Axiomatizations of Game Theoretical Solutions for One-Output Cost Sharing Problems

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Cost sharing rules can be obtained by considering solution concepts on the associated cost games. The Shapley rule, nucleolus rule, and antinucleolus rule commonly satisfy "covariance under strategical equivalence." Covariance, together with "equal treatment of equals" and either additivity or consistency, characterize the Shapley rule. The nucleolus rules are axiomatized analogously by changing the definition of the "reduced cost sharing problem" adequately. In the case of concave cost functions the nucleolus satisfies a strong version of consistency and the antinucleolus rule is a core selector. Cost functions for which the proposed game theoretical solutions coincide with average cost pricing are characterized by a simple functional equation. *Journal of Economic Literature* Classification Numbers: C71, D24.   © 1998 Academic Press

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INTRODUCTION

The problem how to allocate the total production cost of a single homogeneous good is discussed in this paper. A cost function C, i.e. a nondecreasing function on the nonnegative reals which intersects the origin, is used to model the production costs depending on the demanded quantity. Here $C(y)$ should be interpreted as the cost for producing y units of the good. If the agents or players in a finite group N possess the production technology and agent i has demand q_i, then the question how to divide the total cost $C(\sum_{i \in N} q_i)$ among the players is answered in general by proposing any cost sharing rule. Several applications occur in the literature (see, e.g., Shenker, 1989, 1990).

A cost sharing rule assigns an allocation to any cost sharing problem (N, C, q), where q is the demand profile of the agents in N, and C is the

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cost function. This allocation has to be feasible in the sense that the proposed aggregate payment of the players covers the total cost. An example is average cost pricing. With respect to this rule every agent i has to pay her proportional part of the total cost. This means that average cost pricing only depends on the total cost and the demand profile, regardless of the special shape of the cost function. A further cost sharing rule, serial cost sharing, also takes into account the evaluation of the cost function at several distinguished arguments (see Moulin and Shenker, 1992). These authors (see Moulin and Shenker, 1994) showed that both rules can be characterized by certain intuitive properties. They also proposed to consider the cost games generated by the cost sharing problems and apply classical solution concepts for games with transferable utility to these cost games. Indeed, they mentioned the Shapley value and nucleolus as possible cost sharing rules for the cost sharing problems. The present paper, which is organized as follows, discusses the game theoretical solutions in detail.

In Section 1 the necessary notation is presented. It is shown (see Remark 1.3 and Fig. 1) that individual rationality, i.e. every player has to pay not more than her individual cost, cannot be a general property of a cost sharing rule. Indeed, individual costs may not cover total cost. Consequently, individual rationality is dropped as a property for cost sharing rules in general. In view of this fact we define the nucleolus rule to be the prenucleolus of the corresponding cost game. Well-known intuitive properties for cost sharing rules, e.g. Pareto optimality, "equal treatment of equals," the null property (players with zero demands do not have to pay), monotonicity, and additivity properties, are defined formally. Moreover, it is shown that similar well-accepted properties for solution concepts on games imply the properties on cost sharing problems. In order to translate a cost sharing rule into a solution concept on the arising cost games this rule should yield the same result in case it is applied to different cost sharing problems which possess the same induced cost game. A slightly stronger property is "covariance under strategical equivalence." Average cost pricing does not satisfy covariance. The solution concept which arises from a covariant cost sharing rule inherits many properties of the rule as shown in Corollary 1.8. Covariance is the basic property of the game theoretical cost sharing rules discussed in the following sections.

In Section 2 three different axiomatizations of the Shapley rule are presented. The common properties of all characterizations are covariance and equal treatment of equals (i.e., agents with coinciding demands are proposed to pay coinciding amounts). Together with the null property and additivity (i.e., if the cost function is the sum of two cost functions, then everybody has to pay the sum of the fees of both cost sharing problems) or

minmax additivity (a modification of additivity) Theorems 2.1 and 2.2 are similar to the results of Shapley (1953) and Dubey (1975). Here covariance is strongly used and cannot be replaced by Pareto optimality as in the classical context. Especially Theorem 2.2 is not a trivial analogue to Dubey's result, which is emphasized by the fact that in the cost sharing context these properties are not strong enough to uniquely determine the Shapley rule on the restricted class of cost sharing problems which generate monotonic simple cost games. Note that Dubey's result holds for this class of games. In a further axiomatization null property and additivity are replaced by some reduction property which is analogous to that introduced by Hart and Mas-Colell (1989). Clearly, consistency can only be applied to classes of cost sharing problems with varying sets of agents.

In Sections 3 and 4 further versions of consistency are introduced and it is shown that both the nucleolus rule and the antinucleolus rule (i.e., the prenucleolus of the dual cost game) can be axiomatized by some version of consistency, covariance, and equal treatment of equals (see Theorems 3.4 and 4.2). As in the game theoretical context the infinity assumption on the set of potential agents is needed as a prerequisite (see Sobolev, 1975). Moreover, two interesting results concerning cost sharing problems with concave cost functions are presented. A strong version of consistency, together with equal treatment and covariance, uniquely determines the nucleolus even in case of a finite universe of players (see Theorem 3.3). In general, the antinucleolus is, other than the nucleolus rule, not a core selector, even in case that the core is nonempty. In Corollary 4.4 it is proved that the antinucleolus is a core selector in the case of concave cost functions. This fact also shows that the class of induced cost games does not cover, even up to strategical equivalence, the class of all concave games. This is in contrast to the general case; every game is, up to strategical equivalence, an induced game of some cost sharing problem (see Lemma 1.6).

Finally, in Section 5 the cost functions which yield coinciding average cost pricing, Shapley rule, and nucleolus rules, when applied to a cost sharing problem with a fixed aggregate demand, are characterized by a functional equation (see Theorem 5.1). Concerning this class of cost sharing problems any of the proposed cost sharing rules can be characterized by ''separable costs'' (i.e., coincidence with average cost pricing for linear cost functions), monotonicity, and Pareto optimality. For average cost pricing this characterization holds in general (see Moulin and Shenker (1994)). Moreover, Theorem 5.3 shows that linear and parabolic cost functions are the unique functions which globally yield coinciding cost sharing rules. Section 6 presents diagrams which summarize the main results.

1. NOTATION AND PRELIMINARY RESULTS

Let U denote the nonvoid set of potential agents; it can be finite or infinite. A triple (N, C, q) is called **cost sharing problem** (**CSP**), if N is a finite nonvoid subset of U (the set of **agents** of the CSP), C is a nondecreasing function on the nonnegative reals $\mathbb{R}_{\geq 0}$ such that $C(0) = 0$ (the **cost function of** the CSP), and $q = (q_i)_{i \in N} \in \mathbb{R}^N$ is such that $q_i \geq 0$ for $i \in N$ (the **demand profile of** the CSP). Let $\Gamma(U)$ denote the set of all cost sharing problems with the foregoing properties. A **cost sharing rule** (**CSR**) on a subset Γ of $\Gamma(U)$ associates a vector $\sigma(N, C, q) \in \mathbb{R}^N$ satisfying

$$\sum_{i \in N} \sigma_i(N, C, q) \geq C(q(N)) \quad \text{(\textbf{feasibility})}$$

with each CSP $(N, C, q) \in \Gamma$. As usual $x(S) = \sum_{i \in S} x_i$ denotes the aggregate weight of S at $x \in \mathbb{R}^N$ for $S \subseteq N$. Feasibility means that at least the total cost is covered by the agents. We do not require $\sigma_i(N, C, q) \geq 0$, but in all our examples this property automatically holds. Well-known and intuitively justified properties of a CSR σ on Γ are as follows:

(i) **Pareto optimality** (**PO**). $\sum_{i \in N} \sigma_i(N, C, q) = C(q(N))$ for all $(N, C, q) \in \Gamma$.

(ii) **Ranking** (**RAN**). $q_i \leq q_j$ implies $\sigma_i(N, C, q) \leq \sigma_j(N, C, q)$ for all $(N, C, q) \in \Gamma$.

