T)$ but not in R, then the inequality

$$r \geqslant l(\varphi(R)) > l(\varphi^2(T))$$

is satisfied. Thus,

$$M([l(\varphi(T)) + 1, l(R)]) = M([l(\varphi^2(T)) + 1, l(\varphi(R))]) \geqslant M_r ,$$

but

$$\begin{aligned} M([l(\varphi(T)) + 1, l(R)]) &\leqslant M([l(\varphi(T)) + 1, n]) \\ &\leqslant M([r + 1, n]) - M_{r+1} \\ &= M([r + 1, l(S)]) + M([l(S) + 1, n]) - M_{r+1} \\ &< M_r + M_k - M_{r+1} \\ &\leqslant M_r , \end{aligned}$$

a contradiction.

(ii) To verify this assertion, a part of the proof of (i) has to be repeated: start again with Step 1 – only parts (b) and (d) have to be taken into consideration – and observe that all constructed contradictions to (1) are also contradictions to (2), if $t = n$. Steps 2 and 3 can be left unchanged. Q.E.D.

This section is concluded by formulating and proving the explicit results concerning the star-shapedness of the prekernels of homogeneous games.

Proposition 4.1. *The prekernel of a homogeneous game v is star-shaped with center $\mathfrak{c}(v)$.*

Proof. By the Reduction Lemma, parts (i) and (iii), we can assume that v does not possess vetoers, nullplayers, and steps of different types. Let $\mathfrak{c} = \mathfrak{c}(v)$ be the center solution of v (which coincides with the normalized vector of minimal integer weights, since v is assumed to be a standard homogeneous game). Moreover, let τ be the index of the first step, i.e. $\tau = \min\{i \in \Omega \mid i \text{ is a step}\}$. Then

$$\mathfrak{c}_\tau = \mathfrak{c}_{\tau+1} = \cdots = \mathfrak{c}_n$$

is valid by construction (see Remark 2.1(1)). By Theorem 3.2, $\mathfrak{c}$ is a member of the prekernel of v. Let $x \in \mathcal{PK}(v)$.

It suffices to show the following: if $i, j \in \Omega$, $i \neq j$, and $\min\{i, j\} < \tau$, then

$$s_{ij}(\rho x + (1 - \rho)\mathfrak{c}) = \rho \cdot s_{ij}(x) + (1 - \rho) \cdot s_{ij}(\mathfrak{c}) , \quad \text{for all } \rho \in \mathbb{R} \text{ with } 0 \leqslant \rho \leqslant 1 .$$

Lemma 3.3 implies that $\mu(\mathfrak{c}) = e(S, \mathfrak{c})$ for all $S \in W_v^m$. The Technical Lemma directly shows that

$$s_{ij}(\mathfrak{c}) = \mu(\mathfrak{c})$$

and, thus,

$$\mathcal{D}_{ij}(\mathfrak{c}) \subseteq W_v^m .$$

Take any min-win coalition S attaining $s_{ij}(x)$. In view of the Technical Lemma, such a coalition S exists. Then

$$s_{ij}(\rho x + (1-\rho)\mathfrak{c}) = e(S, \rho x + (1-\rho)\mathfrak{c}) = \rho \cdot s_{ij}(x) + (1-\rho)\cdot s_{ij}(y) . \quad \text{Q.E.D.}$$

Two examples are now presented that will demonstrate the 'typical' shape of the prekernel. In what follows it turns out that the center solution is not only a center of the prekernel but also an extreme point of the convex hull of the prekernel (see Theorem 4.3). The examples are superadditive, and hence weakly superadditive. Therefore, the prekernel and the kernel coincide. A game v is *superadditive* if, for every pair (S, T) of disjoint coalitions, $v(S) + v(T) \leq v(S \cup T)$ holds true.

Example 4.2. (1) Kopelowitz (1967) gave examples of weighted majority games with disconnected kernels. Here is one six-person game v, given by the representation $(10; 5, 4, 3, 2, 2, 2)$ or $(\lambda; M) = (20; 10, 8, 6, 4, 4, 4)$. Kopelowitz computed the kernel of this game and came up with $\mathcal{K}(v) = \{x^1, x^2\}$, where

$$x^1 = (2,1,1,1,1,1)/7 , \qquad x^2 = (1,1,0,0,0,0)/2 .$$

The 19-person homogeneous standard game u, minimally represented by

$$(\tilde{\lambda}; \tilde{M}) = (32; 10, 8, 6, 4, 4, 4, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1) ,$$

is a homogeneous game that inherits many properties of v (see Section 6). The kernel of u can be computed as $\mathcal{K}(u) = \mathrm{CH}\{\tilde{x}^1, \mathfrak{c}\} \cup \mathrm{CH}\{\tilde{x}^2, \mathfrak{c}\}$ (see Fig. 1), where 'CH' means 'convex hull of', $\mathfrak{c} = \mathfrak{c}(u) = \tilde{M}/\tilde{M}(\Omega)$, and

$$\tilde{x}^1 = (2,1,1,1,1,1,\underbrace{0,\ldots,0}_{13})/7, \tilde{x}^2 = (1,1,\underbrace{0,\ldots,0}_{17})/2 .$$

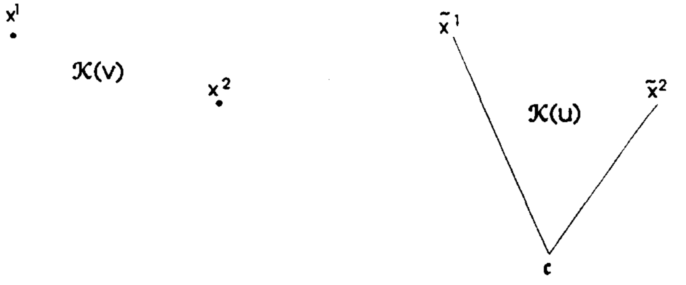

Fig. 1. Kernels.

Clearly the kernel of u possesses a unique center, namely $\mathfrak{c}$.

(2) The second example shows that the nucleolus is not necessarily a center of kernel of a homogeneous game. The *prenucleolus of* a game v is the unique preimputation of v that lexicographically minimizes the non-increasingly ordered vector of excesses. The *nucleolus* is obtained by the same procedure restricted to individually rational Pareto-optimal payoffs. Let v be the homogeneous standard 17-person game, minimally represented by $(\lambda; M)$, where

$$(\lambda; M) = (39; 12, 12, 9, 6, 6, 3, \underbrace{1, \ldots, 1}_{11}).$$

With $\mathfrak{c} = \mathfrak{c}(v)$, $x^1 = (1, 1, 1, 0, 0, 0)/3$, $x^2 = (1, 1, 1, 1, 1, 0)/5$, $x^3 = (4, 4, 3, 2, 2, 1)/16$, and $\tilde{x}^i = (x^i, \underbrace{0, \ldots, 0}_{11})$, the kernel can be described as

$$\mathcal{K}(v) = \text{CH}\{\tilde{x}^1, \tilde{x}^3, \mathfrak{c}\} \cup \text{CH}\{\tilde{x}^2, \tilde{x}^3, \mathfrak{c}\}.$$

Hence a member of the kernel is a center if and only if it belongs to the line segment $\text{CH}\{\tilde{x}^3, \mathfrak{c}\}$ (see Fig. 2). For more details concerning this example, see Section 6. Indeed, in this section it is shown that $\tilde{x}^1$, which clearly is not a center, coincides with the nucleolus of v.

