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The Modified Nucleolus: Properties and Axiomatizations[1]

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Abstract: A new solution concept for cooperative transferable utility games is introduced, which is strongly related to the nucleolus and therefore called modified nucleolus. It takes into account both the "power", i.e. the worth, and the "blocking power" of a coalition, i.e. the amount which the coalition cannot be prevented from by the complement coalition. It can be shown that the modified nucleolus is reasonable, individually rational for weakly superadditive games, coincides with the prenucleolus for constant-sum games, and is contained in the core for convex games. Finally this paper proposes two axiomatizations of this solution concept on the set of games on an infinite universe of players which are similar to Sobolev's characterization of the prenucleolus.

Key Words: Cooperative game, solution concept.

0 Introduction

A new solution concept, the modified nucleolus, for cooperative side payment games with a finite set of players is proposed in this paper.

The expression "modified nucleolus" refers to the strong relationship of this solution to the (pre)nucleolus introduced by Schmeidler (1966).

An imputation belongs to the nucleolus of a game, if it successively minimizes the maximal excesses, i.e. the differences of the worths of coalitions and the aggregated weight of these coalitions with respect to (w.r.t.) the imputation, and the number of coalitions attaining them. For the precise definition Section 1 is referred to. By regarding the excesses as a measure of dissatisfaction the nucleolus obtains an intuitive meaning as pointed out by Maschler, Peleg, and Shapley (1979).

The solution introduced in the present paper constitutes an attempt to treat all coalitions equally as far as this is possible. Therefore it is natural to regard the differences of excesses as a measure of dissatisfaction leading to the following intuitive definition. A preimputation belongs to the modified nucleolus $\Psi(v)$ of

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a game v, if it successively minimizes the maximal differences of excesses and the number of coalition pairs attaining them. The modified nucleolus takes into account both the "power", i.e. the worth, and the "blocking power" of a coalition, i.e. the amount which the coalition cannot be prevented from by the complement coalition. Alike the prenucleolus, which only depends on the worths of the coalitions, the modified nucleolus is a singleton.

To give an example look at the glove game with three players, one of them (player **1**) possessing a unique right hand glove whereas the other players (**2** and **3**) possess one single left hand glove each. The worth of a coalition is the number of pairs of gloves of the coalition (i.e. one or zero). For the explicit definition the end of Section 4 is referred to. If a coalition has positive worth, then **1** is a member of the coalition, i.e. player **1** is a veto player possessing, in some sense, all of the power. Indeed the (pre)nucleolus assigns one to player **1** and zero to the other players. On the other hand both players **2** and **3** together can prevent player **1** from any positive amount by forming a "syndicate". Therefore they together have the same blocking power as player **1** has. The modified nucleolus takes care of this fact and assigns 1/2 to the first and 1/4 to each of the other players.

A further motivation to consider the new solution concept is its behaviour on the remarkable class of weighted majority games. For the subclasses of weighted majority constant-sum games on the one hand and for homogeneous games on the other hand the nucleolus (see Peleg (1968)) and the minimal integer representation (see Ostmann (1987) and Rosenmüller (1987)) respectively can be regarded as canonical representation. Fortunately, the modified nucleolus coincides with the prenucleolus on constant-sum games and, up to normalization, with the weights of the minimal integer representation on homogeneous games. Additionally, it induces a representation for an arbitrary weighted majority game. Therefore the modified nucleolus can be regarded as a canonical representation in the general weighted majority case. For the details which are not specified in this paper Sudhölter (1993b) is referred to.

Section 1 presents the precise definition of the modified solution conept. The dual game v^* of a game v assigns to each coalition the real number which can be given to it if the worth of the grand coalition is shared and the complement coalition obtains its worth. By looking at complements it turns out that the modified nucleoli of v and v^* coincide, this also being a characteristic of the Shapley value.

A certain replication of a game is defined, which allows to reformulate many assertions concerning the prenucleolus for the modified nucleolus. The dual cover of a game arises from a game v with player set N by taking the union of two disjoint copies of N to be the new player set and assigning to a coalition S the maximum of the sums of the worths of the intersections of S with the first copy w.r.t. v and the second copy w.r.t. v^* or, conversely, the first copy w.r.t. v^* and the second w.r.t. v. Hence both, the game and its dual, are totally symmetric ingredients of the dual cover.

A main result of this section, Proposition 1.4, states a strong relationship between the prenucleolus of the dual cover and the modified nucleolus of the initial game. One solution concept arises from the other by the canonical replication or restriction respectively. Therefore, e.g., the modified nucleolus can be computed by each of the well-known algorithms for the calculation of the prenucleolus (see, e.g., Kopelowitz (1967) or Sankaran (1992)) applied to the dual cover.

Section 2 starts applying Kohlberg's characterization of the (pre)nucleolus by balanced collections of coalitions (see Kohlberg (1971)) to the modified nucleolus with the help of Proposition 1.4. It turns out that Ψ can be characterized similarly by balanced collections of coalition pairs (see Theorem 2.2).

The coincidence of the pre- and modified nucleolus whenever possible w.r.t. duality, i.e. whenever the prenucleoli of the game and its dual cannot be distinguished, is the content of Theorem 2.3 and a consequence of Theorem 2.2.

Additionally, it is shown that Ψ satisfies the dummy property (a dummy is any player whose component of the characteristic function behaves additively), weakly respects desirability between players in the sense of Maschler and Peleg (1966), and is reasonable in the sense of Milnor (1952).

Section 3 is devoted to convex games. It turns out that the modified nucleolus is a member of the core in this case.

The prenucleolus is the unique solution concept on the set of games on an infinite universe of players satisfying single valuedness, covariance under strategic equivalence, consistency, and anonymity (see Sobolev (1975)). In Section 4 it is deduced that the modified nucleolus possesses similar axiomatizations, in which consistency is weakened and "strong duality" properties are added. Indeed, all axioms within each characterization including the infinity assumption on the universe of players are shown to be logically independent.

1 Notation and Definitions

A **cooperative game with transferable utility** – a **game** – is a pair $G=(N,v)$, where N is a finite nonvoid set and

$$v:2^N \to \mathbb{R}, \quad v(\emptyset)=0$$

is a mapping. Here $2^N=\{S \subseteq N\}$ is the set of **coalitions** of G.

If $G=(N,v)$ is a game, then N is the **grand** coalition or the set of **players** and v is called **characteristic** (or **coalitional**) **function** of G. Since the nature of G is determined by the characteristic function, v is called game as well.

If $G=(N,v)$ is a game, then the **dual** game (N,v^*) of G is defined by

$$v^*(S)=v(N)-v(N\backslash S)$$

for all coalitions S. The set of **feasible payoff vectors** of G is denoted

$$X^*(N,v) := X^*(v) := \{x \in \mathbb{R}^N \mid x(N) \leq v(N)\},$$

whereas

$$X(N,v) := X(v) := \{x \in \mathbb{R}^N \mid x(N) = v(N)\}$$

is the set of **preimputations** of G (also called set of **Pareto optimal** feasible payoffs of G). Here

$$x(S) := \sum_{i \in S} x_i \, (x(\emptyset) = 0)$$

for each $x \in \mathbb{R}^N$ and $S \subseteq N$. Additionally, let x_S denote the restriction of x to S, i.e.

$$x_S = (x_i)_{i \in S} \in \mathbb{R}^S,$$

whereas

$$A_S := \{x_S \mid x \in A\}$$

for $A \subseteq \mathbb{R}^N$. For disjoint coalitions $S, T \subseteq N$ and $x \in \mathbb{R}^N$ let $(x_S, x_T) := x_{S \cup T}$.

A **solution concept** σ **on** a set Γ of games is a mapping

$$\sigma : \Gamma \to \bigcup_{v \in \Gamma} 2^{X^*(v)}, \sigma(v) \subseteq X^*(v).$$

If $\bar{\Gamma}$ is a subset of Γ, then the canonical restriction of a solution concept σ on Γ is a solution concept on $\bar{\Gamma}$. We say that σ is a solution concept on $\bar{\Gamma}$, too. If Γ is not specified, then σ is a solution concept on every set of games.

Some convenient and well-known properties of a solution concept σ on a set Γ of games are as follows.

σ is **anonymous** (satisfies **AN**), if for each $(N,v) \in \Gamma$ and each bijective mapping $\tau : N \to N'$ with $(N', \tau v) \in \Gamma$

$$\sigma(N', \tau v) = \tau(\sigma(N,v))$$

holds (where $(\tau v)(T) = v(\tau^{-1}(T))$, $\tau_j(x) = x_{\tau^{-1}j}$ $(x \in \mathbb{R}^N, j \in N', T \subseteq N')$).

In this case v and τv are **equivalent** games.

σ is **covariant under strategic equivalence** (satisfies **COV**), if for $(N,v), (N,w) \in \Gamma$ with $w = \alpha v + \beta$ for some $\alpha > 0$, $\beta \in \mathbb{R}^N$

$$\sigma(N,w) = \alpha\sigma(N,v) + \beta$$

holds. The games v and w are called **strategically equivalent**.

σ is **single valued** (satisfies (**SIVA**), if $|\sigma(v)| = 1$ for $v \in \Gamma$.
σ satisfies **nonemptiness** (**NE**), if $\sigma(v) \neq \emptyset$ for $v \in \Gamma$.
σ is **Pareto optimal** (satisfies **PO**), if $\sigma(v) \subseteq X(v)$ for $v \in \Gamma$.

Note that both equivalence and strategical equivalence commute with duality, i.e.

$$(\tau v)^* = \tau(v^*), (\alpha v + \beta)^* = \alpha v^* + \beta,$$

where τ, α, β are chosen according to the definitions given above.

It should be remarked (see Shapley (1953)) that the Shapley value φ – to be more precise the solution concept σ given by $\sigma(v) = \{\varphi(v)\}$ – satisfies all above properties.

Some more notation will be needed. Let (N, v) be a game and $x \in \mathbb{R}^N$. The **excess of** a coalition $S \subseteq N$ **at** x is real number

$$e(S, x, v) := e(S, x) := v(S) - x(S).$$

Let $\mu(x, v) := \mu(x)$ be the **maximal excess** at x, i.e.

$$\mu(x, v) := \max\{e(S, x) \mid S \subseteq N\}.$$

The nucleolus of a game was introduced by Schmeidler (1966). Some corresponding definitions and results are recalled: Let $\vartheta : \bigcup_{n \in \mathbb{N}} \mathbb{R}^n \to \bigcup_{n \in \mathbb{N}} \mathbb{R}^n$ be defined by

$$\vartheta(x) = y \in \mathbb{R}^n \quad (x \in \mathbb{R}^n),$$

where y is the vector which arises from x by arranging the components of x in a non increasing order. The **nucleolus of** v **w.r.t.** X, where $X \subseteq \mathbb{R}^N$, is the set

$$\mathcal{N}(X, v) := \left\{ x \in X \mid \vartheta((e(S, x, v))_{S \subseteq N}) \underset{\text{lex}}{\leq} \vartheta((e(S, y, v))_{S \subseteq N}) \text{ for all } y \in X \right\}.$$

Schmeidler (1966) formulated and proved the following three assertions.

If X is a nonvoid compact set, then $\mathcal{N}(X, v)$ is nonvoid. (1)

If $X$is convex, then $\mathcal{N}(X, v)$ contains at most one vector. (2)

If X is a nonvoid, closed convex subset of $X^*(v)$, then $\mathcal{N}(X, v)$ is a singleton. (3)

The **prenucleolus of** (N, v) is defined to be the nucleolus w.r.t the set of feasible payoff vectors and denoted $\mathcal{PN}(v)$, i.e., $\mathcal{PN}(v) = \mathcal{N}(X^*(v), v)$.

By (3) the prenucleolus of a game is a singleton, and, clearly, the prenucleolus is Pareto optimal. The unique element $v(v)$ of $\mathcal{PN}(v)$ is again called **prenucleolus (point)**.

For completeness reasons we recall that the nucleolus of (N,v) is the set $\mathcal{N}(X,v)$, where $X=\{x \in X(v) \mid x_i \geq v(\{i\})\}$ is the set of **imputations** of v. Maschler, Peleg and Shapley (1979) tried to give an intuitive meaning to the definition of the (pre)nucleolus by regarding the excess of a coalition as a measure of dissatisfaction which should be minimized. If the excess of a coalition can be decreased without increasing larger excesses, this process will also increase some kind of "stability", they argued. Nevertheless, Maschler (1992) asked: "What is more 'stable', a situation in which a few coalitions of highest excess have it as low as possible, or one where such coalitions have a slightly higher excess, but the excesses of many other coalitions is substantially lowered?" Anyone, like the present author, who is not convinced by the first or latter, may try to search for a completely different solution concept. The concept which will be introduced in this paper constitutes an attempt to treat all coalitions equally w.r.t. excesses as far as this is possible. Therefore, instead of minimizing the highest excess, then minimizing the number of coalitions with highest excess, minimizing the second highest excess and so on – the highest difference of excesses is minimized, then the number of pairs of coalitions with highest difference of excesses is minimized... Here is the notation.