(iii) **Separable costs** (**SC**). If $C(y) = \lambda \cdot y$ for $y \geq 0$ and $(N, C, q) \in \Gamma$, then $\sigma(N, C, q) = \lambda \cdot q$.

(iv) **Equal treatment of Equals** (**ET**). $q_i = q_j$ implies $\sigma_i(N, C, q) = \sigma_j(N, C, q)$ for all $(N, C, q) \in \Gamma$.

(v) **Null property** (**NP**). $q_i = 0$ implies $\sigma_i(N, C, q) = 0$ for all $(N, C, q) \in \Gamma$.

(vi) **Anonymity** (**AN**). If $(N, C, q), (N, C, \pi q) \in \Gamma$ for some permutation π of N, then $\sigma_{\pi i}(N, C, \pi q) = \sigma_i(N, C, q)$ for all $i \in N$.

(vii) **Monotonicity** (**MON**). If $(N, C, q), (N, \tilde{C}, q) \in \Gamma$ with $C \leq \tilde{C}$, then $\sigma(N, C, q) \leq \sigma(N, \tilde{C}, q)$.

(viii) **Additivity** (**ADD**). If $(N, C, q), (N, \tilde{C}, q), (N, C + \tilde{C}, q) \in \Gamma$, then $\sigma(N, C, q) + (N, \tilde{C}, q) = (N, C + \tilde{C}, q)$.

(ix) **Minmax additivity** (**MMADD**). If $(N, C, q), (N, \tilde{C}, q), (N, C \wedge \tilde{C}, q), (N, C \vee \tilde{C}, q) \in \Gamma$ (where $\wedge$ denotes "minimum") and $\vee$ denotes "maximum," then $\sigma(N, C, q) + \sigma(N, \tilde{C}, q) = \sigma(N, C \wedge \tilde{C}, q) + \sigma(N, C \vee \tilde{C}, q)$.

EXAMPLE 1.1.   For every CSP (N, C, q) define

$$a_i(N, C, q) = (q_i/q(N)) \cdot C(q(N)) \quad (\text{where } 0/0 = 1).$$

The mapping a is called **average cost pricing (rule)**.

Note that average cost pricing distributes the total cost $C(q(N))$ among the agents according to the Aumann–Shapley unit price (see Aumann and Shapley, 1974). Note, furthermore, that average cost pricing is completely determined by the demand profile and the total cost. There is an example of another CSR, namely **serial cost sharing** (see Moulin and Shenker, 1992), which depends on both the demand profile and the cost function evaluated at n arguments. Moulin and Shenker (1994, Examples 3 and 4 of Section 3) suggested looking at cost sharing rules which mainly depend on the cost function (the demand profile is only used implicitly). They proposed the Shapley value and the nucleolus of a certain induced game as feasible cost sharing rules. To formulate this more explicitly it is necessary to define the **induced (TU) cost game** $(N, v^{C,q})$ **of** an arbitrary CSP (N, C, q). The **coalitional function** $v^{C,q}$ is given by

$$v^{C,q}(S) = C(q(S)) \quad \text{for } S \subseteq N.$$

In general a **game** is a pair (N, v) such that $v: 2^N \to \mathbb{R}$ and $v(\varnothing) = 0$ hold true. The **Shapley (cost sharing) rule** φ assigns to every CSP the Shapley value of its induced game (for the definition of the Shapley value, see Shapley, 1953). There are two cost sharing rules based on the prenucleolus of the induced game or its dual. Here is the precise

DEFINITION 1.2.   1. Let (N, v) be a game (considered as the cost game) and let $X(v) = \{x \in \mathbb{R}^N \mid x(N) = v(N)\}$ be the set of **preimputations of** v. The **prenucleolus** $\bar{v}(v)$ of (N, v) is the unique preimputation which successively minimizes the largest excesses. Here $x(S) - v(S)$ is the **excess of** S at x (**with respect to** v) for $x \in \mathbb{R}^N$. To put it more formally, let

$$Z = \Big\{ x \in X(v) \mid \Theta\big((x(S) - v(S))_{S \subseteq N}\big)$$
$$\underset{\text{lex}}{\leq} \Theta\big((y(S) - v(S))_{S \subseteq N}\big) \text{ for } y \in X(v) \Big\},$$

where Θ applied to a vector $(x(S) - v(S))_{S \subseteq N}$ of excesses at x orders the components of this vector nonincreasingly. The set Z is a singleton and its unique element $\bar{v}(v)$ is the prenucleolus of v (see Maschler, Peleg, and Shapley, 1979). In the definition of the **nucleolus** (see Schmeidler, 1969) $X(v)$ is replaced by the subset of **individually rational** preimputations. (A vector $x \in \mathbb{R}^N$ is individually rational if $x_i \leq v(\{i\})$ for all $i \in N$.)

2. The **nucleolus rule on a** subset Γ of $\Gamma(U)$ assigns to each CSP $(N, C, q) \in \Gamma$ the prenucleolus of its induced game $(N, v^{C,q})$. We denote this cost sharing rule by ν.

3. The **anti-nucleolus rule** ν^* **on** a subset Γ of $\Gamma(U)$ assigns to each CSP $(N, C, q) \in \Gamma$ the negative of the prenucleolus of the negative of its induced game $(N, v^{C,q})$, i.e.,

$$\nu^*(N, C, q) = -\bar{\nu}(-v^{C,q}).$$

Remark 1.3. (i) Note that the nucleolus in the sense of Schmeidler does not necessarily exist; i.e., the set of **imputations** (individually rational preimputations) of a game can be the empty set. This is also true for induced games of cost sharing problems. In Fig. 1 two 2-agent cost sharing problems with demand profiles q and $\tilde{q}$ and the same "S-shaped" cost function C are sketched. Note that S-shaped cost functions (i.e., decreasing marginal costs for "small" aggregate demand and, due to, e.g., weak capacity bounds, increasing marginal costs for "large" aggregate demand) are typical for many economical applications. For the demand profile q feasibility of a CSR σ requires at least one individual fee which is higher than the individual cost whereas in case of $\tilde{q}$ the total cost can be divided among the players in such a way that both pay less than their individual costs. Therefore the version of a nucleolus rule based on individually rational feasible payoffs does not establish a cost sharing rule.

(ii) Using the notion of "interpersonal comparisons of utility" the prenucleolus can be justified for cost games (see Maschler, 1992). Al-
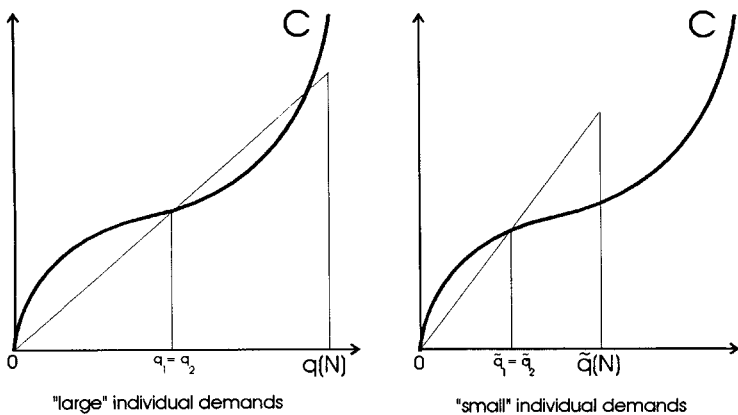

FIG. 1. "S-shaped" cost function.

though the definition of the antinucleolus rule, i.e. successively maximizing minimal excesses, seems to be counterintuitive, there are examples of cost sharing situations for which the antinucleolus rule can be justified (see Potters and Sudhölter, 1995). Therefore, we also analyze the antinucleolus rule ν^* in this paper.