An *extreme point* x of a convex subset of some Euclidean space is an element of this set that is not a convex combination of two elements of this set both differing from x. To show that the center solution is an extreme point of the convex hull of the prekernel (see Theorem 4.4), as suggested by the preceding examples, the following result will be helpful. Moreover, the assertion is also used in Section 5. The idea to prove Theorem 4.4 is very simple. If it can be shown that every element x of the prekernel, which possesses a last component not exceeded by the last component of $\mathfrak{c}(v)$, coincides with the center solution for every homogeneous

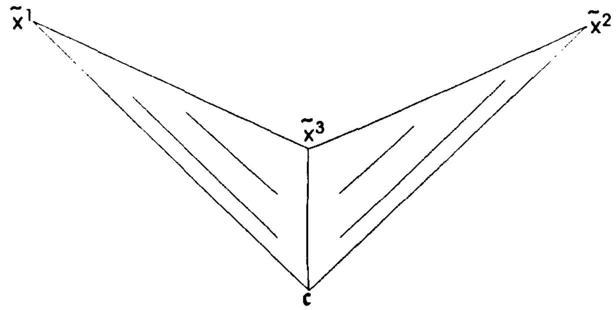

Fig. 2. A 'roof-shaped' kernel.

standard game, then $\mathfrak{c}(v)$ clearly has to be an extreme point by star-shapedness. A generalization of this fact is presented in the following lemma:

Lemma 4.3. *Let* $F: \mathbb{R}^n \to \mathbb{R}^n$ *be defined by* $F_i(x) = \sum_{j=i}^{n} x_j \cdot (j+1-i)!$ *for* $x \in \mathbb{R}^n$ *and* $i \in \Omega$.
Let $\mathfrak{c} = \mathfrak{c}(v)$ *be the center solution of the homogeneous n-person game.*
(i) *Then* $F(x) \leqslant F(\mathfrak{c})$ *componentwise for every* $x \in \mathcal{PK}(v)$.
(ii) *If* $x \in \mathcal{PK}(v)$ *and* $F_i(x) = F_i(\mathfrak{c})$ *for some* $i \in [1, \tau(v)]$, *then* $x = \mathfrak{c}$.

Proof. For simplicity we assume that v is a standard game, which can be done by the Reduction Lemma. It suffices to show that $F_t(x) \geqslant F_t(\mathfrak{c})$ for some $t \in \Omega$ implies $x = \mathfrak{c}$. Take $x \in \mathcal{PK}(v)$ and suppose that there is $t \in \Omega$ satisfying $F_t(x) \geqslant F_t(\mathfrak{c})$. Moreover, take the maximal t with the desired property.

Claim 1. $x(S) \geqslant \mathfrak{c}(S)$ *for every coalition* $S \subsetneq [t, n]$ *satisfying* $t \in S$.
Let $S \subsetneq [t, n]$ *with* $t \in S$ *and* $f^j \in \mathbb{R}^n$ *be defined by* $f^j_k = F_j(e^k)$, *where* e^k *is the kth canonical unit vector of* $\mathbb{R}^n$, *i.e.*

$$f^j_k = \begin{cases} (k+1-j)!\,, & \text{if } k \geqslant j\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

Clearly, the set $\{f^j \mid j \in \Omega\}$ *is a linear basis of* $\mathbb{R}^n$ *and, thus,* 1_S *is a linear combination of the* f^j, *let us say*:

$$1_S = \sum_{j=1}^{n} \rho_j \cdot f^j\,, \quad \text{for some } \rho_j \in \mathbb{R}\,.$$

By construction, $\rho_j = 0$ *for* $j < t$ *and* $\rho_t = 1$. *With the help of an inductive argument, it can be shown that* $\rho_j < 0$ *for* $j \geqslant t+1$. *Indeed, suppose* $\rho_j < 0$ *for every* $j \in [t+1, k]$ *is already verified for some* $k \in [t, n-1]$, *and assume, now, that* $j = k+1$. *By*

$$0 \leqslant \sum_{i=t}^{k} \rho_i f^i_k = \begin{cases} 1\,, & \text{if } k \in S \\ 0\,, & \text{otherwise} \end{cases},$$

i.e.

$$0 \leqslant (k+1-t)! + \sum_{i=t+1}^{k} \rho_i (k+1-i)! \leqslant 1\,,$$

the equality

$$\sum_{i=t}^{k} \rho_i f^i_{k+1} = (k+2-t) f^t_k + \sum_{i=t+1}^{k} \rho_i \cdot f^i_k \cdot (k+2-i) \tag{13}$$

is valid. In the case where $t = k$ (13) *becomes* $\sum_{i=t}^{k} \rho_i f^i_{k+1} = 2$, *hence* $\rho_j < 0$ *in this case. If* $k > t$, *then we come up with*

$$\sum_{i=t}^{k} \rho_i f^i_{k+1} \geq f^t_k + (k+1-t) \cdot \sum_{i=t}^{k} \rho_i f^i_k \geq f^t_k \geq 2 \, ;$$

thus, again, $\rho_j < 0$. *The observation*

$$\begin{aligned} x(S) = \sum_{j=t}^{n} \rho_j \cdot f^j \cdot x &= \sum_{j=t}^{n} \rho_j \cdot F_j(x) \\ &\geq \sum_{j=t}^{n} \rho_j \cdot F_j(\mathfrak{c}) \text{ (by } \rho_j < 0 \text{ for } j > t \text{ and } \rho_t = 1) \\ &= \mathfrak{c}(S) \end{aligned} \tag{14}$$

finishes this part of the proof. Note that (14) *contains a strict inequality in the case where* $t < n$.