Definition 1.1: Let (N,v) be a game. For each $x \in \mathbb{R}^N$ define

$$\tilde{\Theta}(x,v) := \vartheta((e(S,x,v)-e(T,x,v))_{(S,T)\in 2^N \times 2^N}) \in \mathbb{R}^{2^{2|N|}}.$$

The **modified nucleolus** of v is the set

$$\Psi(v) := \left\{ x \in X(v) \mid \tilde{\Theta}(x,v) \underset{\text{lex}}{\leq} \tilde{\Theta}(y,v) \text{ for all } y \in X(v) \right\}.$$

Remark 1.2: Let (N,v) be a game.

(i) If x is any preimputation of the game v, then the following equality holds by definition:

$$\begin{aligned} e(T,x,v^*) = v^*(T) - x(T) &= v(N) - v(N \setminus T) - x(N) + x(N \setminus T) \\ &= x(N \setminus T) - v(N \setminus T) \text{ (by Pareto optimality of } x) \\ &= -e((N \setminus T),x,v). \end{aligned}$$

(ii) With

$$\bar{\Theta}(y,v) := \vartheta((e(S,y,v)+e(T,y,v^*))_{(S,T)\in 2^N \times 2^N})$$

for $y \in \mathbb{R}^N$ Remark 1.2 (i) directly implies for $x \in X(v)$ that

$$\bar{\Theta}(x,v) = \tilde{\Theta}(x,v)$$

holds true. Note that x has to be Pareto optimal for this equation. Nevertheless the modified nucleolus can be redefined as

$$\Psi(v) = \left\{ x \in X^*(v) \,|\, \bar{\Theta}(x,v) \underset{\text{lex}}{\leq} \bar{\Theta}(y,v) \text{ for all } y \in X^*(v) \right\}, \tag{4}$$

since Pareto optimality is, now, automatically satisfied. Indeed, this property can be verified by observing that for every nonvoid coalition both, the excess w.r.t. v and w.r.t. v^*, strictly decrease if all components of a feasible payoff vector can be strictly increased.

(iii) The alternate definition of $\Psi(v)$ in the last assertion (see (4)) directly shows that Ψ satisfies **duality**, i.e. $\Psi(v) = \Psi(v^*)$ holds. Note that the Shapley value also satisfies duality.

(iv) It is straightforward to verify that Ψ satisfies both, anonymity and covariance.

With the help of the next definition and proposition we obtain a relationship between the modified nucleolus of v and the prenucleolus of a game called dual cover of the game. Additionally, it turns out that the modified nucleolus is a singleton.

Definition 1.3: Let (N,v) be a game and $\bar{N} = N \times \{0,1\}$. We identify $N \times \{0\}$ with N and $N \times \{1\}$ with N^* in the canonical way, thus $\bar{N} = N \dot{\cup} N^*$. The game $(N \dot{\cup} N^*, \tilde{v})$, defined by

$$\tilde{v}(S \dot{\cup} T^*) = \max\{v(S) + v^*(T), v(T) + v^*(S)\}$$

for all S, $T \subseteq N$ is the **dual cover** of v.

Proposition 1.4: The modified nucleolus of a game (N,v) is the restriction of the prenucleolus of $(N \cup N^*, \tilde{v})$ to N; i.e. $\psi(v) = v(\tilde{v})_N$. Moreover, $v_i(\tilde{v}) = v_{i^*}(\tilde{v})$ for $i \in N$.

Proof: By the well-known anonymity of the prenucleolus the second assertion is true. Let X be the set of **symmetric** feasible payoff vectors of $\tilde{v}$, i.e.

$$X = \{x \in X^*(\tilde{v}) \,|\, x_i = x_{i^*} \text{ for } i \in N\}.$$

Then – by symmetry of the prenucleolus – we come up with

$$\mathcal{PN}(\tilde{v}) = \left\{ x \in X \,|\, \vartheta((e(S,x,\tilde{v}))_{S \in N \cup N^*}) \underset{\text{lex}}{\leq} \vartheta((e(S,y,\tilde{v}))_{S \in N \cup N^*}) \text{ for } y \in X \right\}.$$

For each $A = \{S \cup T^*, T \cup S^*\} \in D = \{\{S \cup T^*, T \cup S^*\} \,|\, S, T \in N\}$ let $S(A)$ be defined by

$$S(A) = \begin{cases} S \cup T^*, & \text{if } v(S) + v^*(T) \geq v(T) + v^*(S) \\ T \cup S^*, & \text{otherwise} \end{cases},$$

and $\tilde{D} = \{S(A) \mid A \in D\}$. Observe that

$$\mathscr{P}\mathscr{N}(\bar{v}) = \left\{ x \in X \mid \vartheta((e(S,x,\bar{v}))_{S \in \tilde{D}}) \underset{\text{lex}}{\leq} \vartheta((e(S,y,\bar{v}))_{S \in \tilde{D}}) \text{ for } y \in X \right\}$$

holds true. On the other hand – for $x, y \in \mathbb{R}^N$ –

$$\bar{\Theta}(x,v) \underset{\text{lex}}{\leq} \bar{\Theta}(y,v) \text{ iff}$$

$$\vartheta((e(S,x,v)) + e(T,x,v^*))_{S \cup T^* \in \tilde{D}}) \underset{\text{lex}}{\leq} \vartheta((e(S,y,v) + e(T,y,v^*))_{S \cup T^* \in \tilde{D}}),$$

hence the proposition is proved. q.e.d.

In view of Proposition 1.4 the modified nucleolus of a game v is a singleton denoted by $\psi(v)$, i.e.

$$\{\psi(v)\} = \Psi(v).$$

The unique point $\psi(v)$ of $\Psi(v)$ is again called **modified nucleolus (point)**.

With the help of the next definition we obtain a second relationship between the modified nucleolus of v and some nucleolus of a game called dual replication of v.

Definition 1.5: Let (N, v) be a game. The game $(N \dot{\cup} N^*, \bar{v})$, defined by

$$\bar{v}(S \dot{\cup} T^*) = v(S) + v^*(T)$$

for all S, $T \subseteq N$ is the **dual replication of** v.

Corollary 1.6: Let $X = \{x \in X(\bar{v}) \mid x_i = x_{i^*} \text{ for all } i \in N\}$. Then

(i) $\mathscr{N}(X, \bar{v}) = \{y \in \mathbb{R}^{N \cup N^*} \mid y_N \in \Psi(v) \text{ and } y_{i^*} = y_i \text{ for } i \in N\}$,
(ii) $\psi(\bar{v})$ is the replication of $\psi(v)$, i.e. $\psi_i(\bar{v}) = \psi_{i^*}(\bar{v})$ and $\psi(\bar{v})_N = \psi(v)$.

A variant of Property (ii) of this corollary will be used as one characterizing axiom in Section 4.

Proof: A proof of (i) is straightforward and therefore skipped. Assertion (ii) remains to be shown. By definition

$$\bar{v}^*(S \cup T^*) = v(T) + v^*(S) = \bar{v}(T \cup S^*) \text{ for all } S,\ T \subseteq N,$$

thus – by Remark 1.2 (iii), (iv) – $\psi_i(\bar{v}) = \psi_{i^*}(\bar{v})$ for all players $i \in N$. Therefore

$$\Psi(\bar{v})_N = \left\{ x \in X^*(v) \mid \alpha(x) \underset{\text{lex}}{\leq} \alpha(y) \text{ for } y \in X^* \right\}, \text{ where}$$

$$\alpha(x) := \vartheta((e(S,x,v) + e(T,x,v^*) + e(\tilde{S},x,v) + e(\tilde{T},x,v^*))_{(S,T,\tilde{S},\tilde{T}) \in 2^N \times 2^N \times 2^N \times 2^N}).$$

Clearly, $\alpha(x) \underset{\text{lex}}{\leq} \alpha(y)$ iff $\bar{\Theta}(x,v) \underset{\text{lex}}{\leq} \bar{\Theta}(y,x)$. q.e.d.

The Shapley value also satisfies (ii) of the preceding corollary. Moreover, it is verified for completeness reasons that the dual cover of a game uniquely determines the game up to duality.

Lemma 1.7: Let (N, v) be a game. Then

(i) $\varphi_i(\bar{v}) = \varphi_{i^*}(\bar{v})$ and $\varphi(\bar{v})_N = \varphi(v)$,
(ii) If $\tilde{v} = \tilde{w}$ for some game (N, w) then $w \in \{v, v^*\}$.

Proof:

ad (i): The first part of this assertion is guaranteed by duality of the Shapley value. To prove the second part the definition of φ is recalled:

$$\varphi_i(v) = \sum_{i \in S \subseteq N} \frac{(s-1)!(n-s)!}{n!}(v(S) - v(S \setminus \{i\})),$$

where $t = |T|$ denotes the cardinality of T. To each coalition S with $i \in S$ all coalitions of the form $S \dot{\cup} T^*$ $(T \subseteq N)$ can be assigned, yielding the same "marginal value", i.e. $v(S) - v(S \setminus \{i\}) = \bar{v}(S \cup T^*) - \bar{v}(S \setminus \{i\} \cup T^*)$.

Therefore it is sufficient to show that

$$\sum_{t=0}^{n} \binom{n}{t} \frac{(s+t-1)!(2n-s-t)!}{(2n)!} = \frac{(s-1)!(n-s)!}{n!} \tag{5}$$

holds true (since $\binom{n}{t}$ is the number of coalitions $T \subseteq N$ with cardinality t). Equation (5) is equivalent to

$$\sum_{t=0}^{n} \binom{s+t-1}{s-1} \binom{2n-s-t}{n-s} = \binom{2n}{n}, \tag{6}$$

by multiplying (5) with $\frac{(2n)!}{n!(s-1)!(n-s)!}$.

Formula (6) is a special case of

$$\sum_{t=0}^{n}\binom{s+t-1}{s-1}\binom{n+m-s-t}{n-s}=\binom{n+m}{n}, \tag{7}$$

for $m \in \mathbb{N} \cup \{0\}, n \in \mathbb{N}, s \in \mathbb{N}, s \leq n$. Using two inductive arguments the proof of formula (7) is straightforward and therefore skipped.

ad (ii): Let $\tilde{w} = \tilde{v}$, $w \neq v$.

Claim: $\{w(S), w^*(S)\} = \{v(S), v^*(S)\}$ for $S \subseteq N$. (8)

By definition of $\tilde{}$ we have

$$\max\{w(S) + w^*(\emptyset), w^*(S) + w(\emptyset)\} = \max\{v(S) + v^*(\emptyset), v^*(S) + v(\emptyset)\}, \tag{9}$$

and, e.g.,

$$w(S) + w^*(S) = v(S) + v^*(S). \tag{10}$$

Using (10) equality (9) directly implies (8).

Now the proof can be completed. Take any $\bar{S} \subseteq N$ with $w(\bar{S}) \neq v(\bar{S})$, i.e.

$w(\bar{S}) = v^*(\bar{S}) \neq v(\bar{S})$ by (8).

Take any T with $v(T) \neq v^*(T)$. By definition of $\sim$ and (8) we conclude

$$\begin{aligned} \max\{w(T) + w^*(\bar{S}), w(\bar{S}) + w^*(T)\} &= \max\{w(T) + v(\bar{S}), v^*(\bar{S}) + w^*(T)\} \\ = \max\{v(T) + v^*(\bar{S}), v(\bar{S}) + v^*(T)\} &\neq \max\{v(T) + v(\bar{S}), v^*(\bar{S}) + v^*(T)\}, \end{aligned}$$

thus $w(T) = v^*(T)$ again by (7). q.e.d.

2 Properties of the Modified Solution

At first Kohlberg's (1971) characterization of the (pre)nucleolus by balanced collections of coalitions is recalled and applied to the modified nucleolus. It should be remarked that his assumption of zero-normalization – i.e. $v(\{i\}) = 0$ for all players i – can be deleted without destroying the proofs. Moreover, the original results were stated for the nucleolus, but it is easy to formulate analogous properties for the prenucleolus (see Peleg (1988/89)). Some notation is needed.

A finite nonvoid set $X \subseteq \mathbb{R}^N$ is **balanced**, if X possesses a vector of **balancing coefficients** $(\delta_x)_{x \in X}$, i.e.

$$\sum_{x \in X} \delta_x x = 1_N \quad \text{and} \quad \delta_x > 0 \text{ for } x \in X.$$

Here 1_S is the indicator function of S, considered as vector of $\mathbb{R}^N$. A nonvoid subset D of coalitions or $\tilde{D}$ of pairs of coalitions is **balanced** if

$$\{1_S | S \in D\} \quad \text{or} \quad \{1_S + 1_T | (S, T) \in \tilde{D}\} \text{ respectively}$$

is balanced. We say that S and (S, T) respectively is in the **span** of D and $\tilde{D}$ respectively if 1_S and $1_S + 1_T$ is in the span $\{1_S | S \in D\}$ and $\{1_S + 1_T | (S, T) \in \tilde{D}\}$ resp. For $x \in \mathbb{R}^N$, $\alpha \in \mathbb{R}$ define

$$D(x, \alpha, v) = \{S \subseteq N | e(S, x, v) \geq \alpha\},$$
$$\tilde{D}(x, \alpha, v) = \{(S, T) \in 2^N \times 2^N | e(S, x, v) + e(T, x, v^*) \geq \alpha\}.$$

Theorem 2.1 (Kohlberg): Let (N, v) be a game, $\alpha \in \mathbb{R}$, and $x \in X(v)$. Then $x = \nu(v)$ iff each nonvoid $D(x, \alpha, v)$ is balanced.