In order to characterize the game theoretical cost sharing rules (Shapley rule, nucleolus rule, and antinucleolus rule), which are determined by the cost function evaluated at the aggregate demands of the coalitions, one additional property referring to the induced games is needed. Indeed, average cost pricing satisfies all properties (i)–(ix) (see Moulin and Shenker, 1994) and certain reduction properties as shown in the following sections. A game theoretical solution rule is determined by the induced game. A stronger version of this property is presented in

DEFINITION 1.4. A cost sharing rule σ on a set $\Gamma \subseteq \Gamma(U)$ satisfies **covariance** (**COV**), if the following condition holds:

If $(N, C, q) \in \Gamma$ and $\alpha > 0$, $\beta \in \mathbb{R}^N$ are such that there is $(N, \tilde{C}, \tilde{q}) \in \Gamma$ with

$$\tilde{C}(\tilde{q}(S)) = \alpha \cdot C(q(S)) + \beta(S) \quad \text{for } S \subseteq N \ \left(\text{i.e., } \alpha \cdot v^{C,q} + \beta = v^{\tilde{C},\tilde{q}} \right),$$

then $\sigma(N, \tilde{C}, \tilde{q}) = \alpha \cdot \sigma(N, C, q) + \beta$.

Covariance on a set of games is an intuitive property. Two games (N, v) and (N, w) which coincide up to **strategical equivalence** (i.e., $w = \alpha \cdot v + \beta$ for some $\alpha > 0$ and $\beta \in \mathbb{R}^N$) should be treated accordingly by a solution concept for games. In the context of cost sharing problems COV seems less intuitive. Nevertheless, as long as the induced game serves as an adequate description of the cost sharing problem, covariance can be interpreted as in the original game theoretical context. Note that a similar approach can be found in McLean and Sharkey (1994), who use a strong additivity assumption which applies to the games derived by cost sharing problems.

If a CSR σ on a set Γ of cost sharing problems satisfies COV, then there is a unique continuation $\bar{\sigma}$ on the set $\mathscr{G}(\Gamma)$ of games which are strategically equivalent to the induced game of some CSP in Γ. To put it more formally, let $(N, C, q) \in \Gamma$, $\beta \in \mathbb{R}^N$, and $\alpha > 0$. Then $\bar{\sigma}(\alpha \cdot v^{C,q} + \beta) = \alpha \cdot \sigma(N, C, q) + \beta$.

The solution concept $\bar{\sigma}$ on $\mathscr{G}(\Gamma)$ is called **continuation of** σ. A **solution concept** $\bar{\sigma}$ **on** a set $\mathscr{G}$ of cost games associates a vector $\bar{\sigma}(v) \in \mathbb{R}^N$ satisfying feasibility $(\sum_{i \in N} q_i(v) \geq v(N))$ with each game $(N, v) \in \mathscr{G}$. Conversely, if $\mathscr{G}(\Gamma) = \mathscr{G}$, then $\bar{\sigma}$ **induces** a CSR σ on Γ by $\sigma(N, C, q) = \bar{\sigma}(v^{C,q})$. Clearly $\bar{\sigma}$ is feasible, iff σ is feasible. Well-known properties for

a solution concept $\bar{\sigma}$ on $\mathscr{G}$ are as follows (with $(N, v), (N, w) \in \mathscr{G}$):

(a) **Pareto optimality ($\overline{\textbf{PO}}$).** $\Sigma_{i \in N} \bar{\sigma}_i(v) = v(N)$.

(b) **Equal treatment property ($\overline{\textbf{ET}}$).** $v(S \cup \{i\}) = v(S \cup \{j\})$ for $S \subseteq N \setminus \{i, j\}$ (i and j are **interchangeable**) implies $\bar{\sigma}_i(v) \leq \bar{\sigma}_j(v)$.

(c) **Null property ($\overline{\textbf{NP}}$).** $v(S \cup \{i\}) = v(S)$ for $S \subseteq N$ (i is a **nullplayer**) implies $\bar{\sigma}_i(v) = 0$.

(d) **Anonymity ($\overline{\textbf{AN}}$).** If (N, w) arises from (N, v) by a permutation of the players then $\bar{\sigma}$ "respects this permutation."

(e) **Additivity ($\overline{\textbf{ADD}}$).** If $(N, v + w) \in \mathscr{G}$, then $\bar{\sigma}(v) + \bar{\sigma}(w) = \bar{\sigma}(v + w)$.

(f) **MM additivity ($\overline{\textbf{MMADD}}$).** $v \wedge w$, $v \vee w \in \mathscr{G}$ implies $\bar{\sigma}(v) + \bar{\sigma}(w) = \bar{\sigma}(v \wedge w) + \bar{\sigma}(v \vee w)$.

(g) **Covariance ($\overline{\textbf{COV}}$).** If $w = \alpha \cdot v + \beta$ for some $\alpha > 0$, $\beta \in \mathbb{R}^N$, then $\sigma(\alpha v + \beta) = \alpha \sigma(v) + \beta$.

Clearly the following statements for a covariant CSR σ on Γ, together with its continuation $\bar{\sigma}$, hold true:

1. $\bar{\sigma}$ satisfies $\overline{COV}$;
2. $\bar{\sigma}$ satisfies $\overline{PO}$, iff σ satisfies PO;
3. If $\bar{\sigma}$ satisfies $\overline{NP}$, then σ satisfies NP;
4. If $\bar{\sigma}$ satisfies $\overline{AN}$, then σ satisfies AN;
5. If $\bar{\sigma}$ satisfies $\overline{ADD}$, then σ satisfies ADD;
6. If $\bar{\sigma}$ satisfies $\overline{MMADD}$, then σ satisfies MMADD.

Ranking has an analogue for solution concepts, too. Indeed, many well-known solution concepts, e.g. the Shapley value and the proposed versions of nucleoli, **preserve the desirability relation** in the sense of Maschler and Peleg (1966). Preservation of desirability on $\mathscr{G}(\Gamma)$ implies RAN on Γ.

Note that the Shapley rule φ and both nucleolus rules ν and ν^* satisfy COV, PO, RAN, SC, ET, NP, and AN (see, e.g., Peleg, 1989). Moreover, the Shapley rule satisfies ADD and MMADD (see Shapley, 1953, and Dubey, 1975). For completeness reasons we present an example which shows that neither φ nor ν or ν^* satisfy MON.

EXAMPLE 1.5. Let $C(y) = 5 \cdot \sqrt{y}$, $\tilde{C}(y) = \min\{C(y), y\}$, $N = \{1, 2\}$, and $q = (9, 16)$. An easy computation shows that

$$\varphi(N, C, q) = \nu^*(N, C, q) = \nu(N, C, q) = (10, 15),$$

whereas

$$a(N, C, q) = a(N, \tilde{C}, q) = \nu(N, \tilde{C}, q) = \nu^*(N, \tilde{C}, q) = (9, 16).$$

Hence the game theoretical cost sharing rules do not satisfy MON even in the case of concave cost functions.

For every finite nonvoid subset N of U let Γ_N denote the set of cost sharing problems with agent set N. The following assertion shows the "power" of COV.

LEMMA 1.6.   *Every game (N, v) is strategically equivalent to some induced game $(N, v^{C,q})$ of a CSP (N, C, q); i.e. $\mathscr{G}(\Gamma_N) = \{(N, v) \mid v \text{ is a game}\}$.*

The proof of this lemma is constructive and several versions of it will be used.