Claim 2. $x_j \geq \mathfrak{c}_j$ *for* $j \in [1, t]$. *For* $j = t$ *this claim follows immediately from Claim* 1. *For* $j < t$ *we proceed recursively. In the case where* j *is a step,* j *is equivalent to* n *and, thus, to* t, *by the standardness assumption. The prekernel respects the desirability relation, hence* $x_j = x_t \geq \mathfrak{c}_t = \mathfrak{c}_j$. *Therefore, assume that* j *is a sum and* $x_i \geq \mathfrak{c}_i$ *for* $i \in [j+1, t]$. *Take any min-win coalition* S *that attains maximal surplus from* t *over* j *at* x, *i.e.* $S \in W^m_v \cap \mathcal{D}_{tj}(x)$. *Indeed, by the Technical Lemma,* S *exists. Then* $S \cup \{j\} \setminus \{t\}$ *contains a min-win coalition* T. *Clearly, player* j *is a member of* T, *thus* $e(T, x) \leq e(S, x)$ *by the balancedness of* x. *Therefore*

$$\begin{aligned} x_j &\geq x(S \cap [j+1, n] \setminus T) = x(S \cap [j+1, t-1] \setminus T) + x(S \cap [t, n] \setminus T) \\ &\geq \mathfrak{c}(S \cap [j+1, t-1] \setminus T) + \mathfrak{c}(S \cap [t, n] \setminus T) \text{ (by assumption and Claim 1)} \\ &= \mathfrak{c}(S \cap [j+1, n] \setminus T) = \mathfrak{c}_j \text{ (by the definition of } \mathfrak{c}) \end{aligned}$$

implies Claim 2.

Claims 1 and 2 show that $x([1, t-1]) \geq \mathfrak{c}([1, t-1])$ and $x([t, n]) \geq \mathfrak{c}([t, n])$, thus $x(\Omega) \geq \mathfrak{c}(\Omega)$. However, x and $\mathfrak{c}$ are preimputations, thus $x(\Omega) = \mathfrak{c}(\Omega)$. Therefore, $x_j = \mathfrak{c}_j$ for $j < t$. It is sufficient to show that $t = n$ holds true. Indeed, in the case where $t < n$ inequality (14) is strict, thus $x([t, n]) > \mathfrak{c}([t, n])$ and $x(\Omega) > \mathfrak{c}(\Omega)$. Q.E.D.

Theorem 4.4. *The center solution of a homogeneous game is an extreme point of the convex hull of the prekernel of the game.*

Proof. By the Reduction Lemma, we assume, for simplicity, that v is a homogeneous n-person standard game. Let $\mathfrak{c} = \mathfrak{c}(v)$ be the center solution. If $\mathfrak{c}$ is written as a convex combination $\mathfrak{c} = \rho x + (1-\rho) y$ for some $\rho \in \mathbb{R}$ with $0 \leq \rho \leq 1$ and $x, y \in \mathcal{PK}(v)$, then $x_n \geq \mathfrak{c}_n$ or $y_n \geq \mathfrak{c}_n$, let us say $x_n \geq \mathfrak{c}_n$. Then $F_n(x) \geq F_n(\mathfrak{c})$, and hence $x = \mathfrak{c}$. Q.E.D.

5. The center solution as a unique minimizer of a weighted Gini index

To a utility vector $x \in \mathbb{R}^n_{\geq 0}$ its *Lorenz curve* $L(x) = y \in \mathbb{R}^n_{\geq 0}$, defined by

$$y_k = \min\{x(S) \mid S \subseteq \Omega \text{ and } |S| = k\}, \quad \text{for } k \in \Omega,$$

is attached. A *collective utility function* is a continuous map $W: \mathbb{R}^n_{\geq 0} \to \mathbb{R}$ that satisfies:

(i) Anonymity: $W(x) = W(y)$, if y arises from x by a permutation of the components.

(ii) Unanimity: $W(x) \geq W(y)$, if $x \geq y$. Moreover, $W(x) > W(y)$, if $x_i > y_i$ for $i \in \Omega$ for $x, y \in \mathbb{R}^n_{\geq 0}$. A collective utility function W is said to *reduce inequality* if it respects the Lorentz order, i.e. if $x, y \in \mathbb{R}^n_{\geq 0}$ are such that $L(x) \geq L(y)$ and $L_k(x) > L_k(y)$ for some $k \in \Omega$, then $W(x) > W(y)$. A map $G: \mathbb{R}^n_{\geq 0} \to \mathbb{R}$ is called an *inequality index* if there is a collective utility function W reducing inequality such that

$$G(x) = 1 - (n \cdot \alpha(x))/x(\Omega), \quad \text{for } x \neq 0 \text{ and } G(0) = 0,$$

where $\alpha(x) = \sigma \in \mathbb{R}$ is uniquely determined by the condition $W(\sigma \cdot \underbrace{(1, \ldots, 1)}_{n}) = W(x)$ (recall that W is assumed to be continuous). In this case we say that G is the inequality index *induced by W*. Note that G is non-negative and $G(x) = 0$ iff $x_i = x_j$ for $i, j \in \Omega$. Moreover, for $x(\Omega) = y(\Omega)$ it is well known that $L(x) \geq L(y)$ and $L(x) \neq L(y)$ implies $G(y) > G(x)$.

For this notation, more details, and interpretations, see Moulin (1988). For any vector $w = (w_1, \ldots, w_n)$ with increasingly ordered positive components that add up to 1, i.e. $w_1 < \cdots < w_n$ and $w(\Omega) = 1$, the map W, defined by $W^w(x) = \sum_{i=1}^n w_i \cdot x_i^*$, where x^* arises from x by ordering the components of x non-increasingly, is a collective utility function, since $x \leq y$, $x \neq y$ implies $x^* \leq y^*$, $x^* \neq y^*$. Moreover, W^w reduces inequality. This can easily be verified by observing that $L_k(x) = x^*([n+1-k, n])$. Indeed,

$$W^w(x) = w_1 L_n(x) + \sum_{i=2}^n (w_i - w_{i-1}) L_{n+1-i}(x)$$

holds true. The inequality index G^w, induced by W^w, is called a *weighted Gini index*. This function can be redefined as

$$G^w(x) = 1 - n/x(\Omega) \cdot \sum_{i=1}^n w_i \cdot x_i^*, \quad \text{for } x \neq 0. \tag{15}$$

Note that the 'classical' Gini index occurs as a weighted Gini index with weights:

$$w_i = (2i-1)/n^2, \quad \text{for } i \in \Omega$$

(see Moulin, 1988).

The Gini index takes the surface between the 'straight line', i.e. the Lorenz curve of the 'equal treatment vector' $(x(\Omega), \ldots, x(\Omega))/n$ and the Lorenz curve of x as a measure of the inequality of x. Weighted Gini indices may put different weights to different parts of the Lorenz curves (see Fig. 3).

The prekernel on the class of all games can be characterized by intuitive properties (see Peleg, 1986). Therefore, this solution concept possesses a theoretical justification. There is a single valued solution concept in the prekernel, namely the prenucleolus, which was axiomatized by Sobolev (1975). To show that the center solution constitutes another justified proposal as to how to distribute the worth of the grand coalition in a homogeneous game, we present a weighted Gini index that has the center solution as a unique minimizer. Within the prekernel, the center solution is not only Lorenz maximal but it also uniquely maximizes a certain collective utility function that reduces inequality.