For a proof of this theorem Kohlberg (1971) is referred to. The analogon for the modified solution is

Theorem 2.2: Let (N, v) be a game, $\alpha \in \mathbb{R}$, and $x \in X(v)$. Then $x = \psi(v)$ iff each nonvoid $\tilde{D}(x, \alpha, v)$ is balanced.

Proof: Let $z := (x, x^*) \in \mathbb{R}^{N \cup N^*}$, thus z is a preimputation of the dual cover $\tilde{v}$ of v (see Definition 1.3). By the same definition we come up with the following two assertions:

$$\text{If } (S, T) \in \tilde{D}(x, \alpha, v) =: \tilde{D}, \text{ then } S \cup T^* \in D(z, \alpha, \tilde{v}) := D \ni T \cup S^*,$$
$$\text{If } S \cup T^* \in D, \text{ then } (S, T) \in \tilde{D} \text{ or } (T, S) \in \tilde{D}.$$

Particularly, $D \neq \emptyset$ iff $\tilde{D} \neq \emptyset$. Assume that $\tilde{D}$ is nonvoid.

Note that x coincides with $\psi(v)$ iff z coincides with $\nu(\tilde{v})$ by Proposition 1.4. Assume, now, $x = \psi(v)$ and take balancing coefficients $\delta_{S \cup T^*} > 0$ of D, i.e.

$$\sum_{S \cup T^* \in D} \delta_{S \cup T^*} 1_{S \cup T^*} = 1_{N \cup N^*}.$$

For $(S, T) \in \tilde{D}$ define a real number

$$\delta_{(S,T)} = \begin{cases} \frac{1}{2}\delta_{S \cup T^*}, & \text{if } (T, S) \in \tilde{D} \\ \frac{1}{2}(\delta_{S \cup T^*} + \delta_{T \cup S^*}), & \text{otherwise} \end{cases}.$$

Then

$$\sum_{(S,T)\in\tilde{D}} \delta_{(S,T)}(1_S+1_T) = \sum_{\substack{(S,T)\in\tilde{D} \\ \text{and } (T,S)\notin\tilde{D}}} \tfrac{1}{2}(\delta_{S\cup T^*}+\delta_{T\cup S^*})(1_S+1_T)$$
$$+ \sum_{\substack{(S,T)\in\tilde{D} \\ \text{and } (T,S)\in\tilde{D}}} \tfrac{1}{2}\delta_{S\cup T^*}(1_S+1_T) = \sum_{S\cup T^*\in D} \tfrac{1}{2}\delta_{S\cup T^*}(1_S+1_T) = 1_N$$

holds true, thus $\tilde{D}$ is balanced.

Conversely, if $\tilde{D}$ is balanced with balancing coefficient $\delta_{(S,T)} > 0$ for $(S,T)\in\tilde{D}$, then $(\delta_{S\cup T^*})_{S\cup T^*\in D}$, where

$$\delta_{S\cup T^*} = \begin{cases} \delta_{(S,T)} + \delta_{(T,S)}, & \text{if } (S,T)\in\tilde{D}\ni(T,S) \\ \delta_{(S,T)}, & \text{if } (T,S)\notin\tilde{D}\ni(S,T), \\ \delta_{(T,S)}, & \text{if } (S,T)\notin\tilde{D}\ni(T,S) \end{cases}$$

are balancing coefficients for D, q.e.d.

By Remark 1.2 (iii) the modified nucleolus satisfies duality. Nevertheless, ψ coincides with the prenucleolus whenever this is compatible with the duality property. To be more precise, all of $\psi(v)$, $v(v)$, and $v(v^*)$ coincide if the last two vectors coincide. Formally, this assertion is the content of the next

Theorem 2.3: Let (N,v) be a game. If $v(v)=v(v^*)$ holds, then $\psi(v)=v(v)$ is also true.

Proof: In view of Theorem 2.2 it is sufficient to show that $\tilde{D} := \tilde{D}(x,\alpha,v)$ is balanced, whenever $\tilde{D}$ is nonvoid, where $x = v(v)$.

Define

$$A := \{S\subseteq N \mid \text{there is } T\subseteq N \text{ with } (S,T)\in\tilde{D}\}$$

and

$$B := \{T\subseteq N \mid \text{there is } S\subseteq N \text{ with } (S,T)\in\tilde{D}\}$$

and take $S\in A$. Then $(S,T)\in\tilde{D}$, iff $e(T,x,v^*) \geq \alpha - e(S,x,v) =: \alpha(S)$. Since $v(v^*)$ coincides with x by assumption, Theorem 2.1 can be applied; hence the set $D^\alpha_S := D(x,\alpha(S),v^*)$ is balanced, let us say, with balancing coefficients $\delta^*_{(T,S)} > 0$, $T\in D^\alpha_S$. Let β be the maximal excess of coalitions in B w.r.t. v^*, i.e. $\beta = \max\{e(T,x,v^*) \mid T\in B\}$. Then, by definition, $A = D(x,\alpha-\beta,v)$ holds true, thus A is balanced – since $v(v)=x$ – with balancing coefficients $(\delta_S)_{S\in A}$. With

$$c_S := \left(\sum_{T\in D^\alpha_S} \delta^*_{(T,S)}\right)^{-1} \quad \text{and} \quad c := \left(1 + \sum_{S\in A} c_S\delta_S\right)^{-1}$$

the following equation shows the balancedness of $\tilde{D}$:

$$\sum_{(S,T)\in\tilde{D}} c\cdot c_S\delta_S\delta^*_{(T,S)}(1_S+1_T)=c\cdot\sum_{S\in A} c_S\delta_S\sum_{T\in D^\alpha_S}\delta^*_{(T,S)}(1_S+1_T)$$

$$=c\cdot\sum_{S\in A}c_S\delta_S\left(1_N+\sum_{T\in D^\alpha_S}\delta^*_{(T,S)}1_S\right)=c\cdot\left(1_N+\sum_{S\in A}c_S\delta_S 1_N\right)=1_N. \qquad \text{q.e.d.}$$

The prerequisite $v(v)=v(v^*)$ is trivially satisfied for each constant-sum game. Recall that (N,v) is a **constant-sum** game, if $v(S)+v(N\backslash S)=v(N)$ for $S\subseteq N$. Therefore v is a constant-sum game, iff v coincides with the dual game v^*.

Corollary 2.4: For each constant-sum game the pre- and the modified nucleolus coincide.

This corollary can also be proved without using Theorem 2.3 by first observing that the dual cover $\tilde{v}$ of v is a constant-sum game and the prenucleolus of this game arises from the one of the game started with by replication, i.e. $v(\tilde{v})=(v(v),v(v)^*)$.

Up to the end of this section some properties of Ψ are formulated which directly arise from well-known properties of the prenucleolus applied to the dual cover of the game. Indeed, it is shown that the modified solution is reasonable in the sense of Milnor (1952) and satisfies the dummy property. Moreover, the modified nucleolus weakly respects desirability. Some well-known definitions are recalled.

Let (N,v) be a game. Player $i\in N$ is a **dummy** of v, if $v(S\cup\{i\})=v(S)+v(\{i\})$ for all $S\subseteq N\backslash\{i\}$. This player is a **null-player**, if additionally $v(\{i\})=0$ holds. Player $i\in N$ is **at least as desirable as** player $j\in N$, written $i\succsim_v j$, if $v(S\cup\{i\})\geq v(S\cup\{j\})$ for all $S\subseteq N\backslash\{i,j\}$. The arising strict relation is abbreviated $\succ_v$: i is more **desirable than** j, if $i\succsim_v j$ and not $j\succsim_v i$. Player i and j are **interchangeable** $-i\sim_v j-$ if $i\succsim_v j\succsim_v i$. This desirability relation between players was introduced by Maschler and Peleg (1966) and can be generalized to coalitions (see, e.g., Einy (1985)). In the first paper it was shown that the prenucleolus as an element of the prekernel (for the definition of the prekernel Davis and Maschler (1965) and Maschler, Peleg, and Shapley (1972) are referred to) respects desirability – i.e., $v_i(v)\geq v_j(v)$, if $i\succsim_v j$ – and satisfies the dummy property – i.e., $v_i(v)=v(\{i\})$ for each dummy i of v. With the help of the following lemma the same statement can be proved for the modified nucleolus.

Lemma 2.5: Let (N,v) be a game, $\tilde{v}$ be the dual cover of v, and $i,j\in N$.

(i) $i\succsim_v j$, iff $i\succsim_{\tilde{v}} j$.
(ii) i is a dummy of v, iff i is a dummy of $\tilde{v}$.
(iii) $\min\{v(S\cup\{i\})-v(S)\,|\,S\subseteq N\backslash\{i\}\}=\min\{\tilde{v}(\tilde{S}\cup\{i\})-\tilde{v}(\tilde{S})\,|\,\tilde{S}\subseteq(N\backslash\{i\})\cup N^*\}$.
(iv) $\max\{v(S\cup\{i\})-v(S)\,|\,S\subseteq N\backslash\{i\}\}=\max\{\tilde{v}(\tilde{S}\cup\{i\})-\tilde{v}(\tilde{S})\,|\,\tilde{S}\subseteq(N\backslash\{i\})\cup N^*\}$.

Proof: Observe that v^* has the same "desirability structure", the same dummies and so on. To be more precise the exact formulations are as follows – a proof is a straightforward consequence of the definition of v^*:

$i \succsim_v j$ iff $i \succsim_{v^*} j$;
i is a dummy of v iff i is one of v^* and $v(\{i\}) = v^*(\{i\})$;
$a := \min\{v(S \cup \{i\}) - v(S) \mid S \subseteq N \setminus \{i\}\} = \min\{v^*(S \cup \{i\}) - v^*(S) \mid S \subseteq N \setminus \{i\}\}$,
$b := \max\{v(S \cup \{i\}) - v(S) \mid S \subseteq N \setminus \{i\}\} = \max\{v^*(S \cup \{i\}) - v^*(S) \mid S \subseteq N \setminus \{i\}\}$,

ad (i): Let $i \succsim_{\tilde{v}} j$ for some $i, j \in N$ and $S \in N \setminus \{i, j\}$. Defining $\bar{S} = S \cup \{i\}$, $\bar{T} = S \cup \{j\}$ we come up with

$$v(\bar{T}) + v^*(\bar{S}) \leq \tilde{v}(\bar{T} \cup \bar{S}^*) \leq \tilde{v}(\bar{S} \cup \bar{S}^*) = v(\bar{S}) + v^*(\bar{S})$$

by definition and assumption, thus $i \succsim_v j$.
Conversely, if $i \succsim_v j$ and $S \cup T^* \subseteq (N \setminus \{i, j\}) \cup N^*$, let w.l.o.g.

$$\tilde{v}(S \cup \{j\} \cup T^*) = v(S \cup \{j\}) + v^*(T),$$

otherwise exchange the roles of v and v^*. Hence

$$\tilde{v}(S \cup \{j\} \cup T^*) \leq v(S \cup \{i\}) + v^*(T) \leq \tilde{v}(S \cup \{i\} \cup T^*).$$

ad (ii): A proof of this assertion is straightforward using the above observation and therefore we skip it.
ad (iii): Take $S \subseteq N \setminus \{i\}$ and $T \subseteq N$. W.l.o.g. $\tilde{v}(S \cup T^*) = v(S) + v^*(T)$ – otherwise exchange v and v^*. Then

$$\tilde{v}(S \cup \{i\} \cup T^*) - \tilde{v}(S \cup T^*) \geq v(S \cup \{i\}) - v(S) \geq a.$$

Conversely, take $S \subseteq N \setminus \{i\}$ such that $a = v(S \cup \{i\}) - v(S)$. Then

$$\tilde{v}(S \cup \{i\} \cup (S \cup \{i\})^*) - \tilde{v}(S \cup (S \cup \{i\})^*) \leq v(S \cup \{i\}) + v^*(S \cup \{i\}) - (v(S) + v^*(S \cup \{i\})) = a.$$

Finally, assertion (iv) can be proved analogously to (iii). q.e.d.

This last lemma and the well-known properties of the prenucleolus together with Proposition 1.4 directly imply

Corollary 2.6: Let (N, v) be a game and $x \in \Psi(v)$.