Proof.   Let $d(v) = \max\{v(S) - v(T) \mid S, T \subseteq N\}$ be the maximal difference of worths of coalitions. Assume for simplicity reasons $N = \{1, \ldots, n\}$, take $d \geq d(v)$, and define $q, \beta \in \mathbb{R}^N$ by

$$q_i = 2^{i-1}, \beta_i = 2^{i-1} \cdot d \text{ for } i \in N.$$

Moreover, let C be defined by

$$C(y) = \max\{v(S) + \beta(S) \mid S \subseteq N \text{ and } q(S) \leq y\} \text{ for } y \geq 0.$$

Clearly C is a cost function ($C(0) = 0$ and C is nondecreasing). In order to verify that $v^{C,q} = v + \beta$ holds true, it remains to prove

$$v(S) + \beta(S) \leq v(T) + \beta(T) \text{ for } S,$$
$$T \subseteq N \text{ with } S \neq T \text{ and } q(S) \leq q(T). \tag{1}$$

By definition of q there is a unique $i \in N$ such that $i \in T \setminus S$ and $T \setminus \{1, \ldots, i\} = S \setminus \{1, \ldots, i\}$. The definition of β directly shows

$$v(S) + \beta(S) \leq \max_{R \subseteq N} v(R) + \beta(S \cup \{1, \ldots, i-1\}) \tag{2}$$

and

$$v(T) + \beta(T) \geq \min_{R \subseteq N} v(R) + \beta(T \setminus \{1, \ldots, i-1\}). \tag{3}$$

The observation

$$\min_{R \subseteq N} v(R) - \max_{R \subseteq N} v(R) + \beta(T \setminus \{1, \ldots, i-1\})$$

$$- \beta(S \cup \{1, \ldots, i-1\})$$

$$= -d(v) + \beta_i - \beta(\{1, \ldots, i-1\})$$

$$= -d(v) + d\left(2^{i-1} - \sum_{j=0}^{i-2} 2^{j-1}\right)$$

$$= -d(v) + d \geq 0$$

completes the proof. Q.E.D.

A stronger version of this result shows that $\overline{ET}$ and ET are equivalent for a covariant CSR and its continuation.

LEMMA 1.7. *For every game (N, v) there is a vector $\beta \in \mathbb{R}^N$, a demand profile $q \in \mathbb{R}^N_{\geq 0}$, and a cost function C such that $v + \beta$ is the induced game of (N, C, q), interchangeable players of v possess coinciding demands, and nullplayers possess zero demands.*

Proof. Take d as in the last proof, assume that $N = \{1, \ldots, n\}$ and assume that the equivalence classes of interchangeable players are integer intervals, i.e., if i is interchangeable with j and $j > i$, then i and j are interchangeable with k for $k \in \{i, \ldots, j\}$. This can be achieved by just renaming the players. For $j \in N$ define $q_j = 2^{i-1}$ and $\beta_j = 2^{i-1} \cdot d$ in case j is not a nullplayer, where i is the player of minimal index interchangeable with j. Moreover, put $q_j = 0$ for every nullplayer $j \in N$. The proof can be completed analogously to that of Lemma 1.6. Q.E.D.

A direct consequence of this result is

COROLLARY 1.8. *Let σ be a CSR on Γ_N satisfying COV and $\overline{\sigma}$ its continuation on $\mathscr{G}(\Gamma_N)$.*

   (i) *Then σ satisfies ET, if and only if $\overline{\sigma}$ satisfies $\overline{ET}$.*
   (ii) *Then σ satisfies NP, if and only if $\overline{\sigma}$ satisfies $\overline{NP}$.*

2. THE SHAPLEY RULE

In this section we provide axiomatizations for the Shapley rule based on the classical characterizations of the Shapley value due to Shapley (1953), Dubey (1975), and Hart and Mas-Colell (1989). The analogues to Shapley's and Dubey's results are formulated in Theorems 2.1 and 2.2.

THEOREM 2.1.   *The Shapley rule is the unique cost sharing rule on* Γ_N *satisfying ET, NP, ADD, and COV.*

THEOREM 2.2.   *The Shapley rule is the unique cost sharing rule on* Γ_N *satisfying ET, NP, MMADD, and COV.*

Proof of Theorems 2.1 *and* 2.2.   The Shapley value $\overline{\varphi}(\cdot)$ satisfies $\overline{ET}$, $\overline{NP}$, $\overline{COV}$, $\overline{ADD}$, and, thus, $\overline{MMADD}$ (see Shapley, 1953, and Dubey, 1975). Therefore the Shapley rule satisfies ET, NP, COV, ADD, and MMADD. The straightforward observation that ADD implies MMADD shows that it remains to show uniqueness in Theorem 2.2. To this end let σ be a cost sharing rule which satisfies the desired properties, i.e. ET, NP, COV, and MMADD. The continuation $\overline{\sigma}$ satisfies $\overline{COV}$, $\overline{ET}$, and $\overline{NP}$ by Corollary 1.8. In order to show that σ coincides with the Shapley rule, it is useful to verify that $\overline{\sigma}$ satisfies $\overline{MMADD}$. Indeed, let (N, v^1) and (N, v^2) be games. We repeat the construction presented in the proof of Lemma 1.6. Namely, take $d \geq d(v^1) \vee d(v^2)$, define β and q as in the mentioned proof and let C^i be the cost function for $v^i + \beta$, i.e.

$$C^i(y) = \{v^i(S) + \beta(S) \mid S \subseteq N \text{ and } q(S) \leq y\} \quad \text{for } i = 1, 2.$$

MMADD of σ and $\overline{COV}$ of $\overline{\sigma}$ imply

$$
\begin{aligned}
\overline{\sigma}&(N, v^1 \vee v^2) + \overline{\sigma}(N, v^1 \wedge v^2) \\
&= \overline{\sigma}(N, v^1 \vee v^2 + \beta) + \overline{\sigma}(N, v^1 \wedge v^2 + \beta) - 2 \cdot \beta \\
&= \overline{\sigma}(N, (v^1 + \beta) \vee (v^2 + \beta)) \\
&\quad + \overline{\sigma}(N, (v^1 + \beta) \wedge (v^2 + \beta)) - 2 \cdot \beta \\
&= \sigma(N, C^1 \vee C^2, q) + \sigma(N, C^1 \wedge C^2) - 2 \cdot \beta \\
&= \sigma(N, C^1, q) + \sigma(N, C^2, q) - 2 \cdot \beta \\
&= \overline{\sigma}(N, v^1 + \beta) + \overline{\sigma}(N, v^2 + \beta) - 2 \cdot \beta \\
&= \overline{\sigma}(N, v^1) + \overline{\sigma}(N, v^2),
\end{aligned}
$$

and, thus, $\overline{\sigma}$ satisfies $\overline{MMADD}$.

It remains to show that $\overline{\sigma}(v) = \overline{\varphi}(v)$ holds true for every game (N, v). Clearly, this is true for the **flat** game $v = 0$ by $\overline{NP}$. By $\overline{COV}$ (of $\overline{\varphi}$ and $\overline{\sigma}$) we can assume that v is monotonic, i.e. $v(S) \leq v(T)$ for $S \subseteq N$. If $v = u_T$ is the **unanimity** game of T for some $\varnothing \neq T \subseteq N$ (i.e., $u_T(S) = 1$, if $T \subseteq N; 0$, otherwise for $S \subseteq N$), then we proceed by induction on $|T|$. If $|T| = 1$, then $\overline{\sigma}(v) = \overline{\varphi}(v)$ by $\overline{COV}$, because v is strategically equivalent to the flat game. If $|T| \geq 2$, then take $i, j \in T$, $i \neq j$, observe that $v = v_{T \setminus \{i\}}$

$\wedge\ v_{T\setminus\{j\}}$, and, thus, $\overline{\sigma}(v) = \overline{\varphi}(v)$ by $\overline{ET}$, $\overline{NP}$, $\overline{MMADD}$, feasibility, and the inductive hypothesis applied to $v_{T\setminus\{i\}}$ and $v_{T\setminus\{j\}}$. We proceed by induction on $t(v) = |\{v(S) \mid S \subseteq N\}|$. If $t(v) = 1$, then v is the flat game. If $t(v) = 2$, then v is a positive multiple of the maximum of finitely many unanimity games, thus $\overline{\sigma}(v) = \overline{\varphi}(v)$ by $\overline{MMADD}$ and the fact that the minimum of finitely many unanimity games is a unanimity game. If $t(v) = t \geq 3$, then let a, b, c denote the three highest worths of coalitions:

$$c = \max\{v(S) \mid S \subseteq N\}, \qquad b = \max\{v(S) \mid v(S) < c, S \subseteq N\},$$

$$a = \max\{v(S) \mid v(S) < b, S \subseteq N\}.$$

Define three monotonic games v^1, v^2, w by

$$v^1(S) = \begin{cases} v(S), & \text{if } v(S) \neq b \\ a, & \text{otherwise} \end{cases},$$

$$v^2(S) = \begin{cases} v(S), & \text{if } v(S) \neq c \\ b, & \text{otherwise} \end{cases},$$

$$w(S) = v^1 \wedge v^2 \quad \text{for } S \subseteq N$$

and observe that $v = v^1 \vee v^2$. The inductive hypothesis can be applied to v^1, v^2, and to w, thus $\overline{MMADD}$ completes the proof.   Q.E.D.