Theorem 5.1. *Let v be a homogeneous n-person game and $w \in \mathbb{R}^n$ be defined by*

$$w_i = (i!) \Big/ \left(\sum_{j=1}^{n} j! \right), \quad \text{for } i \in \Omega .$$

Then the center solution $\mathfrak{c} = \mathfrak{c}(v)$ is the unique minimizer of the weighted Gini index G^w within the prekernel of v.

Proof. Recall that $W^w(x) = \sum_{i=1}^n w_i \cdot x_i$ for $x \in \mathcal{PK}(v)$, since $x = x^*$. Indeed, x possesses non-increasingly ordered components because the prekernel respects the desirability relation. Inserting the concrete w we come up with

$$W^w(x) = h \cdot \sum_{i=1}^{n} i! \cdot x_i = h \cdot F_1(x) ,$$

where $h = 1/\sum_{i=1}^n i!$ and F is defined as in Lemma 4.3. The application of this lemma proves the assertion. Q.E.D.

The difference between the classical Gini index, G, and the weighted Gini index, G^w, can be explained with the help of an example. Let w be defined as in the preceding example, let $x = (14, 2, 2)$, and $y = (9, 9, 0)$. A straightforward computation shows that with respect to W, y is preferred to x, whereas x is preferred to y with respect to G^w. The shaded areas of Fig. 3 represent the Gini indices (left part of the figure) and the weighted Gini indices, respectively, up to normalization. The horizontal distance between consecutive players i and $i-1$ equals $w_i - w_{i-1}$. This can be seen in general by rewriting (15) for $x \neq 0$ as

$$G^w(x) = (n/x(\Omega)) \cdot \sum_{i=2}^{n} (w_i - w_{i-1}) \cdot ((x(\Omega) \cdot (n+1-i))/n - x^*([i, n])) .$$

In our special case, $w_i - w_{i-1}$ is proportional to $i \cdot i!$, whereas all differences of consecutive weights for the classical Gini index coincide. Hence, G^w puts 'larger

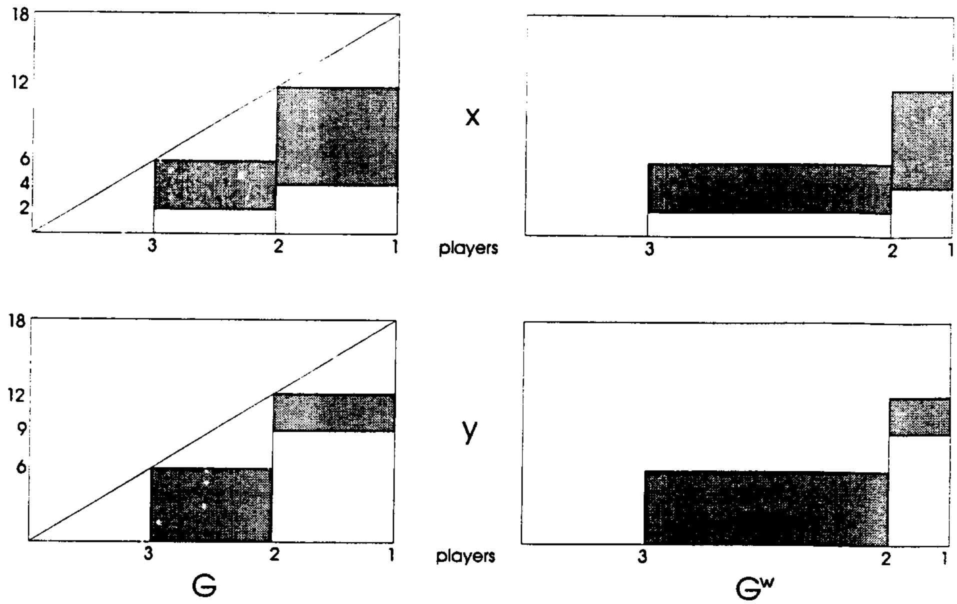

Fig. 3. Lorenz curves.

weights to poorer people'. Thus, small players are treated as well as possible within the prekernel from the center solution.

In Example 4.2(2) the center solution $c(v)$ in Lorenz-maximal within the kernel, but it is not a Lorenz-*maximum*. Indeed, $L_{14}(\tilde{x}^2) = 3/5 > 35/59 = L_{14}(c(v))$. Moreover, this example shows that the nucleolus (i.e. $\tilde{x}^1$) of v is not even Lorenz-maximal, since it is strictly Lorenz-dominated by both $\tilde{x}^2$ and the center solution. In many other examples the center solution is the unique Lorenz-maximal preimputation of the prekernel of a homogeneous game, but a classification of games with this property is not known.

Finally, it should be remarked that $c(v)$ is the unique minimizer within the prekernel of the mapping that assigns the negative weight $-x_{\tau(v)}$ of player $\tau(v)$ (the last player equivalent to the first step) to any preimputation x. Unfortunately, this mapping only *weakly* respects the Lorenz order. Moreover, it depends on the game v (i.e. on $\tau(v)$), whereas G^w is defined independently of the game. Only the number of players has to be known.

6. The general weighted majority case

It is the aim of this section to show that the prekernel of each weighted majority game is strongly related to the prekernels of certain derived homogeneous games, called homogeneous extensions. This yields a new justification for considering the special and small subclass of homogeneous games within the class of arbitrary weighted majority games. Theorem 6.6, the main result, is based on the following:

Lemma 6.1. *Let v be a homogeneous n-person game, $\mathfrak{c} = \mathfrak{c}(v)$ its center solution, and $x \in \mathcal{PK}(v)$ be a further element of the prekernel. If $\mathfrak{c} \neq x$, then there exists a unique $\tilde{x} \in \mathcal{PK}(v)$ on the straight line intersecting x and $\mathfrak{c}$ with $\tilde{x}_{\tau(v)} = 0$.*

Proof. By the Reduction Lemma, we assume that v is a homogeneous standard game, i.e. $\tau = n$. In view of Lemma 4.3, we know that $x_n < \mathfrak{c}_n$. Let $\rho > 0$ be the real number, which is defined by $(1+\rho)\cdot x_n - \rho \cdot \mathfrak{c}_n = 0$, i.e. $\rho = x_n/(\mathfrak{c}_n - x_n)$. It suffices to show that $x^\alpha = (1+\alpha)\cdot x - \alpha \cdot \mathfrak{c}$ belongs to the prekernel of v for $0 \leq \alpha \leq \rho$. Let α be maximal such that $x^\alpha \in \mathcal{PK}(v)$. Clearly, $\alpha \leq \rho$. Therefore, it remains to show that $\alpha = \rho$ is valid. Suppose, on the contrary, that $\alpha < \rho$. Then all components of x^α are positive; thus, by the Technical Lemma, part (ii), $s_{ij}(x^\alpha) > \mu(x^\alpha) > \mu(x^\alpha) - x_i^\alpha$ for every pair (i, j) of different players that are not both equivalent to n. Therefore, $\mathcal{D}_{ij}(x^\alpha) \subseteq W_v^m$ and, thus, there exist $\varepsilon > 0$ satisfying $\mathcal{D}_{ij}(x^{\alpha+\varepsilon}) = \mathcal{D}_{ij}(x^\alpha)$; hence, $x^{\alpha+\varepsilon} \in \mathcal{PK}(v)$ by the balancedness of this vector. The last observation contradicts the maximality of α. Q.E.D.