(i) $x_i(v) \geq x_j(v)$, if $i \succsim_v j$;
(ii) $x_i = v(\{i\})$ for each dummy i of v;

(iii) $x_i \geq \min\{v(S \cup \{i\}) - v(S) | S \subseteq N \backslash \{i\}\}$ for $i \in N$;
(iv) $x_i \leq \max\{v(S \cup \{i\}) - v(S) | S \subseteq N \backslash \{i\}\}$ for $i \in N$.

Assertion (iv) is called **reasonableness** (in the sense of Milnor (1952)). A solution concept satisfying (iii) and (iv) for each of its elements is called **reasonable on both sides**.

Remark 2.7:

(i) For technical reasons the following assertion is needed. A proof which is straightforward is skipped.

Let N be a finite nonvoid set, D and $\tilde{D}$ be balanced collections of coalitions and pairs of coalitions respectively. Then every subset E and $\tilde{E}$ with $D \subseteq E \subseteq 2^N$, $\tilde{D} \subseteq \tilde{E} \subseteq 2^N \times 2^N$ such that E and $\tilde{E}$ are in the span of D and $\tilde{D}$ respectively are balanced.

(ii) Recall that a game (N, v) is **weakly superadditive**, if

$$v(S) \cup \{i\}) \geq v(S) + v(\{i\}),$$

for $i \in N$ and $S \subseteq N \backslash \{i\}$. Reasonableness on both sides directly implies that $\psi(\cdot)$ is **individually rational** for every weakly superadditive game v, i.e. $\psi_i(v) \geq v(\{i\})$ for $i \in N$.

3 The Modified Nucleolus in the Convex Case

A game (N, v) is **convex**, if

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$$

for all coalitions $S, T \subseteq N$.

Convex games were introduced by Shapley and have interesting economic applications (see Shapley (1971)).

For the class of convex games all well-known solution concepts are nonvoid subsets of the core and many coincide or are singletons (see, e.g., Maschler, Peleg, Shapley (1972)). Here, the core of a game v is the set

$$\mathscr{C}(v) = \{x \in X(v) | e(S, x, v) \leq 0 \text{ for all } S \subseteq N\}.$$

It is the aim of this section to show that the modified nucleolus also belongs to the core for convex games. Moreover, illustrating examples are presented at the end of this section.

Theorem 3.1: Let (N, v) be a convex game. Then $\psi(v) \in \mathscr{C}(v)$.

Proof: Let $\psi := \psi(v)$, $\mu := \mu(\psi, v)$, $\mu^* := \mu(\psi, v^*)$, $\mathscr{D}_0 := D(\psi, \mu, v)$, and $\mathscr{D}_1 := D(\psi, \mu^*, v^*)$ – for the definition of $D(\cdot, \cdot, \cdot)$ Section 2 is referred to.

It remains to show $\mu \leq 0$.

Assume, on the contrary, $\mu > 0$. Note that $\mu^* \geq \mu$ holds by $v^*(S) \geq v(S)$ for all $S \subseteq N$, i.e., by convexity.

Recall that $\mathscr{D}_0$ is a near-ring in the sense of Maschler, Peleg, Shapley (1971): Indeed, for S, $T \subseteq N$, we have

$$e(S, \psi, v) + e(T, \psi, v) \leq e(S \cap T, \psi, v) + e(S \cup T, \psi, v) \tag{1}$$

by convexity. Hence, using the positivity of μ – both, $S \cup T$ and $S \cap T$ are members of $\mathscr{D}_0$ if S and T are ($\mathscr{D}_0$ is a **near-ring**). This closedness w.r.t. intersection and union shows

$$S^0, S^1 \in \mathscr{D}_0, \text{ where } S^0 = \bigcap_{T \in \mathscr{D}_0} T, S^1 = \bigcup_{T \in \mathscr{D}_0} T.$$

Note that $\emptyset \neq S^0 \subseteq S^1 \neq N$ holds, since ψ is a preimputation.

We claim that each $T \in \mathscr{D}_1$ intersects S^0: Inequality (1) applied to an arbitrary coalition $S \subseteq N \backslash S^0$ and to $T = S^0$, together with $\mu > 0$ directly implies $N \backslash (S \cup S^0) \notin \mathscr{D}_1$. By definition

$$\mathscr{D}_0 \times \mathscr{D}_1 = \tilde{D}(\psi, \mu + \mu^*, v) =: \tilde{D}.$$

Take balancing coefficients $\alpha_{(S,T)}$ for $\tilde{D}$ and player $i \in N \backslash S^1$. Then equality

$$\sum_{(S,T) \in \tilde{D}} \alpha_{(S,T)}(1_S + 1_T) = 1_N,$$

applied to i, shows

$$\sum_{(S,T) \in \tilde{D}} \alpha_{(S,T)} \geq 1, \tag{2}$$

since there is no $S \in \mathscr{D}_0$ containing i. Take $\bar{T} \in \mathscr{D}_1$ and $j \in \bar{T} \cap S^0$. Then the same equality applied to j, i.e.

$$\sum_{(S,T) \in \tilde{D}} \alpha_{(S,T)}(1_S + 1_T)(j) = 1,$$

directly implies – using $S^0 \subseteq S$ for $S \in \mathscr{D}_0$ –

$$\sum_{(S,T) \in \tilde{D}} \alpha_{(S,T)} + \sum_{S \in \mathscr{D}_0} \alpha_{(S,\bar{T})} \leq 1,$$

which contradicts (2). q.e.d.

Let $\mathscr{LC}(v)$ denote the **least core of** v, i.e.,

$$\mathscr{LC}(v) = \{x \in X^*(v) | e(S,x,v) \leq \max\{e(T,\nu(v),v) | \emptyset \neq T \neq N\}, \emptyset \neq S \neq N\},$$

if N is no singleton (see Maschler, Peleg, Shapley (1979)). If N is a singleton, then Pareto optimality is explicitely presumed and can easily be deduced otherwise. At the end of this section (see Example 3.2(i)) a convex game v is presented which shows that the prenucleolus of v^* does not necessarily coincide with the modified nucleolus. Moreover, w.r.t. this convex game the modified nucleolus is no member of the least core of v.

In this example the modified nucleolus is a member of the least core of the dual game v^* of v. Since $\psi(v)$ minimizes the largest sum of excesses w.r.t. v and v^*, the modified nucleolus is contained in $\mathscr{LC}(v^*)$ in the convex case, as long as this least core intersects the core of the original game v. The question whether this intersection is nonvoid for any convex game is answered negatively by Example 3.2(iii). Additionally, it is shown that nonemptiness of the core of an arbitrary game w does not necessarily imply $\psi(w) \in \mathscr{C}(w)$, even if the Shapley value is a member of the core (see Example 3.2(ii)).

Example 3.2:

(i) Let $N = \{1,2,3,4\}$ and v be defined by

$$v(S) = \begin{cases} 0, & \text{if } S \in \{\emptyset, N\} \\ -1, & \text{if } S = \{2,4\} \\ -2, & \text{if } S \in \{\{2,3,4\},\ \{1,2,4\}\} \\ -6, & \text{if } S = \{3\} \\ -7, & \text{if } S \in \{\{4\},\ \{2\}\} \\ -10, & \text{if } S \in \{\{3,4,\},\ \{1,2\}\} \\ -8, & \text{otherwise} \end{cases}.$$

The proof of convexity of v is straightforward and skipped.

With $x = (-1, 1, -1, 1)$ we claim

$$\nu(v) = 2x, \nu(v^*) = \frac{x}{2}, \quad \psi(v) = \psi(v^*) = \frac{2x}{3}.$$

A sketch of the proof of the last equation and of $\nu(v^*) \neq \frac{2x}{3} =: z$ is presented.

The first equality can be verified analogously. For an illustration of the game v, of v^*, and of arising excesses Fig. 1 is referred to. The columns of the matrix are the lexicographically ordered coalitions. The additional rows of this figure represent v, v^* and the excess at z w.r.t. v and v^* and the excess at $2x$ w.r.t. v.

	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
$v(\cdot)$	0	−7	−6	−10	−7	−1	−8	−2	−8	−8	−8	−8	−10	−2	−8	0
$v^*(\cdot)$	0	8	2	10	8	8	8	8	2	8	1	7	10	6	7	0
$3\cdot e(\cdot,z,v)$	0	−23	−16	−30	−23	−7	−24	−8	−22	−24	−20	−22	−30	−8	−22	0
$3\cdot e(\cdot,z,v^*)$	0	22	8	30	22	20	24	22	8	24	7	23	30	16	23	0
$e(\cdot,2x,v)$	0	−9	−4	−10	−9	−5	−8	−4	−6	−8	−4	−6	−10	−4	−6	0

Fig. 1

With $\alpha \in \mathbb{R}$ define the matrix

$$I_\alpha := (1_S + 1_T)_{(S,T)\in \tilde{D}(\alpha)},$$

where $\tilde{D}(\alpha) := \tilde{D}(z,\alpha,v)$. Clearly (see Fig. 1), $\tilde{D}(\alpha) = \varnothing$ for $\alpha > 10$.

Moreover, it can directly be seen that

$$I_\alpha = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 1 & 1 & 2 & 2 \end{bmatrix}, \quad \text{if } 8 < \alpha \le 10;$$

$$I_\alpha = \begin{bmatrix} \multicolumn{4}{c}{I_{10}} \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 2 & 1 & 1 & 2 \\ 1 & 2 & 2 & 1 \end{bmatrix}, \quad \text{if } \frac{23}{3} < \alpha \le 8;$$

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix} \quad \text{is a submatrix of } I_{\frac{23}{3}}.$$

Hence $\tilde{D}_\alpha$ is balanced for $\frac{23}{3} < \alpha \le 10$ – there are trivial balancing coefficients. Clearly, (1, 1, 2, 2, 1, 1)/6 is a vector of balancing coefficients for the pairs of coalitions corresponding to the matrix A. The rows of A span the Euclidean space, thus $\tilde{D}(\alpha)$ is balanced for $\alpha \le \frac{23}{3}$ by Remark 2.7(i). Hence $\psi(v) = z$.

In order to show that $z \neq \nu(v^*)$ it is sufficient to verify that $\mathscr{D} = D\left(z, \frac{23}{3}, v^*\right)$ is nonvoid and not balanced. Indeed, the first four rows of the corresponding matrix

$$I_{\mathscr{D}} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

are linearly independent and, thus, span $\mathbb{R}^4$. Moreover,

$$\begin{aligned}
(1\ 0\ 0\ 1) &= (1\ 1\ 0\ 0) - (1\ 1\ 1\ 0) + (1\ 0\ 1\ 1),\\
(0\ 1\ 1\ 0) &= (0\ 0\ 1\ 1) + (1\ 1\ 1\ 0) - (1\ 0\ 1\ 1),\\
(1\ 1\ 1\ 1) &= (1\ 1\ 0\ 0) + (0\ 0\ 1\ 1),
\end{aligned}$$

thus A and $\mathscr{D}$ cannot be balanced. Hence $z \neq \nu(v^*)$. Besides, a further element of the core – the center – is the Shapley value $\varphi(v)$ being no member of the straight line to x, i.e. $\varphi_1(v) \neq -\varphi_2(v)$. Indeed, $\varphi(v) = (-23, 19, -15, 19)/12$ holds true.

Figure 1 also shows that

$$\max\{e(S,z,v) \mid \emptyset \neq S \neq N\} = -7/3 > -4 = \max\{e(S,2x,v) \mid \emptyset \neq S \neq N\},$$

hence $z \notin \mathscr{LC}(v)$.

(ii) Let $N = \{1,2,3\}$ and games v, w be given by Fig. 2. Define two vectors x and y by $x = (8,2,2)$, $y = (6,3,3)$.

	0	0	0	0	1	1	1	1
	0	0	1	1	0	0	1	1
	0	1	0	1	0	1	0	1
$v(\cdot)$	0	-2	-2	-2	-2	10	10	12
$e(\cdot,x,v)$	0	-4	-4	-6	-10	0	0	0
$e(\cdot,y,v)$	0	-5	-5	-8	-8	1	1	0
$w(\cdot)$	0	-6	-6	6	6	6	6	12
$e(\cdot,x,w)$	0	-8	-8	2	-2	-4	-4	0
$e(\cdot,y,w)$	0	-9	-9	0	0	-3	-3	0

Fig. 2

It is easy to verify $\psi(v) = \psi(w) = y$ and $\varphi(v) = \varphi(w) = x$. Clearly, $y \notin \mathscr{C}(v) \ni x$. Conversely, $x \notin \mathscr{C}(w) \ni y$.