To prove that both characterizations are axiomatizations, we present examples which show the logical independence of both sets of properties. A weighted Shapley value (see Kalai and Samet, 1988) satisfies $\overline{NP}$, $\overline{COV}$, $\overline{ADD}$, and $\overline{MMADD}$ but does not satisfy $\overline{ET}$ in general for $n \geq 2$. Therefore the corresponding CSR possesses all desired properties except ET. The CSR σ defined by

$$\sigma_i(N, C, q) = v(\{i\}) + \left(v(N) - \sum_{j \in N} v(\{j\})\right) \Big/ n,$$

where $v = v^{C,q}$ satisfies all axioms except NP for $n \geq 2$. For $n \geq 3$ the nucleolus rule does not satisfy ADD or MMADD, but it possesses all other properties. Finally, average cost pricing can be used as an example which shows the independence of COV.

Remark 2.3.   Dubey (1975) used $\overline{MMADD}$ to characterize the Shapley value on monotone simple games. A **monotone simple game** (N, v) is a game such that $v(S) \in \{0, 1\}$, $v(N) = 1$ (the game is simple) and $v(S) \leq v(T)$ for $S \subseteq T \subseteq N$ (the game is monotone). The analogue does not hold in the cost sharing context. Indeed, on the set

$$\Gamma = \{(N, C, q) \mid C \text{ is a cost function with } C(y) \in \{0, 1\}\}$$

of cost sharing problems possessing cost functions with domain $0, 1$ the Shapley rule is not uniquely determined by ET, NP, MMADD, and COV, even together with PO. To see this, define

$$\sigma_i(N, C, q) = \begin{cases} 0, & \text{if } i \text{ is a nullplayer of the induced game} \\ 1/k, & \text{otherwise,} \end{cases}$$

where k denotes the number of nonnullplayers of the induced game. Clearly σ satisfies NP, ET, and PO. MMADD and COV are not strong enough to rule out the CSR σ on this small set of cost sharing problems. One reason can be seen in the fact that the class of induced games is too small. Indeed, every induced game is not only a monotone simple game but also a weighted majority game.

In the end of this section it is shown that NP and ADD (or MMADD) can be replaced by a reduction property in the sense of Hart and Mas-Colell. Certainly Γ_N has to be replaced by some richer class of cost sharing problems, namely by $\Gamma(U)$. The next definition recalls the notions of a "reduced game" and "consistency." Moreover, the analogues for cost sharing problems are presented.

DEFINITION 2.4. Let σ be a CSR on $\Gamma(U)$ and $\bar{\sigma}$ be a solution concept on $\mathscr{G}(U) = \mathscr{G}(\Gamma(U))$.

(i) Let $(N, v) \in \mathscr{G}(U)$ and $\varnothing \neq S \subseteq N$. The $\bar{\sigma}$-**reduced game** $(S, v_{\bar{\sigma}, S})$ (on S with respect to $\bar{\sigma}$) is defined by

$$v_{\bar{\sigma}, S}(T) = v(T \cup N \setminus S) - \sum_{i \in N \setminus S} \bar{\sigma}_i(T \cup N \setminus S, v) \text{ for } \varnothing \neq T \subseteq N$$

and

$$v_{\bar{\sigma}, S}(\varnothing) = 0.$$

(ii) $\bar{\sigma}$ satisfies $\bar{\sigma}$-**consistency** ($\overline{\text{CON}}$) **in the sense of Hart-Mas-Colell**, if the restriction to S of the solution is a solution of the $\bar{\sigma}$-reduced game; i.e.,

$$\bar{\sigma}(S, v_{\bar{\sigma}, S}) = \bar{\sigma}(N, v)_S \quad \text{for every game } (N, v) \text{ of } \mathscr{G}(U).$$

(iii) For a CSP (N, C, q) and $\varnothing \neq S \subseteq N$ define the σ-**reduced function** $C_{\sigma, S} \colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ by

$$C_{\sigma, S}(y) = C(y + q(S^c))$$

$$- \min\Big\{ \sum_{i \in S^c} \sigma_i(T \cup S^c, C, q_{T \cup S^c}) \mid T \subseteq S \text{ and}$$

$$q(T) = \max\{q(R) \mid R \subseteq S, q(R) \leq y\}\Big\}$$

for $y > 0$ and $C_{\sigma, S}(0) = 0$.

Here $S^c = N \setminus S$ is the complement of S. Note that the σ-reduced function is not necessarily a cost function (monotonicity of C is not guaranteed).

(iv) The CSR σ satisfies σ-**consistency** (**CON**), if for $(N, C, q) \in \Gamma(U)$ and $\varnothing \neq S \subseteq N$ the following condition holds: If $C_{\sigma, S}$ is a cost function and for every $\varnothing \neq T \subseteq S$

$$C_{\sigma, S}(q(T)) = C(q(T) + q(S^c)) - \sum_{i \in S^c} \sigma_i(T \cup S^c, C, q_{T \cup S^c})$$

is valid, then $\sigma(S, C_{\sigma, S}, q_S) = \sigma(N, C, q)_S$.

Definition 2.4(i),(ii) is due to Hart and Mas-Colell (1989). They showed that the Shapley value is uniquely determined on $\mathscr{G}(U)$ by $\overline{CON}$ and some weak versions of $\overline{ET}$, $\overline{COV}$, and $\overline{PO}$. It is obvious that $\overline{CON}$, together with $\overline{COV}$ (on $\mathscr{G}(U)$), implies CON for the associated CSR on $\Gamma(U)$. If the reduced function $C_{\sigma, S}$ is a cost function, then the arising aggregate costs for the coalitions can be interpreted as follows. Coalition $T \subseteq S$ determines its new cost (supposing all members of S^c agree on the cost sharing rule σ) to be the total cost for the aggregate demand of itself and S^c diminished by the aggregate fee which will be paid by S^c in the new situation. It should be noted that coalition T imagines a situation in which only its own agents and the agents of S^c are present. A different reduced situation will be discussed in the following two sections.