Remark 6.2. It is well known (see, fore example, Maschler and Peleg, 1966) that the prekernel of a game is a finite union of polytopes (i.e. compact convex polyhedral sets) $\bigcup_{j=1}^r P^j$. If v is a homogeneous game, then P^j can be chosen to contain the center solution as one extreme point (see Proposition 4.1 and Theorem 4.4). The preceding lemma shows that any other extreme point (if it exists) can be replaced by an extreme point with a zero component at $\tau(v)$ (see Fig. 4). To be more precise, let x be any extreme point of P^j other than $\mathfrak{c} = \mathfrak{c}(v)$. Then x can be replaced by $\tilde{x}$ as defined in Lemma 6.1, i.e. $\mathrm{CH}\, P^j \cup \{\tilde{x}\}$ is a

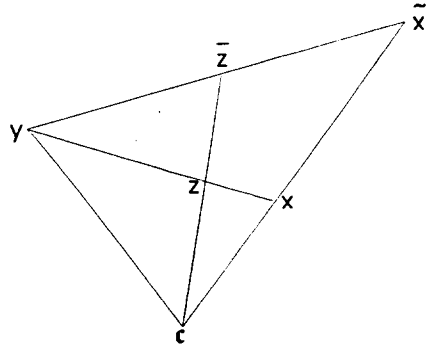

Fig. 4. A polytope.

subset of the prekernel. Indeed, for $y \in P^j$ and $\bar{z} \in \mathrm{CH}\{\tilde{x}, y\}$, there is $z \in \mathrm{CH}\{x, y\}$ such that $\bar{z}$ is on the straight line intersecting z and $\mathfrak{c}$; thus $\bar{z}$ is a member of the prekernel by Lemma 6.1. For a sketch, see Fig. 4.

For the sake of completeness, it will be shown that the prekernel of a homogeneous game is related, in an easy way, to that of a certain homogeneous game without steps. A homogeneous game is said to be a game *without steps* if only the last non-nullplayer is a step. This derived game arises from the truncated game (in which only one equivalence class of steps is present) by collecting all steps into one player. The precise formulation of this reduction result is the content of the following theorem:

Theorem 6.3. *Let v be a homogeneous n-person game and $\bar{\tau} = \tau(v)$ be the last player equivalent to the first step τ of v. Let w be the τ-person game defined by*

$$w(S) = \begin{cases} v(S \cup [\bar{\tau}+1, n]), & \text{if } \tau \notin S \\ v(S \cup [\tau+1, n]), & \text{if } \tau \in S \end{cases} \text{ for } S \subseteq [1, \tau].$$

(i) *w is a homogeneous game without steps and without nullplayers.*
(ii) $\mathcal{PK}(v) = \{x \in \mathbb{R}^n \mid x_\tau = \cdots = x_{\bar{\tau}}, x_j = 0$ *for* $j \in [\bar{\tau}+1, n], (x_1, \ldots, x_{\tau-1}, x([\tau, \bar{\tau}])) \in \mathcal{PK}(w)\}$.

Proof. Assertion (i) is valid, since all players $\tau, \ldots, \bar{\tau}$ occur in every minimal winning coalition of v either simultaneously or not at all (by 'steps rule their followers'). This means, if $(\lambda; M)$ is the minimal representation of the truncated $\bar{\tau}$-person game v', then

$$(\lambda; M_1, \ldots, M_{\tau-1}, M([\tau, \bar{\tau}])) \tag{16}$$

is the minimal representation of w. By the Reduction Lemma it can be assumed without loss of generality that v is already standard, i.e. $\bar{\tau} = n$. The following two assertions are valid:

(i) If $x \in \mathcal{PK}(v)$ and $x_n = 0$, then $(x_1, \ldots, x_\tau) \in \mathcal{PK}(w)$.
(ii) If $x \in \mathcal{PK}(w)$ and $x_\tau = 0$, then $(x, \underbrace{0, \ldots, 0}_{n-\tau}) \in \mathcal{PK}(v)$.

A proof of these assertions is straightforward and therefore skipped. By (16)

$$\mathfrak{c}(w) = (\mathfrak{c}_1, \ldots, \mathfrak{c}_{\tau-1}, \mathfrak{c}([\tau, n])),$$

where $\mathfrak{c} = \mathfrak{c}(v)$. In view of Lemma 6.1

$$\mathcal{PK}(v) = \bigcup_{\substack{x \in \mathcal{PK}(v) \\ x_n = 0}} \mathrm{CH}\{x, \mathfrak{c}\},$$

and

$$\mathcal{PK}(w) = \bigcup_{\substack{x \in \mathcal{PK}(w) \\ x_\tau = 0}} \mathrm{CH}\{x, \mathfrak{c}(w)\}.$$

Thus the proof is finished. Q.E.D.

Definition 6.4. Let u be a weighted majority n-person game. A homogeneous k-person game v is a homogeneous extension of u, if the following conditions are satisfied:

(i) $k > n, n+1 \sim_v \cdots \sim_v k$,
(ii) $v(S) \leq u(S \cap \Omega)$ (where $\Omega = [1,n]$) for $S \subseteq [1,k]$, and
(iii) $u(S) = v(S \cup [n+2,k])$ for $S \subseteq \Omega$.

It should be remarked that a weighted majority n-person game u is a reduced game of each of its k-person homogeneous extensions with respect to Ω and all preimputations x of v satisfying $x_{n+1} = \cdots = x_k = 0$. Indeed,

$$u(S) = v(S \cup [n+2,k]) \leq \max_{Q \subseteq [n+1,k]} v(S \cup Q) = v(S \cup [n+1,k]) \leq u(S)\,, \quad \text{for } S \subseteq \Omega\,.$$

The proof of the following lemma is constructive and shows that it is very easy to construct a homogeneous extension of a weighted majority game in the case where a representation of this game is given.