(iii) Let $N = \{1, 2, 3, 4, 5\}$ and v be defined by

$$v(S) = \begin{cases} 1, & \text{if } S \in \{\{2, 3, 4, 5\}, \{1, 3, 4, 5\}, \{1, 2, 4, 5\}\} \\ 4, & \text{if } S \in \{\{1, 2, 3\}, \{1, 2, 3, 5\}, \{1, 2, 3, 4\}\} \\ 5, & \text{if } S = N \\ 0, & \text{otherwise} \end{cases} .$$

Then v is clearly convex. Using Remark 2.7(i) it can easily been seen that $\psi(v) = \frac{x}{6}$ holds true, where

$$x = (8, 8, 8, 3, 3).$$

Hence

$$\mu(\psi(v), v^*) \geq v^*(\{3, 4\}) - \frac{11}{6} = \frac{19}{9} > 3.$$

Taking $y = (1, 1, 1, 1, 1)$, the maximal excess w.r.t. v^* strictly decreases, i.e.

$$\mu(y, v^*) = 3.$$

Therefore the modified nucleolus is no member of the least core of v^*. Besides, it can be verified that

$$\nu(v^*) = y, \quad \nu(v) = (13, 13, 13, 3, 3)/9.$$

4 Axiomatizations

In this section the characterizing axioms for the prenucleolus will be recalled and the fact that the modified nucleolus can be axiomatized by similar axioms will be shown. First the necessary definitions concerning the prenucleolus are given.

Definition 4.1:

(i) For a set U let $\Gamma_U = \{(N, v) | N \subseteq U\}$ denote the set of games with player set contained in U.

(ii) Let (N, v) be a game, $x \in \mathbb{R}^N$, and $\bar{S}$ be a nonvoid coalition of N. The game $(\bar{S}, v^{\bar{S},x})$, where

$$v^{\bar{S},x}(S) = \begin{cases} v(N) - x(N \backslash \bar{S}), & \text{if } S = \bar{S} \\ 0, & \text{if } S = \varnothing \\ \max\{v(S \cup Q) - x(Q) | Q \subseteq N \backslash \bar{S}\}, & \text{otherwise} \end{cases},$$

is the **reduced game** of v w.r.t x and $\bar{S}$.

(iii) A solution concept σ on a set Γ of games satisfies **consistency** (**CONS**) if $(N,v)\in\Gamma$, $x\in\sigma(v)$, $\emptyset \neq \bar{S} \neq N$ implies $(\bar{S}, v^{\bar{S},x})\in\Gamma$ and $x_{\bar{S}}\in\sigma(\bar{S}, v^{\bar{S},x})$.

The notion of a reduced game was introduced by Davis and Maschler (1965). For the axiom CONS – also called **reduced game property** – and for the following axiomatization of the prenucleolus Sobolev (1975) is referred to. Note that the condition $(\bar{S}, v^{\bar{S}x})\in\Gamma$ in the definition of the reduced game can be dropped in Sobolev's result since the considered set of games (Γ_U) is rich enough, i.e. each reduced game w.r.t. each feasible payoff vector automatically is an element of this set.

Theorem 4.2 (Sobolev): If U is an infinite set, then there exists a unique solution concept on Γ_U satisfying SIVA, AN, COV, CONS; and it is the prenucleolus.

For the definition of SIVA, AN, COV Section 1 is referred to.

Proposition 1.4 together with the single valuedness of the prenucleolus and the definition show that the modified nucleolus also satisfies SIVA, AN, and COV. Moreover, Ψ does not satisfy CONS on Γ_U, since it does not coincide with ν (see, e.g., Example 3.2 (i)). In what follows it turns out that the modified nucleolus can be characterized by replacing the reduced game property and the anonymity by two additional axioms. Before stating and proving the main results of this section (Theorem 4.10), some notation and assertions concerning the coincidence of ν and ψ are needed.

Definition 4.3: Let (N,v) be a game.

(i) The game (N,w) is a **shift** game of v if there is a real number $\alpha\in\mathbb{R}$ such that

$$w(S) = \begin{cases} v(S)+\alpha, & \text{if } \emptyset \neq S \neq N \\ v(S) & , \text{ otherwise} \end{cases}.$$

In this case w is the **α-shift** game of v, denoted ${}^{\alpha}v$.

(ii) For $x\in\mathbb{R}^N$ let $\Lambda(x,v)$ be defined by

$$\Lambda(x,v) = \min\{v(T) - v^*(T) \mid \emptyset \neq T \neq N\} - \underline{\mu}(x,v),$$

where $\underline{\mu}(x,v) = \max\{e(S,x,v) \mid \emptyset \neq S \neq N\}$ denotes the maximal nontrivial excess at x. Here $\min\emptyset = \infty$ and $\max\emptyset = -\infty$ as usual and, in addition, $\Lambda(x,v) = 0$ for a 1-person game.

(iii) The game v has the **large excess difference property** (**LED**) w.r.t. $x\in\mathbb{R}^N$, if $\Lambda(x,v)\geq 0$.

For a verbal interpretation of LED the paragraph following Definition 4.6 is referred to.

Lemma 4.4: Let (N,v) be a game and $x\in\mathbb{R}^N$ for some $|N|\geq 2$.

(i) $\Lambda(x,v) = \min\{\min\{e(S,x,v), e(T,x,v)\} - e(S,x,v) - e(T,x,v^*) | \emptyset \neq S, T \neq N\}$,
(ii) $\Lambda(x, {}^{\alpha}v) = \Lambda(x,v) + \alpha$ for $\alpha \in \mathbb{R}$,
(iii) The shift game ${}^{\alpha}v$ of v satisfies LED w.r.t. x, iff $\alpha \geq -\Lambda(x,v)$.

Proof: Assertion (ii) is a direct consequence of the corresponding definition, whereas (iii) is implied by (ii).

ad (i): Let $\emptyset \neq S, T \neq N$ be two coalitions. If $e(S,x,v) \geq e(T,x,v)$, then we come up with

$$\begin{aligned}&\min\{e(S,x,v), e(T,x,v)\} - e(S,x,v) - e(T,x,v^*)\\&\quad = e(T,x,v) - e(T,x,v^*) - e(S,x,v)\\&\quad = v(T) - v^*(T) - e(S,x,v) \geq \Lambda(x,v).\end{aligned} \tag{1}$$

If $e(S,x,v) < e(T,x,v)$, we conclude

$$\begin{aligned}&\min\{e(S,x,v), e(T,x,v)\} - e(S,x,v) - e(T,x,v^*)\\&\quad = -e(T,x,v^*) = v(T) - v^*(T) - v(T) + x(T)\\&\quad \geq v(T) - v^*(T) - \underline{\mu}(x,v) \geq \Lambda(x,v).\end{aligned} \tag{2}$$

Conversely, let $\underline{\mu}(x,v)$ be attained by $\bar{S}$ for some $\emptyset \neq \bar{S} \neq N$ and $\min\{v(T) - v^*(T) | \emptyset \neq T \neq N\}$ be attained by $\bar{T}$ for some $\emptyset \neq \bar{T} \neq N$. Then

$$\min\{e(\bar{S},x,v), e(\bar{T},x,v)\} - e(\bar{S},x,v) - e(\bar{T},x,v^*) = \Lambda(x,v)$$

can be verified directly. q.e.d.

Lemma 4.5: Let (N,v) be a game and $\nu := \nu(v)$ be the prenucleolus of v.

(i) The prenucleolus of each shift game of v coincides with ν.
(ii) If v satisfies LED w.r.t. ν, then $\nu = \psi(v)$.

Proof: Assertion (i) is a trivial consequence of the corresponding definition. It remains to verify

(ii) By Theorem 2.3 it is sufficient to show that $\tilde{D} := \tilde{D}(\nu, \alpha, v)$ is balanced, if $\tilde{D} \neq \emptyset$. Applying LED we come up with

$$e(S,\nu,v) + e(S,\nu,v^*) \leq e(S,\nu,v)$$

for each $\emptyset \neq S \neq N$ (by Lemma 4.4 (i)), and thus with

$$e(S,\nu,v^*) \leq 0 \text{ for all } S \subseteq N. \tag{3}$$

Moreover, inequality (3) and Remark 1.2 (i) imply

$$e(S, v, v) \geq 0 \text{ for all } S \subseteq N. \tag{4}$$

In view of these considerations the following assertions are obvious. Let $D := D(v, \alpha, v)$. Then

$$S \in D \text{ for all } (S, T) \in \tilde{D}, \tag{5}$$

$$T \in D \text{ for all } (S, T) \in \tilde{D} \text{ with } \emptyset \neq T \neq N, \tag{6}$$

$$(S, \emptyset) \in \tilde{D} \text{ iff } (S, N) \in \tilde{D} \text{ iff } S \in D, \tag{7}$$

$$(S, \emptyset) \in \tilde{D} \text{ if } (S, T) \in \tilde{D} \text{ for some } T \subseteq N. \tag{8}$$

With $\tilde{D}_1 := \{(S, \emptyset) \in \tilde{D}\}$, $\tilde{D}_2 = \{(S, N) \in \tilde{D}\}$, $\tilde{D}_{12} = \tilde{D}_1 \cup \tilde{D}_2$, and using (7), (8) we obtain

$$\tilde{D}_1 = \{(S, \emptyset) \mid S \in D\} \text{ and } \tilde{D}_2 = \{(S, N) \mid S \in D\}.$$

The balancedness of D (Theorem 2.1, D is nonvoid by (5)) directly implies that both $\tilde{D}_1$ and $\tilde{D}_2$ are balanced, thus $\tilde{D}_{12}$ is. Clearly, each pair $(S, T) \in \tilde{D} \backslash \tilde{D}_{12}$ is in the span of $\tilde{D}_{12}$; hence Remark 2.7(i) completes the proof. q.e.d.

In the following definition some notation is introduced leading to axiomatizations of the modified nucleolus.

Definition 4.6: Let σ be a solution concept on a set Γ of games, let (N, v) be a game and $x \in \mathbb{R}^N$.

(i) Define a game (N, v^x) by

$$v^x(S) = \begin{cases} v(S), & \text{if } S \in \{\emptyset, N\} \\ \max\{v(S) + \mu + 2\mu^*, v^*(S) + \mu^* + 2\mu\}, & \text{otherwise} \end{cases}$$

for $S \subseteq N$, where $\mu = \mu(x, v)$ and $\mu^* = \mu(x, v^*)$.

(ii) σ satisfies **excess comparability (EC)**, if $v \in \Gamma$, $x \in \sigma(v)$, and $v^x \in \Gamma$ imply $x \in \sigma(v^x)$.

(iii) σ satisfies the **dual replication property (DRP)**, if the following is true: If $v \in \Gamma$, $\tau: N \cup N^* \to \tilde{N}$ is a bijection such that $(\tilde{N}, w) \in \Gamma$, where $w = \tau\bar{v}$ (for the definition $\bar{v}$ of the dual replication of v Section 1 is referred to), $x \in \sigma(v)$, then $\tau(x, x^*) \in \sigma(w)$.

(iv) The **difference vector $d^v \in \mathbb{R}^N$ of maximal and minimal marginal contributions w.r.t. v and** x is defined by

$$d_i^v = \max\{v(S \cup \{i\}) - v(S) \mid S \subseteq N \backslash \{i\}\} - \min\{v(S \cup \{i\}) - v(S) \mid S \subseteq N \backslash \{i\}\}$$

for $i \in N$ (and $d_i^v = 0$ if $|N| = 1$).

(v) σ satisfies the **dual cover property (DCP)**, if the following is valid: If $(N, v) \in \Gamma$, $\tau: N \cup N^* \to \tilde{N}$ is a bijection such that $(\tilde{N}, w) \in \Gamma$, where $w = {}^{\alpha}(\tau\tilde{v})$ (for the definition of the dual cover $\tilde{v}$ of v Section 1 is referred to) with $\alpha = 6d^v(N)$, and if $x \in \sigma(v)$, then $\tau(x, x^*) \in \sigma(w)$.

(vi) σ satisfies **large excess difference consistency (LEDCONS)**, if $(N, v) \in \Gamma$, $x \in \sigma(v)$, and v satisfies LED w.r.t. x implies $(S, v^{S,x}) \in \Gamma$ and $x_S \in \sigma(v^{S,x})$.

The idea of the game v^x is as follows. Assume that x is Pareto optimal, i.e. x constitutes a rule how to share $v(N)$. Moreover, assume that the players agree that this rule should take into account the worth $v(S)$ of each coalition S and the amount which S can be given, if the complement coalition $N \backslash S$ obtains its own worth $v(N \backslash S)$. Now the problem to compare these numbers $v(S)$ and $v^*(S)$ is solved here by adding constants to both, $v(S)$ and $v^*(S)$, such that the arising modified maximal excesses w.r.t. v and v^* coincide (as long as both initial maximal excesses are attained by nontrivial coalitions). Excess comparability now means that the solution x has not to be changed if the game v is replaced by v^x, i.e. by a game which contains v and its dual as totally symmetric ingredients in its definition such that the coalitions with maximal initial excesses possess coinciding new excesses (except if one maximal excess is attained by the empty and grand coalition only). If $x = \nu(\tilde{v})_N$ is the restriction of the prenucleolus of the dual cover of the game, then v^x coincides – up to a shift – with the reduced game of the dual cover w.r.t. the initial player set and the prenucleolus, hence $x = \nu(v^x)$ by Lemma 4.5(i) in this case. Moreover, v^x satisfies LED w.r.t. x, hence x coincides with the modified nucleolus of v^x. Therefore $\psi(v^{\psi(v)}) = \psi(v)$ holds true by Proposition 1.4.