THEOREM 2.5. *The Shapley rule is the unique cost sharing rule on $\Gamma(U)$ satisfying ET, COV, and CON.*

Proof. It suffices to show the uniqueness part. Let σ be a CSR with the desired properties. Then the continuation $\bar{\sigma}$ on $\mathscr{G} = \mathscr{G}(U)$ of σ satisfies $\overline{COV}$ and $\overline{ET}$ by Corollary 1.8. Moreover, $\bar{\sigma}$ is $\overline{PO}$ on 1-agent cost sharing problems by $\overline{COV}$. The proof is finished as soon it is verified that $\bar{\sigma}$ satisfies $\overline{CON}$ and $\overline{PO}$. In order to show $\overline{CON}$ and $\overline{PO}$ take $(N, v) \in \mathscr{G}$, $\varnothing \neq S \subseteq N$, and assume that $\bar{\sigma}(R, v)$ is Pareto optimal for every proper subset $\varnothing \neq R \subsetneq N$. First of all it is proved that there is an induced game (N, w) of some CSP $(N, \tilde{C}, q)$ which is strategically equivalent to (N, v) such that the $\bar{\sigma}$-reduced game $w_{\bar{\sigma}, S}$ is the induced game of the σ-reduced CSP $(N, \tilde{C}_{\sigma, S}, q_S)$. Indeed, assume without loss of generality that $N = \{1, \ldots, n\}$ and $S = \{n + 1 - s, \ldots, n\}$. Moreover, take $q = (2^{i-1})_{i \in N}$, β, and C as defined in the proof of Lemma 1.6, i.e. $(N, C, q) \in \Gamma(U)$ such that $u = v + \beta = v^{C, q}$. Define $\tilde{\beta}_i = 0$ for $i \in S^c$ and $\tilde{\beta}_i = 2^i \cdot d$ for $i \in S$,

where $d = d(u) + \max_{R \subseteq S} \sum_{i \in S^c} \sigma_i(R \cup S^c, C, q_{R \cup S^c})$. For $y \geq 0$ define

$$\tilde{C}(y) = \max\{w(T) \mid T \subseteq N \text{ and } q(T)$$
$$= \max\{q(R) \mid R \subseteq N, q(R) \leq y\}\},$$

where $w = u + \tilde{\beta}$. Clearly $\tilde{C}(y) = C(y)$ for $y \leq q(S^c)$ and $\tilde{C}$ is nondecreasing, thus a cost function. The straightforward observation that

$$\tilde{C}(q(T)) = C(q(T)) + \tilde{\beta}(T) \quad \text{for } T \subseteq N$$

shows that $v^{\tilde{C}, q} = w$ holds true. By construction, Pareto optimality on proper subproblems and COV of σ, the σ-reduced function $\tilde{C}_{\sigma, S}$ is a cost function and the induced game of $(S, \tilde{C}_{\sigma, S}, q_S)$ coincides with $w_{\bar{\sigma}, S}$; thus $\bar{\sigma}$ satisfies $\overline{CON}$ by $\overline{COV}$. Property $\overline{PO}$ is a direct consequence of $\overline{CON}$ applied to some proper S.   Q.E.D.

For completeness reasons examples are presented which show the logical independence of ET, COV, and CON. There is a positively weighted Shapley value on $\mathscr{G}(U)$ which does not satisfy $\overline{ET}$ in case there are at least two potential agents. (Note that the notion of positively weighted Shapley values introduced for the set of games with **fixed** player set (see Kalai and Samet, 1988) can easily be generalized to $\mathscr{G}(U)$ (see Potters and Sudhölter, 1995).) By definition weighted Shapley values satisfy $\overline{COV}$. The proof that positively weighted Shapley values satisfy $\overline{CON}$ is straightforward and skipped. In view of these considerations the associated CSR on $\Gamma(U)$ shows the independence of ET. In view of the definition it can easily be seen that average cost pricing satisfies CON, thus independence of COV is guaranteed for $|U| \geq 2$. Finally the nucleolus rule shows that CON cannot be dropped as a prerequisite of Theorem 2.5 in case $|U| \geq 3$.

3. THE NUCLEOLUS RULE

There is a characterization of the prenucleolus on the family of all games with the player set contained in an infinite universe due to Sobolev (1975). He needs a consistency property based on a certain reduced game introduced by Davis and Maschler (1965). The corresponding notion and its analogue in the cost sharing situation is content of

DEFINITION 3.1.   Let σ be a CSR on $\Gamma \subseteq \Gamma(U)$ and $\bar{\sigma}$ be a solution concept on $\mathscr{G} \subseteq \mathscr{G}(U)$.

(i) For every game (N, v), every nonvoid coalition $S \subseteq N$, and every $x \in \mathbb{R}^N$ the **reduced game** $(S, v_{S,x})$ **of** v (**at** x **with respect to** S) is defined by

$$v_{S,x}(T) = \begin{cases} 0, & \text{if } T = \varnothing \\ v(N) - x(S^c), & \text{if } T = S \text{ for } T \subseteq N \\ \min\{v(T \cup R) - x(R) \mid R \subseteq S^c\}, & \text{otherwise.} \end{cases}$$

(ii) $\bar{\sigma}$ is **consistent** (satisfies $\overline{\text{CONS}}$), if $v_{S, \sigma(v)} \in \mathscr{G}$ and $\sigma(S, v_{S, \sigma(v)}) = \sigma(N, v)_S$ for $v \in \mathscr{G}$ and $\varnothing \neq S \subseteq N$.

(iii) For every CSP (N, C, q), every nonvoid coalition $S \subseteq N$, and every $x \in \mathbb{R}^N$ the **reduced function** $C_{S,x} \colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ is defined by

$$C_{S,x}(y) = \begin{cases} \min\{C(y + q(R)) - x(R) \mid R \subseteq S^c\}, & \text{if } 0 \leq y < q(S) \\ C(y + q(S^c)) - x(S^c), & \text{if } y \geq q(S). \end{cases}$$
$$(1)$$

(Note that the reduced function, although it is nondecreasing, it need not be a cost function, since $C_{S,x}(0)$ might be negative.)

(iv) σ is **strongly consistent** (satisfies **SCONS**), if $(S, C_{S,x}, q_S) \in \Gamma$ and $\sigma(S, C_{S,x}, q_S) = x_S$, where $x = \sigma(N, C, q)$ for $(N, C, q) \in \Gamma$.

(v) σ is **consistent** (satisfies **CONS**), if for $(N, C, q) \in \Gamma$, $\varnothing \neq S \subseteq N$, and $x = (N, C, q)$ the following condition is satisfied:

If $C_{S,x}(q_i) \geq 0$ for $i \in S$, then $(S, (C_{S,x})_+, q_S) \in \Gamma$ and $x_S = \sigma(S, (C_{S,x})_+, q_S)$. (Here y_+ denotes the positive part of the real number y.)

An interpretation of CONS is similar to that of CON (see Section 2). The main difference is that a coalition T takes an "optimistic" view of the world; every coalition R of members of S^c (who agreed upon the proposal given by σ) can be taken as a coalition of partners. Therefore the total cost generated by the aggregate demand of the union $T \cup R$ can be "reduced" by the fee which will be paid by R. Moreover, it should be noted that T imagines a situation with respect to the grand coalition in which all members of S^c have already paid their fees. The "pessimistic" view of the world will be discussed in Section 4.

Remark 3.2. (i) A solution concept $\bar{\sigma}$ on $\mathscr{G}(U)$ which satisfies $\overline{\text{CONS}}$ induces a CSR σ on $\Gamma(U)$ which satisfies CONS but not necessarily SCONS, since $C_{S,x}(0) = 0$ is not necessarily true for $x = \sigma(N, C, q)$. Indeed, the prenucleolus and thus the nucleolus rule are consistent. Nevertheless, the nucleolus rule does not satisfy SCONS as examples

show. The objection that the definition of the reduced function (see (1))
should be modified by, e.g., taking the positive part of this function, can be
countered as follows. In the interpretation presented above it is hard to
justify that only certain coalitions take the optimistic view of the world and
others throw away money. Moreover, the idea that (at least) the nucleolus
rule should satisfy CONS excludes this modification as the following
example shows:

Let $N = \{1, 2, 3\}$, $q = (1, 2, 4)$, $C(y) = y^2$. The nucleolus rule ν applied
to this CSP yields $x = \nu(N, C, q) = (7, 14, 28)$.

With $S = \{1, 2\}$ the reduced function satisfies $C_{S, x}(1) = -3$, $C_{S, x}(2) = 4$,
and $C_{S, x}(3) = 21$; thus the positive part of C determines the game (S, v)
defined by

$$v(\{1\}) = v(\varnothing) = 0, \quad v(\{2\}) = 4, \quad v(S) = 21.$$

Clearly $\bar{\nu}(v) = (8, 13)/2 \neq (7, 14) = x_S$.

(ii) It should be noted that it is not known whether there is any
Pareto optimal CSR on $\Gamma(U)$ satisfying SCONS, unless $|U| \leq 2$. SCONS is
only used in the context of cost sharing problems with concave cost
functions (see Theorem 3.3).