Lemma 6.5. *Every weighted majority game possesses a superadditive homogeneous extension.*

Proof. Let $(\tilde{\lambda}, \tilde{M})$ be an integer representation of the weighted majority n-person game u. Recall that $\tilde{\lambda} = \min\{\tilde{M}(S) \mid S \in W_u\}$ is presumed. Let r be the last non-nullplayer of u. Define $M \in \mathbb{R}^n$ by $M_i = 2 \cdot n \cdot \tilde{M}_i$ for $i \leq r$, $M_i = 2$ for $i > r$, and $\lambda = 2 \cdot n \cdot \tilde{\lambda}$. Then $(\lambda; M)$ is another integer representation of u. This representation only possesses even components. With $k = M(\Omega) - \lambda + n + 1$ define $(\bar{\lambda}; \bar{M})$ by

$$(\bar{\lambda}; \bar{M}) = (M(\Omega); M, \underbrace{1, \ldots, 1}_{k-n})\,.$$

Clearly, $(\bar{\lambda}; \bar{M})$ is a homogeneous representation of a weighted majority game v. The following observation shows that v is superadditive:

$$\bar{M}([n+1,k]) = k - n = M(\Omega) - \lambda + 1 < M(\Omega) \text{ (since } \lambda \text{ is even)}$$

The homogeneous game v is a homogeneous extension of u. This fact is a direct consequence of Definition 6.4 and the construction of v. Q.E.D.

Note that the homogeneous extension v of u described in the proof of Lemma 6.5 is a game without nullplayers, in the case where u possesses at least one non-nullplayer that has no vetoer. In the case where u does not possess any vetoers at all, v is a game without steps (since every player $i \in \Omega$ can be replaced by a tail within the lex-max, min-win coalition of v; moreover, any player i with $n < i < k$ is contained in a min-win coalition that does not contain k; hence, i can be replaced by k using this coalition). The prekernel of a directed game with vetoers is well known by the Reduction Lemma. Therefore, we restrict our attention to weighted majority games without vetoers.

The prekernel of any homogenous k-person extension of a weighted majority n-person game u without vetoers arises from that of u by taking the convex hull of the center solution and every point of the prekernel of u. Here, the prekernel of u is considered as a subset of $\mathbb{R}^k$ by considering $\mathbb{R}^n$ as the canonical n-dimensional subset of $\mathbb{R}^k$. The precise statement is given in

Theorem 6.6. *Let v be any homogeneous k-person extension of the weighted majority n-person game u without vetoers. Then*

$$\mathcal{PK}(v) = \bigcup_{x \in \mathcal{PK}(u)} \mathrm{CH}\{(x, \underbrace{0, \ldots, 0}_{k-n}), \mathfrak{c}(v)\}\ .$$

The assertion of Theorem 6.6 can be written as

$$\mathcal{PK}(u) = \{x \in \mathbb{R}^n \mid (x, \underbrace{0, \ldots, 0}_{k-n}) \in \mathcal{PK}(v)\}\ .$$

Proof. Let $(\bar{\lambda}; \bar{M})$ be the minimal integer representation of the homogeneous k-person extension v of u. By Definition 6.4 $(\lambda; M)$, where $\lambda = \bar{M}(\Omega) - k + n + 1$ and M is the restriction of $\bar{M}$ to Ω, represents u. It is straightforward (even without use of the reduced game property of the prekernel) to show that if $\tilde{x} = (x, \underbrace{0, \ldots, 0}_{k-n}) \in \mathcal{PK}(v)$, then $x \in \mathcal{PK}(u)$. It remains to verify the converse: if $x \in \mathcal{PK}(u)$ and $\tilde{x} = (x, \underbrace{0, \ldots, 0}_{k-n})$, then $\tilde{x} \in \mathcal{PK}(v)$. By Lemma 3.4, $s_{ij}(x, u)$ is attained by a winning coalition $S \subseteq \Omega$ for $i, j \in \Omega$ with $i \neq j$. By Definition 6.4:

$$e(S, x, u) = e(S \cup [n + 2, k], \tilde{x}, v)\ ;$$

thus $s_{ij}(x, u) \le s_{ij}(\tilde{x}, v)$. For any coalition $T \subseteq \Omega_k$:

$$e(T, \tilde{x}, v) \le e(T \cap \Omega, x, u)\ ,$$

thus $s_{ij}(x, u) \ge s_{ij}(\tilde{x}, v)$. Hence, $\tilde{x}$ balances i and j. For $i, j > n$ and $i \neq j$, $\tilde{x}$ balances i and j because $\tilde{x}_i = \tilde{x}_j = 0$ and $i \sim_v j$. Therefore, it suffices to show that for $i \in \Omega$, $j > n$ the maximal surplus from i over j and the maximal surplus from j over i at $\tilde{x}$ with respect to v coincides with the maximal excess $\mu(\tilde{x}, v)$. Notice that $\mu(\tilde{x}, v) = \mu(x, u)$ is trivially satisfied. Take any min-win coalition $S \in W_u^m$ satisfying $e(S, x, u) = \mu(x, u)$. If $i \left\{\begin{smallmatrix} \in \\ \notin \end{smallmatrix}\right. S$, then there is a coalition $T \in W_u^m$ satisfying $e(T, x, u) = \mu(x, u)$ and $i \left\{\begin{smallmatrix} \notin \\ \in \end{smallmatrix}\right. T$. This is true by the absence of vetoers and by the balancedness of x. In each case there are min-win coalitions of maximal excess such that one of them contains, and the other does not contain, i. Assume without loss of generality $i \in S$ and $i \notin T$. The observations

$$e(S \cup [n+1,k] \setminus \{j\}, \tilde{x}, v) = e(S,x,u) = \mu(x,u) = \mu(\tilde{x}, v)$$

and

$$e(S \cup [n+1,k], \tilde{x}, v) = e(S,x,u) = \mu(x,u) = \mu(\tilde{x}, v)\,,$$

complete the proof. Q.E.D.

An example of a weighted majority game u with nullplayers, and a homogeneous extension as constructed in the existence proof, is as follows. Let u be the three-person majority game with two additional nullplayers, i.e. u can be represented by $(\tilde{\lambda}; \tilde{M}) = (2;\ 1\ 1\ 1\ 0\ 0)$. The representations $(\lambda; M)$ of u and $(\bar{\lambda}; \bar{M})$ presented in the proof of Lemma 6.5 are given by

$$(20;\ 10\ \ 10\ \ 10\ \ 2\ \ 2) \quad \text{and} \quad (34;\ 10\ \ 10\ \ 10\ \ 2\ \ 2\ \underbrace{1\ldots1}_{15})\,,$$

respectively. The nullplayers 4 and 5 of u are no longer nullplayers of the homogeneous extension v. Note that the kernel of u consists of the unique member $(1,1,1,0,0)/3$, whereas the kernel of v is the convex hull of $(1,1,1,\underbrace{0,\ldots,0}_{17})/3$ and $\mathfrak{c}(v) = \bar{M}/49$. Note that a homogeneous extension cannot coincide with u even for a homogeneous u (as in this case) because a homogeneous extension must possess strictly more players. Moreover, it should be remarked that there are infinitely many homogeneous extensions of a given weighted majority game. This can be seen as follows. If $(\bar{\lambda}; \bar{M})$ is constructed according to the existence proof, then $(\bar{\lambda} + r; \bar{M}, \underbrace{1,\ldots,1}_{r})$ is a minimal representation of a homogeneous extension for $r \in \mathbb{N}$. In our special example there are further homogeneous extensions, e.g. $(2;\ 1\ 1\ 1\ 0\ 0\ 0)$.