The large excess difference property can be interpreted with the help of v^x as follows. If v satisfies LED w.r.t. the Pareto optimal vector x, then $\mu(x, v^*) = 0$ (see (3), applied to x). Due to definition of LED we obtain $v(S) - v^*(S) - \mu(x, v) \geq 0$, thus

$$v(S) + 2\mu(x, v^*) + \mu(x, v) \geq v^*(S) + 2\mu(x, v) + \mu(x, v^*) \text{ for } \emptyset \neq S \subsetneq N.$$

Therefore v^x coincides (up to a shift) with v and, hence, v is the only significant ingredient of v^x in this case. If the coalitions agree to the "comparability principle" (i.e. to the replacement of v by v^x), then each coalition should argue with its excess w.r.t. the original game instead of switching to the dual game v^*.

In view of Corollary 1.6(ii) and Remark 1.2(iv) the modified nucleolus clearly satisfies the dual replication property.

As shown in the next lemma SIVA and both DRP or DCP imply AN on Γ_U for infinite U. Therefore every solution concept satisfying one of these pairs of axioms also satisfies anonymity.

Lemma 4.7: Let σ be a single valued solution concept on Γ_U for some infinite U.

(i) If σ satisfies DRP, then σ is anonymous.
(ii) If σ satisfies DCP, then σ is anonymous.

Proof: Assertion (ii) can be proved analogously to (i), hence we concentrate on this case. Let σ be a solution concept on Γ_U satisfying the desired properties. Let (N, v) and $(\bar{N}, \pi v)$ be equivalent games in Γ_U, i.e. $\pi: N \to \bar{N}$ is bijective. Since U is infinite, there exists a game in Γ_U which is equivalent to the dual replication of v, say $(\tilde{N}, \tau\tilde{v}) \in \Gamma_U$ for some bijective mapping $\tau: N \cup N^* \to \tilde{N}$ and some $\tilde{N} \subseteq U$. The mapping $\tilde{\pi}: N \cup N^* \to \bar{N} \cup \bar{N}^*$ defined by

$$\tilde{\pi}_i = \begin{cases} \pi_i, & \text{if } i \in N \\ \pi_{i^*}, & \text{if } i \in N^* \end{cases}$$

is a bijection. Moreover, the composition $\tau\tilde{\pi}^{-1}: \bar{N} \cup \bar{N}^* \to \tilde{N}$ is a bijection. Let x be the unique element of $\sigma(v)$ and $y = \pi x$. The obvious equality

$$\tau\tilde{\pi}^{-1}(y, y^*) = \tau\tilde{\pi}^{-1}\tilde{\pi}(x, x^*) = \tau(x, x^*) \in \sigma(\tau\tilde{v})$$

directly implies $\pi x \in \sigma(\pi v)$. q.e.d.

The following two lemmata are useful.

Lemma 4.8: Let (N, v) be a game, $\emptyset \neq \bar{S} \subseteq N$, and $x \in X^*(v)$.

(i) If v satisfies LED w.r.t. x, then $v^{\bar{S},x}$ satisfies LED w.r.t. $x_{\bar{S}}$.
(ii) If x is reasonable on both sides (w.r.t. v), then the shifted dual cover ${}^{\alpha}\tilde{v}$, where $\alpha = 6d^v(N)$, satisfies LED w.r.t. the replicated vector $(x, x^*) \in \mathbb{R}^{N \dot{\cup} N^*}$.
(iii) If v satisfies LED w.r.t. x, then v satisfies LED w.r.t. $\nu(v)$.

Proof:

(i) Let $v, \bar{S}, x$ have the desired properties and let w denote the reduced game, i.e. $w = v^{\bar{S},x}$. Let $\emptyset \neq S,\ T \subsetneqq \bar{S}$ such that $\Lambda(x_{\bar{S}}, w) = w(T) - w^*(T) - e(S, x_{\bar{S}}, w)$. The observation $w^*(T) = \min\{v^*(T \cup R) - x(R) \mid R \subseteq N \setminus \bar{S}\}$ shows that there are coalitions $R_1, R_2, R_3 \subseteq N \setminus \bar{S}$ such that

$$\begin{aligned} w(T) &= v(T \cup R_1) - x(R_1), w(S) = v(S \cup R_2) - x(R_2), \\ w^*(T) &= v^*(T \cup R_3) - x(T \cup R_3) \leq v^*(T \cup R_1) - x(R_1), \text{ thus} \\ 0 \leq \Lambda(x, v) &\leq v(T \cup R_1) - v^*(T \cup R_1) - e(S \cup R_2, x, v) \\ &= e(T \cup R_1, x, v) - e(T \cup R_1, x, v^*) - e(S \cup R_2, x, v) \\ &\leq e(T \cup R_1, x, v) - e(T \cup R_3, x, v^*) - e(S \cup R_2, x, v) \\ &= e(T, x_{\bar{S}}, w) - e(T, x_{\bar{S}}, w^*) - e(S, x_{\bar{S}}, w) \\ &= w(T) - w^*(T) - e(S, x_{\bar{S}}, w) = \Lambda(x_{\bar{S}}, w). \end{aligned}$$

(ii) Let v, x have the desired properties, let $\alpha = 6d^v(N)$, and let $w = {}^{\alpha}\tilde{v}$. An inductive argument on the cardinality of S shows – by using reasonableness of x on both sides (see Corollary 2.6) – that

$$-d^v(S) \le e(S, x, v) \le d^v(S) \text{ for } S \subseteq N, \tag{9}$$

hence

$$-d^v(S) \le e(S, x, v^*) \le d^v(S) \text{ for } S \subseteq N, \tag{10}$$

holds true. Nonnegativity of d^v implies that $d^v(S)$ can be replaced by $d^v(N)$ in (9) and (10) whenever it occurs. Therefore we have $-2 \cdot d^v(N) \le e(S, y, \tilde{v}) \le 2 \cdot d^v(N)$ for $S \subseteq \tilde{N} = N \dot{\cup} N^*$, thus – with $y = (x, x^*)$ –

$$4 \cdot d^v(N) \le e(S, y, w) = -e(\tilde{N} \backslash S, y, w^*) \le 8 \cdot d^v(N) \text{ for } \varnothing \neq S \subsetneqq \tilde{N}. \tag{11}$$

In view of

$$\begin{aligned} \Lambda(y, w) &= \min_{\varnothing \neq T \subsetneqq \tilde{N}} (w(T) - w^*(T)) - \underline{\mu}(y, w) \\ &= \min_{\varnothing \neq T \subsetneqq \tilde{N}} (e(T, y, w) - e(T, y, w^*)) - \underline{\mu}(y, w) \\ &\ge \min_{\varnothing \neq T \subsetneqq \tilde{N}} e(T, y, w) - \max_{\varnothing \neq T \subsetneqq \tilde{N}} e(T, y, w^*) - \underline{\mu}(y, w) \ge 0 \\ & \qquad \text{(by (11))} \end{aligned}$$

w satisfies LED w.r.t. y.

(iii) Clearly $\underline{\mu}(\nu(v), v) \le \underline{\mu}(x, v)$ for $x \in X^*(v)$, hence this assertion is a consequence of the definition of $\Lambda(\cdot, v)$. q.e.d.

Due to the last lemma the modified nucleolus satisfies DCP on every set Γ of games and satisfies LEDCONS on Γ_U for every set U, since U is rich enough in the sense that any reduced game of a game in this set again belongs to this set of games. Therefore both pairs of axioms, DRP and SIVA on the one hand and DCP and SIVA on the other hand side, imply duality and can, thus, be regarded as strong versions of duality.

Lemma 4.9: Let σ be a solution concept on a set Γ_U of games satisfying SIVA, COV, and LEDCONS. Then the following assertions hold true:

(i) If σ satisfies EC, then σ satisfies Pareto optimality (PO).
(ii) If U is infinite and σ satisfies DCP, then σ satisfies PO.

Proof: We proceed analogously to Peleg's (1988/89) proof of PO, if SIVA, COV, and CONS are satisfied. In fact CONS is only used concerning reduced games w.r.t. 1-person coalitions. Indeed, for 1-person games SIVA and COV imply PO. Let σ be a solution concept on Γ_U with the desired properties.

ad (i): It remains to verify that $w = v^x$ satisfies LED w.r.t. x for $x \in \sigma(v)$, because any 1-person reduced game of w coincides with the 1-person reduced game of v w.r.t. the same feasible payoff vector. Let $(N, v) \in \Gamma$ for some $|N| \geq 2$, $x \in \sigma(v)$, and $w = v^x$. Take nontrivial coalitions $\emptyset \neq S,\ T \subsetneq N$ such that $\underline{\mu}(x, w) = e(S, x, w)$, $w(T) - w^*(T) = \min\{w(T) - w^*(T) \mid \emptyset \neq S,\ T \subsetneq N\}$, and observe that

$$w^*(T) = \min\{v(T) - 2\mu - \mu^*, v^*(T) - 2\mu^* - \mu\},$$

where $\mu = \mu(x, w)$, $\mu^* = \mu(x, w^*)$, holds by definition of w. Assume $w(T) = v(T) + 2\mu^* + \mu$ (otherwise exchange the roles of v and v^*). We come up with

$$\begin{aligned} \Lambda(x, w) &= w(T) - w^*(T) - e(S, x, w) = v(T) + 2\mu^* + \mu - w^*(T) - e(S, x, v) \\ &\geq v(T) + 2\mu^* + \mu - (v(T) - 2\mu - \mu^*) - e(S, x, v) \geq 3(\mu + \mu^*) - 2(\mu + \mu^*) \\ &\geq \mu + \mu^* \geq 0. \end{aligned}$$

The last inequality is guaranteed since $\mu + \mu^* \geq e(\emptyset, x, v) + e(\emptyset, x, v^*) \geq 0$ is true.
ad (ii): This assertion follows from the following four steps.

Step 1: If $(N, v) \in \Gamma_U$ is an additive game, i.e. $v = x$ for some $x \in \mathbb{R}^N$, then $x \in \sigma(v)$.
Let $y \in \sigma(v)$. Then, by covariance, both $2y$ and $y + x$ are members of $\sigma(2v) = \sigma(v + x)$. By SIVA $2y = y + x$, thus $x = y$.

In view of Step 1 it is sufficient to prove that the solution is Pareto optimal for an arbitrary nonadditive game (N, v) of Γ_U. Moreover, we can assume $v(N) = 0$ by COV. Let $(N, v) \in \Gamma_U$ be a nonadditive game such that $v(N) = 0$ (hence $|N| \geq 2$, $d^v(N > 0)$. Let $x \in \sigma(v)$ and $((N_r, v_r))_{r \in \mathbb{N}}$ be the sequence of successively shifted dual covers of $(N_0, v_0) = (N, v)$, i.e. $v_r = {}^{6d_{r-1}}v_{r-1}$, $N_r = N_{r-1} \cup N^*_{r-1}$, where $d_{r-1} = d^{v_{r-1}}(N_{r-1})$ denotes the aggregated maximal difference of marginal contributions of the players in N_{r-1} for $r \in \mathbb{N}$. Note that $v_r(N_r) = 0$ for $i \in \mathbb{N}$.

Step 2: $d_r \geq 2^r \cdot d_0$ for $r \in \mathbb{N}_0$.
Indeed, $d_r \geq 2^r \cdot d_0$ for $r = 0$. Suppose the assertion is verified for some $r \geq 0$. Take some player $i \in N_r$ and observe that

$$\begin{aligned} d_{r+1} \geq d_i^{v_{r+1}} &\geq (v_{r+1}(\{i\}) - v_{r+1}(\emptyset)) - (v_{r+1}(N_{r+1}) - v_{r+1}(N_{r+1} \backslash \{i\})) \\ &\geq (v_r(\{i\}) + 6d_r) + v_r(N_r) + v_r^*(N_r \backslash \{i\}) + 6d_r \text{ (by definition)} \\ &\geq 12d_r \geq 2d_r \geq 2^{r+1} d_0 \text{ (by the inductive hypothesis).} \end{aligned}$$

Step 3: $v_r(\{i\}) \geq c + (2^r - 1)d_0 \leq v_r(N_r \backslash \{i\})$ for $i \in N_r$ and $r \in \mathbb{N}_0$, where

$$c = \min\{v(\{i\}) \mid i \in N\} \cup \{v(N \backslash \{i\}) \mid i \in N\}.$$

Again we proceed by induction observing that the assertion is valid for $r = 0$.

For $i \in N_r$ we come up with

$$v_{r+1}(\{i\}) \geq v_r(\{i\}) + 6d_r \geq c + (2^r - 1)d_0 + 2^r d_0 \text{ (by the hypothesis and Step 2)}$$
$$\geq c + (2^{r+1} - 1)d_0$$

and analogously with

$$v_{r+1}(N_{r+1} \backslash \{i\}) \geq c + (2^{r+1} - 1)d_0.$$

For $i \in N^*_{r+1}$ the assertion follows from symmetry reasons.