(iii) Note that average cost pricing satisfies CONS on $\Gamma(U)$. A proof
of this property is straightforward and skipped.

For any set Γ of cost sharing problems let Γ^* denote the subset of cost
sharing problems with **concave** cost functions within Γ.

THEOREM 3.3.   *The nucleolus rule is the unique cost sharing rule on $\Gamma^*(U)$
satisfying ET, COV, and SCONS.*

Proof.   Clearly each induced game (N, v) of some CSP (N, C, q) with
concave C is **concave**:

$$v(S) + v(T) \geq v(S \cup T) + v(S \cap T) \text{ for } S, T \subseteq N.$$

Therefore the nucleolus $\bar{\nu}(v)$ belongs to the **core**

$$\mathscr{C}(v) = \left\{x \in \mathbb{R}^N \mid x(N) = v(N) \text{ and } x(S) \leq v(S) \text{ for } S \subseteq N\right\}.$$

This consideration shows that any reduced function is itself a cost function
in case ν is the CSR. Moreover, it is not necessary to distinguish whether
y exceeds or does not exceed $q(S)$ in equality (1). The first row of (1) can
be taken as definition in this special case. Therefore reducing does not
destroy concavity. Up to now we have shown that CONS and SCONS
cannot be distinguished for the nucleolus rule on cost sharing problems

with concave cost functions. By the choice of the set of cost sharing problems every reduced CSP is a member of $\Gamma^*(U)$; thus Sobolev's (1975) result shows that the nucleolus rule possesses the desired properties.

To show uniqueness let σ be any CSR with the desired properties and $\bar{\sigma}$ its continuation. By COV σ is Pareto optimal on 1-agent cost sharing problems. CONS guarantees PO in general. The proof is complete as soon as it is verified that $\bar{\sigma}(v)$ belongs to the prekernel of v for every $v \in \mathscr{G}(\Gamma^*(U))$. (Recall that the **prekernel of** a game (N, v) is the set

$$\left\{ x \in X(v) \mid s_{ij}(x) = s_{ji}(x) \quad \text{for } i, j \in N, i \neq j \right\},$$

where $s_{ij}(x) = \max\{x(S) - v(S) \mid j \notin S \ni i\}$ is the **maximal surplus from** i **over** j **at** x.) Indeed, Maschler, Peleg, and Shapley (1972) showed that the prekernel of a concave game is a singleton consisting of the prenucleolus only. If (N, C, q) is any CSP with a concave cost function and $n \geq 2$, take different agents i and j of N, take $S = \{i, j\}$, and consider the reduced CSP $(S, C_{S,x}, q_S)$, where $x = \sigma(N, C, q)$. The vector x is a member of the prekernel of $v^{C,q}$, if and only if x_S is the **standard solution** for $v = v^{S, C_{S,x}}$ for $i, j \in N$ with $i \neq j$. (Recall that $\bar{\sigma}$ is called **standard** on $\mathscr{G}$, if for every game $(N, v) \in \mathscr{G}$ with $n = 2$ $x_i = (v(N) + v(\{i\}) - v(\{j\}))/2$ is true, where $x = \sigma(v)$ and $N = \{i, j\}$.) To show that our continuation $\bar{\sigma}$ is standard, let (N, C, q) be a 2-agent CSP in $\Gamma^*(U)$ (i.e. $|U| \geq 2$ is assumed implicitly; otherwise the proof is already finished). Without loss of generality let $q_i \leq q_j$. Define $\tilde{q}_k = q_i$ for $k \in N$ and

$$\tilde{C}(y) = \begin{cases} C(y), & y \leq C(q_i) \\ C(q_i) + (y - q_i) \cdot \big(C(q(N)) - C(q_j) \big)/q_i, & y \geq C(q_i), \end{cases}$$

where $0/0 = 1$ (if $q_i = 0$) in this context. Since C is concave, $C(q_i) + C(q_j) \geq C(q(N))$, thus $\tilde{C}$ is concave. Therefore $(N, \tilde{C}, \tilde{q}) \in \Gamma^*(U)$. Moreover, $v^{C,q} + \beta = v^{\tilde{C}, \tilde{q}}$, where $\beta_i = 0$ and $\beta_j = C(q_i) - C(q_j)$. By ET $\sigma_i(N, \tilde{C}, \tilde{q})$ coincides with $\sigma_j(N, \tilde{C}, \tilde{q})$. COV and PO complete the proof of standardness.   Q.E.D.

Note that this characterization is in fact a characterization of the prekernel which can be generalized to the set of concave cost games with player sets contained in U.

Although the analogue of Lemma 1.6 does not hold (i.e., it is not true that every concave cost game coincides, up to strategical equivalence, with the induced game of some CSP with a concave cost function (see Section 4)), COV cannot be dropped, because average cost pricing satisfies ET and SCONS but it does not coincide with the nucleolus rule if U contains

different agents. Sudhölter (1993) presents examples which show that ET is independent unless U is a singleton. The Shapley rule satisfies ET and COV but SCONS fails for $|U| \geq 3$.

It should be remarked that it is not known whether CONS is strong enough to replace SCONS in Theorem 3.3. Nevertheless, the present approach shows that within this restricted family of cost sharing problems strong consistency (which seems to be more intuitive than CONS) can easily be satisfied. The general case requires CONS. SCONS is not satisfied even for average cost pricing.

THEOREM 3.4. *If U is infinite, then the nucleolus rule is the unique cost sharing rule on $\Gamma(U)$ satisfying ET, COV, and CONS.*

Proof. It is sufficient to show uniqueness. Let σ be a CSR with the desired properties and let $\bar{\sigma}$ be its continuation. Sobolev (1975) showed that $\bar{\sigma}$ has to coincide with $\bar{\nu}$, if $\bar{\sigma}$ satisfies $\overline{COV}$, $\overline{AN}$, and $\overline{CONS}$. Orshan (1993) showed that $\overline{AN}$ can be replaced by $\overline{ET}$. In view of Corollary 1.8 it suffices to show that $\bar{\sigma}$ satisfies $\overline{CONS}$. A further modification of the proof of Lemma 1.6 (which is similar to that of Theorem 2.5) shows that $\bar{\sigma}$ is consistent. Indeed, if $(N, v) \in \mathcal{G}(U)$ and S is a nonvoid coalition in N, then there is a demand profile q and a cost function C such that $v^{C, q} = v + \beta$ and $C_{S, x}$ $(x = \sigma(N, C, q))$ is a cost function. COV completes the proof.   Q.E.D.

There are examples which show the logical independence of the properties as well as the necessity of the infinity assumption (see Sudhölter, 1993).

4. THE ANTINUCLEOLUS RULE

In this section we present an axiomatization of the antinucleolus by ET, COV, and dual consistency. Moreover, it is shown that the antinucleolus rule is in the core, if the cost function is concave.

DEFINITION 4.1. Let σ be a CSR on $\Gamma \subseteq \Gamma(U)$ and $\bar{\sigma}$ be a solution concept on $\mathcal{G} \subseteq \mathcal{G}(U)$.

(i) For every game (N, v), every nonvoid coalition $\bar{\sigma} \subseteq N$, and every $x \in \mathbb{R}^N$ the **dual reduced game** $(S, {}^*v_{S, x})$ of v (**at** x **with respect to** S) is

defined by

$$*v_{S,x}(T) = \begin{cases} 0, & \text{if } T = \varnothing \\ v(N) - x(S^c), & \text{if } T = S \text{ for } T \subseteq N \\ \max\{v(T \cup R) - x(R) \mid R \subseteq S^c\}, & \text{otherwise.} \end{cases}$$

(ii) $\bar{\sigma}$ is **dual consistent** (satisfies $\overline{\text{DCONS}}$), if $*v_{S,\sigma(v)} \in \mathscr{G}$ and $\sigma(S, *v_{S,\sigma(v)}) = \sigma(N, v)_S$ for $v \in \mathscr{G}$ and $\varnothing \ne S \subseteq N$.