To give a non-homogeneous example, look at the four-person game minimally represented by $(3;\ 2\ 2\ 1\ 1)$. Then the game represented by $(8;\ 4\ 4\ 2\ 2\ 1\ 1\ 1)$ as well as $(12;\ 4\ 4\ 2\ 2\ 1\ 1\ 1\ 1\ 1\ 1)$ is a homogeneous extension. The second one is the homogeneous extension constructed in the existence proof.

Going back to Example 4.2(1) it is obvious that u is a homogeneous extension of v (though not the one constructed in the existence proof). Therefore, the kernel of u arises from the kernel of v as sketched in Fig. 1.

To compute the kernel of v in Example 4.2(2), again, Kopelowitz's list can be used. Indeed, v is a homogeneous extension of the game u represented by

$$(15;\ 12\ \ 12\ \ 9\ \ 6\ \ 6\ \ 3)$$

and minimally represented by

(5; 4 4 3 2 2 1).

Kopelowitz computed the kernel of u as

$$\mathcal{K}(u) = \mathrm{CH}\{x^1, x^3\} \cup \mathrm{CH}\{x^2, x^3\},$$

and the nucleolus as x^1. In view of Theorem 6.6, the nucleolus of v is either $\tilde{x}^1$ or a convex combination of this vector and the center solution, since the other extreme points of the convex hull of the kernel possess larger maximal excesses. By $\mu(\tilde{x}^1, v) = 1/3$ and $\mu(c, v) = 20/59$, it is clear that the nucleolus of v coincides with $\tilde{x}^1$.

7. Remarks and examples

Peleg (1966) showed that the kernels of certain homogeneous constant-sum games (a game on Ω is a constant-sum game if $v(S) + v(\Omega \setminus S) = v(\Omega)$ for $S \subseteq \Omega$), called partition games, are star-shaped. Partition games were introduced by Isbell (1956, 1958). He observed that a simple constant-sum game (without nullplayers) has at least as many min-win coalitions as players. And, up to one famous exception, the partition games are exactly those with this minimal number. The exception is the projective seven-person game, introduced by Richardson (1956). This game has a very symmetric kernel with equal treatment of the players in the center – the center being no extreme point of the convex hull.

In this paper it will be shown that the prekernel of a partition game is not only star-shaped, by homogeneity, but a singleton. We start by recalling the definition of partition games. Let $n \geq 4$ in this section.

Definition 7.1. The game v is an n-person *partition game* if its minimal representation $(\lambda; M)$ satisfies the following conditions:

(i) $M_{n-1} = M_n = 1, M_2 = M_3, M_1 = 1 + M([4, n])$, and

(ii) $M_i \in \{M_{i+1}, 1 + M([i+2, n])\}$ for $i \in \Omega \setminus \{1, 2, n-1, n\}$.

Note that there are precisely 2^{n-4} partition games. The following list shows the minimal representations of all seven-person partition games:

(6; 5, 1, 1, 1, 1, 1, 1)	(9; 5, 4, 4, 1, 1, 1, 1)
(10; 7, 3, 3, 3, 1, 1, 1)	(11; 7, 4, 4, 3, 1, 1, 1)
(9; 7, 2, 2, 2, 2, 1, 1)	(12; 7, 5, 5, 2, 2, 1, 1)
(11; 8, 3, 3, 3, 2, 1, 1)	(13; 8, 5, 5, 3, 2, 1, 1)

Theorem 7.2. *The (pre)kernel of a partition game is a singleton.*

Proof. Let v be a partition game with n persons and $(\lambda; M)$ its minimal representation. Then $\mathfrak{c} = \mathfrak{c}(v) = M/M(\Omega)$ is the center solution of v. It suffices to show that $x = \mathfrak{c}$ for every $x \in \mathcal{K}(v)$. To do this, let $x \in \mathcal{K}(v)$ and break Ω into successive disjoint intervals $T_1, \ldots, T_r$ of cardinalities $t_1, \ldots, t_r$ such that the intervals represent the equivalence classes of players, i.e.

$$T_1 = \{1\}, M_i = M_j\,, \quad \text{for } i, j \in T_k \text{ and } k \in \Omega_r\,,$$

and

$$M_i > M_j\,, \quad \text{for } i \in T_p \quad \text{and} \quad j \in T_k\,, \quad \text{for } p < k\,.$$

For any $S \subseteq \Omega$ define $\tilde{S} \in \mathbb{R}^r$ by $\tilde{S}_i = |S \cap T_i|$. Isbell showed that $S \in W_v^m$, iff

$$\tilde{S} \in \mathcal{M} = \{\tilde{S}^j \mid j = 1, \ldots, r\}\,,$$

where

$$\tilde{S}_i^j = \begin{cases} t_i\,, & \text{if } j - i \equiv 0 \bmod 2 \text{ and } i \leq j\,, \\ 0\,, & \text{if } (j - i \equiv 1 \bmod 2 \text{ and } i < j) \text{ or } i > j + 1\,, \\ 1\,, & \text{otherwise}\,. \end{cases}$$

Define $\mathcal{D} = \{\tilde{S} \mid S \in \mathcal{D}(x) \cap W_v^m\}$.

Claim. $\mathcal{D} = \mathcal{M}$.

As soon as this last equality is shown, the proof is finished, since $\mathfrak{c}$ is the unique vector x for which both $x(S) = \text{constant}$ for $S \in W_v^m$ and $x(\Omega) = 1$. Indeed, a homogeneous constant-sum game does not possess different steps.

Note that all players of T_i are interchangeable and thus obtain equal weights according to x. By the star-shapedness of the kernel, $x_n > 0$ can be assumed without loss of generality (otherwise replace x by a non-trivial convex combination of x and $\mathfrak{c}$).

Since $x_n > 0$, there is $\tilde{S} \in \mathcal{D}$ with $\tilde{S}_r > 0$, but $\tilde{S}^{r-1}$ and $\tilde{S}^r$ are the only elements of $\{\tilde{S} \in \mathcal{M} \mid \tilde{S}_r > 0\}$. If $\tilde{S}^{r-1} \in \mathcal{D}$, then $s_{ij}(x) = \mu(x)$ for $i \in T_{r-1}$, $j \in T_r$ since $\tilde{S}_{r-1}^{r-1} > 0$ and $\tilde{S}_r^{r-1} = 1 < t_r$ hold true. By the balancedness of x we have $s_{ji}(x) = \mu(x)$, implying the existence of $\tilde{S} \in \mathcal{D}$ with $\tilde{S}_r > 0$, $\tilde{S}_{r-1} < t_{r-1}$ and thus $\tilde{S} = \tilde{S}^r$. Therefore, in each case, $\tilde{S}^r \in \mathcal{D}$ is valid.