Let $x^r = (x^{r-1}, (x^{r-1})^*) \in \mathbb{R}^{N_r}$ be the r-fold replicated vector of $x^0 = x$. In view of Step 3 and $d_0 > 0$ there is $\bar{r} \in \mathbb{N}$ such that $x^{\bar{r}-1}$ is reasonable on both sides for $v_{\bar{r}-1}$.

Step 4: $v_{\bar{r}}$ satisfies LED w.r.t. $x^{\bar{r}}$.

This assertion follows from the following inequalities:

$$\Lambda(x^{\bar{r}}, v_{\bar{r}}) = \min_{\emptyset \neq S \subsetneq N_{\bar{r}}} (v_{\bar{r}}(S) - v^*_{\bar{r}}(S)) - \underline{\mu}(x^{\bar{r}}, v_{\bar{r}})$$
$$\geq \min_{\emptyset \neq S \subsetneq N_{\bar{r}}} e(S, x^{\bar{r}}, v_{\bar{r}}) - \max_{\emptyset \neq S \subsetneq N_{\bar{r}}} e(S, x^{\bar{r}}, v^*_{\bar{r}}) - \max_{\emptyset \neq S \subsetneq N_{\bar{r}}} e(S, x^{\bar{r}}, v_{\bar{r}})$$
$$\geq 4d_{\bar{r}} + 4d_{\bar{r}} - 8d_{\bar{r}} \text{ (by reasonableness on both sides and (11)).}$$

Let $(\tilde{N}, w) \in \Gamma_U$ be some game equivalent to $(N_{\bar{r}}, v_{\bar{r}})$ (w exists because the cardinality of U is assumed to be infinite), say $w = \tau v_{\bar{r}}$ for some bijection $\tau: N_{\bar{r}} \to \tilde{N}$. Then, by Step 4, w satisfies LED w.r.t. $y = \tau x^{\bar{r}}$. Moreover, in view of DCP, y is a member of $\sigma(w)$. Applying LEDCONS we come up with $y_{\{i\}} \in \sigma(w^{\{i\},y})$ for $i \in \tilde{N}$, thus y is Pareto optimal for w by Step 1. The conclusion $v(N) = x(N)$ is, now, obvious. q.e.d.

Theorem 4.10: Let U be an infinite set.

(a) There is a unique solution concept on Γ_U satisfying SIVA, COV, LEDCONS, EC, and DRP.
(b) There is a unique solution concept on Γ_U satisfying SIVA, COV, LEDCONS, and DCP.

In both cases ((a) and (b)) the unique solution concept is the modified nucleolus.

Proof: The modified nucleolus satisfies SIVA, COV, LEDCONS, EC, DRP, and DCP (see Section 1, Lemma 4.5, and Lemma 4.8).

It remains to show that the modified nucleolus is uniquely determined by both systems of axioms. We proceed analogously to Peleg's (1988/89) adaptation of

Sobolev's proof of the corresponding direction of Theorem 4.2. Let σ be a solution concept on the given set of games with the desired properties. Let (N, v) be a game in Γ_U and $x \in \sigma(v)$ (i.e. $\sigma(v) = \{x\}$ by SIVA), $y = \psi(v)$. W.l.o.g. $y = 0$ can be assumed by COV. In view of AN (see Lemma 4.7) and the infinity assumption on the cardinality u of U it can be assumed that the dual replication $(N \dot{\cup} N^*, \bar{v})$ is a member of Γ_U in case (a) and the shifted dual cover $(N \dot{\cup} N^*, w)$, where $w = {}^{6d^v(N)}\bar{v}$, is a member of Γ_U in case (b). In case (a) we proceed by defining $w = \bar{v}^{(x,x^*)}$. Observe that $w = {}^{\alpha}(\bar{v}^{(y,y^*)})$ for some nonnegative real α in (a), since the modified nucleolus (y, y^*) of $\bar{v}$ minimizes the sum of maximal excesses w.r.t. the game and its dual. The game w satisfies LED w.r.t. $(y, y^*) = \nu(w)$ by Lemma 4.5 and 4.8 in both cases (a) and (b). In view of SIVA it is sufficient to show $(y, y^*) \in \sigma(w)$ in case (a) (by DRP and EC) and in case (b) (by DCP). Sobolev constructs a game $(\tilde{N}, \omega) \in \Gamma_U$ with $N \dot{\cup} N^* \subseteq \tilde{N}$ satisfying

$$\omega^{N \dot{\cup} N^*, z} = w, \text{ where } z = 0 \in \mathbb{R}^{\tilde{N}}; \tag{12}$$

$$\omega(S) \geq \min_{\varnothing \neq S \subsetneq N} w(S) \text{ for } \varnothing \neq S \subsetneq \tilde{N}; \tag{13}$$

$$w \text{ is } \textbf{transitive}\text{, i.e.:} \begin{cases} \text{for } i,\ j \in \tilde{N} \text{ there is a permutation } \pi \text{ on} \\ \tilde{N} \text{ leaving } \omega \text{ unchanged and mapping } i \text{ to } j \end{cases}. \tag{14}$$

Therefore, by AN and PO (see Lemma 4.7 and 4.9), $z \in \sigma(\omega)$ by (14). The proof is finished (by LEDCONS and (12)) as soon it is shown that ω satisfies LED w.r.t. z. To do this note that

$$\begin{aligned}
\Lambda(z, \omega) &= \min_{\varnothing \neq S \subsetneq \tilde{N}} (\omega(S) - \omega^*(s)) - \max_{\varnothing \neq S \subsetneq \tilde{N}} \omega(S) \\
&\geq \min_{\varnothing \neq S \subsetneq \tilde{N}} \omega(S) - \max_{\varnothing \neq S \subsetneq \tilde{N}} \omega^*(S) - \max_{\varnothing \neq S \subsetneq \tilde{N}} \omega(S) \\
&\geq 2 \cdot \min_{\varnothing \neq S \subsetneq \tilde{N}} \omega(S) - \max_{\varnothing \neq S \subsetneq \tilde{N}} \omega(S) \text{ (by } \omega(\tilde{N}) = 0 \text{ (see (12))} \\
&\geq 2 \cdot \min_{\varnothing \neq S \subsetneq N \dot{\cup} N^*} \omega(S) - \max_{\varnothing \neq S \subsetneq N \dot{\cup} N^*} \omega(S) \text{ (by (13))}
\end{aligned} \tag{15}$$

holds true. In case (b), (15) and (11) directly imply $\Lambda(z, w) \geq 0$. In case (a), define $\mu = \mu(y, v)$ and $\mu^* = \mu(y, v^*)$. In this case (15) can be rewritten as

$$\begin{aligned}
\Lambda(z, w) &\geq 2 \cdot \min_{S, T \subseteq N} (v(S) + v^*(T) + 3(\mu + \mu^*)) - \max_{S, T \subseteq N} (v(S) + v^*(T) + 3(\mu + \mu^*)) \\
&\geq 2 \cdot ((-\mu^* - \mu) + 3(\mu + \mu^*)) - (\mu + \mu^* + (3(\mu + \mu^*)) = 0.
\end{aligned}$$

q.e.d.

The rest of this section is devoted to prove that both characterizations of the modified nucleolus given by Theorem 4.10 are axiomatizations, i.e. examples of solutions are presented which show that both systems of axioms are logically independent. Moreover, the infinity assumption on the cardinality of the universe of players is discussed. Finally it is remarked that Theorem 4.10 remains valid, if DRP and DCP respectively are replaced by certain weaker version of these axioms.

Let U be an infinite set. Then define for $(N, v) \in \Gamma_U$:

$\sigma^1(v) = \emptyset$ (the **empty** solution);
$\sigma^2(v) = \{x \in \mathbb{R}^N \mid x_i = v(N)/n \text{ for } i \in N\}$ (the **equal split** solution);
$\sigma^3(v) = \{\varphi(v)\}$ (the Shapley value);
$\sigma^4(v) = \mathscr{PN}(v)$ (the prenucleolus).

Obviously the empty solution does not satisfy SIVA but all other properties of (a) and (b), the equal split solution does not satisfy COV but all other axioms, and the prenucleolus satisfies SIVA, COV, CONS (thus LEDCONS) and does not coincide with Ψ in general (see Example 3.2), hence does not satisfy DCP. The Shapley value satisfies SIVA, COV, and (by anonymity and Lemma 1.7) DRP. Moreover, σ^3 satisfies DCP and EC, hence does not satisfy LEDCONS (indeed, ψ and φ do not coincide in general as seen via the game ω below), as shown in the following

Lemma 4.11: Let (N, v) be a game, $c, d \in \mathbb{R}$, and (N, w) be defined by

$$w(S) = \begin{cases} v(S), & \text{if } S \in \{\emptyset, N\} \\ \max\{v(S) + c, v^*(S) + d\}, & \text{otherwise} \end{cases} \quad \text{for } S \subseteq N.$$

Then $\varphi(v) = \varphi(w)$ is true.

Proof: Let v, c, d, w be chosen according to the prerequisites of the lemma. For $\emptyset \neq S \subsetneqq N$ the equivalence

$$w(S) = v(S) + c \text{ if and only if } w(N \backslash S) = v(N \backslash S) + c \tag{16}$$

and the formula

$$v(T) - v(N \backslash T) = v^*(T) - v^*(N \backslash T) \text{ for } T \subseteq N \tag{17}$$

can be deduced directly. With the help of (16) and (17) we come up with

$$\begin{aligned}(w(S) &- w(S \backslash \{i\})) + (w(N \backslash (S \backslash \{i\})) - w(N \backslash S)) \\ &= (v(S) - v(S \backslash \{i\})) + (v(N \backslash (S \backslash \{i\})) - v(N \backslash S)) \text{ for } S \subseteq N \backslash \{i\}\end{aligned} \tag{18}$$

holds true for any $i \in N$. Applying (18) to a formula defining the Shapley value yields

$$\begin{aligned}
\varphi_i(w) &= \sum_{i\in S\subseteq N} \frac{(s-1)!(n-s)!}{n!}(w(S)-w(S\backslash\{i\})) \\
&= \frac{1}{2}\cdot \sum_{i\in S\subseteq N} \frac{(s-1)!(n-s)!}{n!}((w(S)-w(S\backslash\{i\}))+(w(N\backslash(S\backslash\{i\}))-w(N\backslash S))) \\
&= \frac{1}{2}\cdot \sum_{i\in S\subseteq N} \frac{(s-1)!(n-s)!}{n!}((v(S)-v(S\backslash\{i\}))+(v(N\backslash(S\backslash\{i\}))-v(N\backslash S))) \\
&= \varphi_i(v).
\end{aligned}$$

q.e.d.

Clearly the Shapley value is not sensitive against shifting of games, hence Lemma 1.7(i), the definition of v^x (Definition 4.6(i)), and Lemma 4.11 directly imply that σ^3 satisfies DCP and EC.

In order to define a solution concept σ^5 which shows that DRP is independent within the first system of properties assume that $N=\{1, 2, 3\}\subseteq U$ (otherwise rename the players of U). Let (N,ω) be the glove game discussed in the introduction defined by

$$\omega(S)=\begin{cases}1, & \text{if } 1\in S \text{ and } s\geq 2\\ 0, & \text{otherwise}\end{cases} \quad \text{for } S\subseteq N.$$

It is easy to verify that $\psi(\omega)=(2,1,1)/4$, $x:=\nu(\omega)=(1,0,0)$, and $\varphi(\omega)=(4,1,1)/6$. For any game w not equivalent to a game strategically equivalent to ω define

$$\sigma^5(w)=\Psi(w).$$

Otherwise, define

$$\sigma^5(\tau(\alpha\omega+\beta))=\{\tau(\alpha x+\beta)\},$$

if $\alpha>0$, $\beta\in\mathbb{R}^N$, and τ is a bijection $\tau\colon N\to\tilde{N}$ for some $\tilde{N}\subseteq U$. It is straightforward to show that this solution satisfies SIVA and COV. In order to prove EC it suffices to verify $\psi(\omega^x)=x$ which can be done by observing $\omega^x={}^{\alpha}\omega$ for some $\alpha>0$ and applying Lemma 4.5(ii).

To show that σ^5 satisfies LEDCONS it suffices to prove that ω does not occur as a reduced game of some game v satisfying LED w.r.t. $\psi(v)$ (since "reducing commutes with equivalence and strategical equivalence", i.e.

$$(\tau(\alpha v+\beta))^{S,\tau(\alpha x+\beta)}=\tau(\alpha v^{\tau^{-1}S,x}+\beta_{\tau^{-1}S})$$

for any bijection $\tau\colon N\to\tilde{N}$, $\alpha>0$, $x,\beta\in\mathbb{R}^N$, $\emptyset\neq S\subseteq\tau(N)$). In view of the fact that a reduced game of a game v satisfying LED w.r.t. $\psi(v)$ satisfies LED w.r.t. the restricted vector (see Lemma 4.8) it is sufficient to verify that w satisfies LED neither w.r.t. x nor w.r.t. $\psi(w)$, which is straightforward and therefore left to the reader. Consequently σ^5 satisfies SIVA, COV, LEDCONS, and EC and does not coincide with Ψ, hence violates DRP (and DCP).

To complete the independence part a solution satisfying SIVA, COV, LEDCONS, and DRP has to be constructed which does not coincide with Ψ. Some notation is needed. Let (N, v) be a game. A vector $x \in \mathbb{R}^N$ is **symmetrically weakly balanced (*sw*-balanced) w.r.t.** (v, v^*) if there are nonvoid balanced subsets $A \subseteq \{S \subseteq N \mid e(S, x, v) = \mu(x, v)\}$ and $B \subseteq \{S \subseteq N \mid e(S, x, v^*) = \mu(x, v^*)\}$ such that there are balancing coefficients $(\gamma_S)_{S \in A}$ and $(\delta_T)_{T \in B}$ with $\sum_{S \in A} \lambda_S = \sum_{T \in B} \delta_T$. Using the duality theorem of linear programming it is easy to see that if a Pareto optimal x is *sw*-balanced, then

$$x \in (\mathscr{L}\mathscr{C}(v) \cup \mathscr{C}(v)) \cap (\mathscr{L}\mathscr{C}(v^*) \cup \mathscr{C}(v^*)).$$

Define

$$\sigma^6(v) = \begin{cases} \Psi(v), & \text{if } \psi(v) \text{ is } sw\text{-balanced w.r.t. } (v, v^*) \\ \{\varphi(v)\}, & \text{otherwise} \end{cases}.$$

Clearly σ^6 satisfies SIVA, COV, and AN. The following lemma together with AN and Lemma 1.7 directly implies that σ^6 satisfies LEDCONS and DRP.

Lemma 4.12: Let (N, v) be a game, $x = \psi(v)$, and $y = \varphi(v)$.

(i) If v satisfies LED w.r.t. y, then v satisfies LED w.r.t. x.
(ii) If v satisfies LED w.r.t. x, then x is *sw*-balanced w.r.t. (v, v^*).
(iii) x is *sw*-balanced w.r.t. (v, v^*), iff (x, x^*) is *sw*-balanced w.r.t. $(\bar{v}, \bar{v}^*)$.

Proof:

(i) If v satisfies LED w.r.t. y, then v satisfies LED w.r.t. $v(v)$, since

$$\underline{\mu}(y, v) \geq \underline{\mu}(v(v), v)$$

holds true by definition of the prenucleolus. In this case $v(v) = x$ by Lemma 4.5(ii).

(ii) If v satisfies LED w.r.t. x, then $e(S, x, v^*) \leq 0 \leq e(S, x, v)$ for each coalition S in N (see (3), (4); only Pareto optimality of v is used). Therefore $x \in \mathscr{C}(v^*)$ and thus, $B = \{\varnothing, N\}$ is a balanced subset of $\{S \subseteq N \mid e(S, x, v^*) \neq \mu(x, v^*)\}$ and $A = D(x, \mu(x, v), v)$ (for the definition of this set Section 2 is referred to) is a balanced subset of $\{S \subseteq N \mid e(S, x, v) = \mu(x, v)\}$ by Theorem 2.1. Let $(\gamma_S)_{S \in A}$ be balancing coefficients for A. The obvious fact $\sum_{S \in A} \gamma_S \geq 1$ implies *sw*-balancedness, since $\delta_N = 1, \delta_\varnothing = \sum_{S \in A} \gamma_S - 1$ (or 1 for $\{N\}$ in case equality holds) are balancing coefficients for B.

(iii) Let x be sw-balanced w.r.t. (v, v^*) and let $A, B (\gamma_S)_{S\in A}, (\delta_T)_{T\in B}$ be defined as above. Then

$$\tilde{A} = \{S \dot\cup T^* \mid S\in A, T\in B\} \subseteq \{S \subseteq N \dot\cup N^* \mid e(S, x, x^*), \bar{v}) = \mu((x, x^*), \bar{v})\},$$
$$\tilde{B} = \{T \dot\cup S^* \mid S\in A, T\in B\} \subseteq \{S \subseteq N \dot\cup N^* \mid e(S, x, x^*), \bar{v}^*) = \mu((x, x^*), \bar{v}^*)\}.$$

With $c = \sum_{S\in A}\gamma_S = \sum_{T\in B}\delta_T$ it is straightforward to verify that $\tilde{\gamma}_{S\dot\cup T^*} = (\gamma_S\delta_T)/c$ and $\tilde{\delta}_{T\dot\cup S^*} = \tilde{\gamma}_{S\dot\cup T^*}$ $(S\in A,\ T\in B)$ are balancing coefficients for $\tilde{A}$ and $\tilde{B}$ respectively, thus (x, x^*) is sw-balanced w.r.t. $(\bar{v}, \bar{v}^*)$.

Conversely, if $\tilde{A} \subseteq \{S \subseteq N \dot\cup N^* \mid e(S, (x, x^*), v) = \mu((x, x^*), v)\}$ is balanced with balancing coefficients $(\tilde{\gamma}_S)_{S\in\tilde{A}}$, then

$$A = \{S \subseteq N \mid \text{there is } T \subseteq N \text{ with } S \dot\cup T^* \in \tilde{A}\} \text{ and}$$
$$B = \{T \subseteq N \mid \text{there is } S \subseteq N \text{ with } S \dot\cup T^* \in \tilde{A}\}$$

and balanced with balancing coefficients

$$\gamma_S = \sum_{S\dot\cup T^*\in\tilde{A}} \tilde{\gamma}_{S\dot\cup T^*} (S\in A) \text{ and } \delta_T = \sum_{S\dot\cup T^*\in\tilde{A}} \tilde{\gamma}_{S\dot\cup T^*} (T\in A) \text{ respectively.}$$

Clearly A and B consist of coalitions of maximal excess w.r.t. v and v^*.

q.e.d.

It remains to verify that σ^6 does not coincide with Ψ on Γ_U, which can be done by noting that this set of games contains a game equivalent to ω (for the definition of the glove game ω the paragraph is referred to in which σ^5 is defined). Indeed, $\psi(\omega)$ is not sw-balanced w.r.t. (ω, ω^*) and $\varphi(\omega) \neq \psi(\omega)$.

Let us now consider a finite universe U of players. If $|U| = u \geq 3$, let us say $U = \{1, \ldots, u\}$, two cases may be distinguished.

If u is odd, define a game (w, U) by

$$w(S) = \begin{cases} 1, & \text{if } 1\in S \text{ and } s \geq u-1 \\ 0, & \text{otherwise} \end{cases} \quad \text{for } S \subseteq U,$$

hence

$$w^*(S) = \begin{cases} 1, & \text{if } 1\in S \text{ and } s \geq 2 \\ 0, & \text{otherwise} \end{cases} \quad \text{for } S \subseteq U,$$

We claim $\psi(v) = x$, where

$$x_i = \begin{cases} 2/(u+1), & \text{if } i = 1 \\ 1/(u+1), & \text{otherwise} \end{cases} \quad \text{for } i\in U. \text{ Indeed,}$$

$$D(x, \mu(x, w), w) = \{S \subseteq U \mid s = u-1 \text{ and } 1\in S),$$
$$D(x, \mu(x, w^*), w^*) = \{S \subseteq U \mid (s = 2 \text{ and } 1\notin S) \text{ or } S = \{1\}\},$$

thus $\widetilde{D}(x, \mu(x,w) + \mu(x,w^*), w)$ is balanced and spans $2^U \times 2^U$, hence $\psi(w) = x$ by Remark 2.7(i). Define $y \in \mathbb{R}^U$ by $y_1 = 1$ and $y_i = 0$ for $i \in U \backslash \{1\}$. It should be noted that x is the prenucleolus of w and w^x coincides with a shift game of w. Analogously to σ^5 define σ^7 on Γ_U to be the solution given by $\sigma^7(v) = \Psi(v)$ for any game not being equivalent to any game strategically equivalent to w and $\sigma^7(\tau(\alpha w + \beta)) = \{\tau(\alpha x + \beta)\}$ otherwise. σ^7 satisfies SIVA, COV, LEDCONS, and EC. Indeed, a proof of these properties is straightforward and skipped. Moreover, no game in the domain of σ^7 can have a dual replication or shifted dual cover which is equivalent to any game being strategically equivalent to w, because such a game possesses an even number of players. Therefore σ^7 satisfies, additionally, DRP and DCP.

If u is even, a similar procedure can be used: Consider the game $(U \backslash \{u\}, w)$, where w is defined as above for the universal player set $U \backslash \{u\}$ with an odd number of members. The analogously defined σ^7 satisfies SIVA, COV, EC, DRP, and DCP. Since w cannot occur as a reduced game of a game v satisfying LED w.r.t. the proposed solution, σ^7 satisfies LEDCONS.

If $u = 1$, Theorem 4.10 remains valid (though the properties are obviously not logically independent). It remains to discuss the case $u = 2$, let's say $U = \{1, 2\}$. A solution concept σ on Γ_U satisfying SIVA, COV, LEDCONS, and EC satisfies PO by Lemma 4.9(i).

In order to prove that σ coincides with Ψ it is sufficient to show that σ is a **standard** solution, i.e. coincides with the Shapley value (pre- and modified nucleolus) for 2-person games. Indeed, if (U, v) is additive, then COV and SIVA directly imply $v \in \sigma(v)$. If (U, v) is not additive, then we can assume $v(\{i\}) = 0$ for $i \in U$ by COV, hence $v(U) \neq 0$. Let $x \in \sigma(v)$. Then

$$x \in \sigma(v^x), \quad \text{(by EC)}$$

$$v^x(\{i\}) = c \text{ for } i \in U \text{ for some } c \in \mathbb{R} \text{ with } v(U) \langle c \rangle 0,$$

hence $x - \tilde{c} \in \sigma(v^x - \tilde{c})$ by COV, where $\tilde{c}_i = c$ for $i \in U$. On the other hand

$$v^x - \tilde{c} = d \cdot v, \quad \text{where } d = (v(U) - 2c)/v(U).$$

In case $v(U) < 0$ we have $d > 0$ and thus (by COV and SIVA) $x - \tilde{c} = d \cdot x$. Consequently $x_i = v(U)/2$ holds for $i \in U$ in this case. If $v(U) > 0$ we have $d < 0$. Analogous considerations (replacing v by $v^x - \tilde{c}$) imply $(x - \tilde{c})_i = (v^x - \tilde{c})(U)/2$, hence $x_i = v(U)/2$ for $i \in U$ in this case.

The system of axioms in (b) does not guarantee AN, as the following example shows: $\sigma^8(v) = \{x^v\}$ for any game (U, v), where $x_1^v = v(\{1\})$, $x_2^v = v(U) - v(\{1\})$, and $\sigma^8(v)$ is the Pareto optimal solution for 1-person games. Clearly σ^8 satisfies SIVA, COV, and CONS (hence LEDCONS). Moreover, $d^v(N) = 0$, hence ${}^{6d^v(N)}\tilde{v}$ (and the dual replication $\tilde{v}$) is additive for any 1-person game (N, v). Therefore σ^8 satisfies DCP (and DRP).

The preceding examples show that both systems of axioms of Theorem 4.10 are logically independent if U is infinite. In the finite case the properties of (a) uniquely determine a solution concept if and only if $|U| \leq 2$, whereas the properties of (b) uniquely determine a solution if and only if $|U| \leq 1$.

Remark 4.13: It should be noted that DRP and DCP can be replaced by some weaker versions ("there is $y \in \sigma(w)$ such that $x_i = y_{\tau i}$ for $i \in N$" in Definition 4.6(iii) and (v) respectively) in Theorem 4.10. If the axioms are formulated as "there is $y \in \sigma(w)$ such that $x_i = y_i$ for $i \in N$, if $\tau_i = i$ for $i \in N$", then it is, unfortunately, not known how to deduce anonymity (and Pareto optimality). On the other hand Theorem 4.10 remains valid if DRP and DCP respectively is replaced by the weak version and AN. In some sense this variant of Theorem 4.10 can be regarded as "the best" analogon on Sobolev's characterization of the prenucleolus. Indeed, compare the new version of (b) with the classical axiomatization of the prenucleolus. SIVA, COV, and AN apply in both characterizations whereas CONS occurs now in its weak form LEDCONS. The additional property (the weak version of DCP) together with SIVA and AN guarantees that duality is satisfied. This last property is crucial for the modified nucleolus. Unfortunately it is not known whether this best analogon constitutes an axiomatization, i.e. whether AN is logically independent. It should be noted that AN is independent in Sobolev's characterization, as shown in Sudhölter (1993a), but it is not known how the examples presented in this paper can be modified in order to solve the new problem of independence.

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