Now, $s_{ij}(x) = \mu(x)$ shows that there is $\tilde{S} \in \mathcal{D}$ with $\tilde{S}_{r-1} > 0$, $\tilde{S}_r < T_r$; thus $\tilde{S} \in \{\tilde{S}^{r-1}, \tilde{S}^{r-2}\}$. Assume that $\tilde{S} = \tilde{S}^{r-2}$ (i.e. $r \geq 3$); thus, $\tilde{S}_{r-2} > 0$ and $\tilde{S}_r = 0$. As a consequence, we obtain

$$s_{ij}(x) = \mu(x)\,, \quad \text{for } i \in T_{r-2}, j \in T_r\,,$$

and, by the balancedness, $s_{ji}(x) = \mu(x)$. We conclude that there is $\tilde{S} \in \mathcal{D}$ with $\tilde{S}_r > 0$, $\tilde{S}_{r-2} < t_{r-2}$; thus $\tilde{S} = \tilde{S}^{r-1}$. Up to now we have proved $\tilde{S}^{r-1}$, $\tilde{S}^r \in \mathcal{D}$.

Assume $\tilde{S}^r, \tilde{S}^{r-1}, \ldots, \tilde{S}^{\alpha+1} \in \mathcal{D}$ from some $\alpha < r - 2$. If $\alpha = 0$, then the proof is complete. Therefore, assume $\alpha \geq 1$. Again, since $\tilde{S}_{\alpha+2}^{\alpha+1} > 0$ and $\tilde{S}_\alpha^{\alpha+1} = 0$ there is

$\tilde{S} \in \mathcal{D}$ with $\tilde{S}_\alpha > 0$ and $\tilde{S}_{\alpha+2} < t_{\alpha+2}$. Observe that $\tilde{S} = \tilde{S}^\alpha$, if $\alpha = 1$. Therefore, assume that $\alpha > 1$. Then, clearly, $\tilde{S} \in \{\tilde{S}^\alpha, \tilde{S}^{\alpha+1}\}$ is valid. Assume that $\tilde{S} = \tilde{S}^{\alpha-1}$; thus $\tilde{S}_{\alpha-1} > 0$ and $\tilde{S}_{\alpha+1} = 0$ and, by balancedness, there is $T \in \mathcal{D}$ with $\tilde{T}_{\alpha+1} > 0$ and $\tilde{T}_{\alpha-1} < t_{\alpha-1}$. Consequently, $\tilde{T} = \tilde{S}^\alpha$ holds true. Q.E.D.

Finally, some examples are presented showing the following assertions.

(i) The kernel of a homogeneous constant-sum game need not be a singleton or even convex.

(ii) An element of the kernel of a homogeneous game, even in the constant-sum case, need not satisfy the condition that the maximal surplus of player i over j coincides with the maximal excess, even for non-interchangeable players i and j.

(iii) The least core of a homogeneous game need not be contained in the kernel of the game. The *least core of* a game is the convex set of preimputations that minimize the maximal excess of non-trivial coalitions.

(iv) An element of the kernel of a weighted majority game need not satisfy the property that the maximal surplus of player i over player j is attained by a min-win coalition for non-interchangeable players i and j.

Note that a possible example showing (iii) has to be a non-constant-sum game, because otherwise the least core is a singleton consisting of the nucleolus as Peleg (1968) showed. Nevertheless, this assertion may be surprising because prekernel, nucleolus and least core behave in the same way with respect to homogeneous games with steps of different type, which was shown by Rosenmüller and Sudhölter (1994) and Peleg et al. (1994).

Example 7.3.

(a) Let v be the homogeneous 11-person, constant-sum game minimally represented by

$$(\lambda; M) = (16; 10, 6, 4, 2, 2, 2, 1, 1, 1, 1, 1)\,.$$

Define

$$x^1 = (2, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0)/7, x^2 = (1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0)/2\,,$$
$$x^3 = (6, 3, 3, 2, 2, 2, 0, 0, 0, 0, 0)/18 \quad \text{and} \quad \mathfrak{c} = \mathfrak{c}(v) = M/31\,.$$

It is easy to verify that

$$\mathcal{K}(v) = \text{CH}\{x^1, x^3, \mathfrak{c}\} \cup \text{CH}\{x^2, x^3, \mathfrak{c}\}\,.$$

There are only two 'types' to min-win coalitions in T_{32}, the first consists of players 1, 3, and one additional player in [4, 6] and the second consists of playes 1, 3 and two additional players in [7, 11]. The excess at x^3 is 7/18 and 1/2, respectively. The maximal excess at x^3 is attained by, e.g. $\{1, 4\} \cup [7, 10]$, and equals 10/18. This example thus shows assertions (i) and (ii).

(b) Let $(\lambda; M) = (8; 4, 3, 2, 2, 1)$ represent the five-person weighted majority game v. Then $x = M/12 \in \mathcal{K}(v)$. Indeed, $M/12$ is the nucleolus of the game. Moreover, $s_{52}(x)$ cannot be attained by a min-win coalition because there is no min-win coalition containing player 5 but not player 2. As a consequence, we have assertion (iv) and, additionally, $s_{52}(x) < \mu(x)$.

(c) Let v be the homogeneous seven-person game, represented by

$$(\lambda; M) = (14; 6, 5, 3, 3, 2, 1, 1) .$$

Define $x^1 = (2, 2, 1, 1, 0, 0, 0)/6$ and $x^2 = (2, 1, 1, 1, 1, 0, 0)/6$. Then the least core is the convex hull of x^1, x^2 and $\mathfrak{c}(v) = M/21$. Moreover, the nucleolus of v can be computed as $(7/11) \cdot \mathfrak{c}(v) + (2/11) \cdot (x^1 + x^2)$. It can easily be verified that $x^1, x^2 \not\in \mathcal{K}^*(v)$. With the help of a computer it was checked that this is the only seven-person example, showing assertion (iii).

Final remarks.

(1) Kopelowitz (1967) presented an algorithm that computes the (pre)kernels of weighted majority games. This method is strongly based on Maschler and Peleg's (1966) theoretical results on the kernel for general games. Though Lemmas 3.3 and 3.4 might be used to slightly simplify this method in the directed simple case, it is not intended to suggest that star-sharpedness of the prekernels for the homogeneous extension is a property that will help in the computational context. Moreover, it is not known whether star-shapedness has an impact on the possible shapes of the prekernels for general weighted majority games.

(2) The main results on the prekernel can also be formulated for the kernel. Clearly, Proposition 4.1, Theorems 4.4, 5.1, 6.3, and 6.6 remain valid for the kernel if the absence of winning players is additionally assumed. Recall that prekernel and kernel coincide for weakly superadditive games. If exactly one winning player occurs, then the center solution is no member of the kernel unless this winning player is a vetoer and, thus, all other members of the grand coalition are nullplayers. Nevertheless, Proposition 4.1 can be weakened to read: the kernel of a homogeneous game is star-shaped or empty.

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