(iii) For every CSP (N, C, q), every nonvoid coalition $S \subseteq N$, and every $x \in \mathbb{R}^N$ the **dual reduced function** $C^*_{S,x} \colon \mathbb{R}_{\ge 0} \to \mathbb{R}$ is defined by

$$C^*_{S,x}(y) = \begin{cases} 0, & \text{if } y = 0 \\ C(q(N)) - x(S^c), & \text{if } y > \bar{y} \\ \max\{C(y + q(R)) - x(R) \mid R \subseteq S^c\}, & \text{if } 0 < y \le \bar{y}, \end{cases}$$

where $\bar{y} = \max\{q(T) \mid T \subsetneqq S\}$. (Note that the dual reduced function need not be a cost function, since $C^*_{S,x}(\bar{y})$ might be larger than $C^*_{S,x}(\epsilon + q(S))$.)

(iv) σ is **dual consistent** (satisfies **DCONS**), if for $(N, C, q) \in \Gamma$, $\varnothing \ne S \subseteq N$, and $x = \sigma(N, C, q)$ the following condition is satisfied:
If $C^*_{S,x}$ is nondecreasing, then $(S, C^*_{S,x}, q_S) \in \Gamma$ and $x_S = \sigma(S, C^*_{S,x}, q_S)$.

Note that nondecreasing in (iv) only means $C^*_{S,x}(\bar{y}) \le C(q(N)) - x(S^c)$. The main difference between CONS and DCONS is that the "optimistic" view is replaced by the "pessimistic" view of coalition $T \subseteq S$. Every coalition R of members of S^c (who agreed upon the proposal given by σ) has to be considered as possible coalition of partners of T. Therefore the total cost generated by the aggregate demand of the union $T \cup R$ can be "reduced" by the fee which will be paid by R in the "worst" case.

Note that $\overline{DCONS}$ for induced games implies DCONS for the cost sharing problems. The proof of the next theorem is similar to that of Theorem 3.4. Mainly games have to be replaced by their "duals". Recall that if (N, v) is a game, then (N, v^*), defined by $v^*(S) = v(N) - v(S^c)$ for $S \subseteq N$ is its dual.

THEOREM 4.2. *If U is infinite, then the antinucleolus rule is the unique cost sharing rule on $\Gamma(U)$ satisfying ET, COV, and DCONS.*

Proof. By duality (the antiprenucleolus coincides with the prenucleolus of the dual game) it is sufficient to show uniqueness. Therefore we only need to show that $\bar{\sigma}$ satisfies $\overline{DCONS}$. The proof of $\overline{DCONS}$ is very similar to the proof of $\overline{CONS}$ in Theorem 3.4. Therefore we skip the details and apply Sobolev's and Orshan's results. Q.E.D.

In order to show the independence of the properties again Sudhölter (1993) is referred to.

To provide the second result of this section the definition of the least core of a cost game (N, c) is recalled. The **least core of** (N, v) is the set

$$\mathscr{LC}(v) = \{ x \in X(v) \mid x(S) - v(S) \leq \max\{ y(T) - v(T) \mid \varnothing \neq T \subsetneqq N \},$$
$$\varnothing \neq S \subsetneqq N \text{ for } y \in X(v) \}$$

of preimputations which minimize the maximal nontrivial excess. As a matter of fact the nucleolus $\bar{\nu}(v)$ is a member of the least core.

LEMMA 4.3.   *If C is a concave cost function and (N, C, q) is a CSP, then the least core of the dual of its induced game is contained in its core, i.e.,*

$$\mathscr{LC}(v^*) \subseteq \mathscr{C}(v), \quad \text{where } v = v^{C, q}.$$

Proof.   Let x be any preimputation of v which does not belong to the core of v. Then there is a coalition S in N with positive excess; i.e, $v(S) - x(S) < 0$. Define $\mathscr{D} = \mathscr{D}(x)$ to be the set of maximal excess coalitions (i.e. $\mathscr{D} = \{ S \subseteq N \mid v(S) - x(S) \leq v(T) - x(T) \text{ for } T \subseteq N \}$). The set $\mathscr{D}$ possesses the "near-ring property" in the sense of Maschler, Peleg, and Shapley (1972); thus $\bar{S} = \bigcup_{S \in \mathscr{D}} S \in \mathscr{D}$. Conversely, let $\mathscr{M} = \{ S \subseteq N \mid v(S) - x(S) \geq v(T) - x(T) \text{ for } T \subseteq N \}$ denote the set of minimal excess coalitions. For $i \in \bar{S}^c$ we easily obtain

$$x_i < v(\bar{S} \cup \{i\}) - v(\bar{S}). \tag{1}$$

Claim.   For $S \in \mathscr{M}$ either $\bar{S}^c \subseteq S$ or $S \subseteq \bar{S}^c$ is true.

If $q(S) \leq q(\bar{S})$, then assume $S \cup \bar{S} \neq N$ and take $i \in (S \cup \bar{S})^c$. The fact

$$x_i < v(\bar{S} \cup \{i\}) - v(\bar{S}) \leq v(S \cup \{i\}) - v(S),$$

which is true by concavity of C and (1), implies

$$(v(S \cup \{i\}) - x(S \cup \{i\})) - (v(S) - x(S)) > 0,$$

which establishes a contradiction to the fact $S \in \mathscr{M}$. Therefore $\bar{S}^c \subseteq S$ in this case.

If $q(S) > q(\bar{S})$, then take such S that $|S|$ is minimal. If $\bar{S} \cap S \neq \varnothing$, take $i \in \bar{S} \cap S$ and observe that

$$v(S) - v(S \setminus \{i\}) \leq v(\bar{S}) - v(\bar{S} \setminus \{i\}) \quad \text{(by concavity of } C\text{)}$$

is valid; hence

$$(v(S) - x(S)) - (v(S \setminus \{i\}) - x(S \setminus \{i\}))$$
$$\leq (v(\bar{S}) - x(\bar{S})) - (v(\bar{S} \setminus \{i\}) - x(\bar{S} \setminus \{i\}))$$
$$\leq 0.$$

Thus equality holds (remember that $S \in \mathcal{M}$, $\bar{S} \in \mathcal{D}$). Minimality of S and the first part of this proof show that $q(S \setminus \{i\}) \leq q(\bar{S})$ and $\bar{S}^c \subseteq S$. If $S \cap \bar{S} = \emptyset$, then $S \subseteq \bar{S}^c$. Now the proof can be finished using Kohlberg's (1971) characterization of the nucleolus by balanced collections of coalitions which can easily be weakened for the least core (see, e.g., Sudhölter, 1997). Indeed, if y is a preimputation of v, then y belongs to the least core of v^*, if and only if $\mathcal{D}(y)$ is **weakly balanced**. Weakly balanced means that the barycenter of the grand coalition is in the convex hull of the barycenters of the maximal-excess coalitions; i.e., there are coefficients $\gamma_S \geq 0$ such that $\sum_{S \in \mathcal{D}} \gamma_S \cdot 1_S = 1_N$. By our claim x cannot be a member of the least core of v^*. Q.E.D.

This last lemma shows that the set $\mathcal{G}(\Gamma^*(U))$ of games strategically equivalent to the induced game of some CSP with concave cost function is a "small" subset of the set of all concave games with player set contained in U. (Note that concavity is closed under strategical equivalence; i.e., the game is concave, iff every game, which is strategically equivalent to the initial game, is concave.) This can be seen with the help of examples. There is a concave game such that the least core of its dual does not even intersect the core of the game (see Sudhölter, 1997, Example 3.2.3).

COROLLARY 4.4. *The antinucleolus of a CSP (N, C, q) with concave C is in the core of its induced cost game $v^{C,q}$.*

5. COINCIDENCE OF COST SHARING RULES

This section serves to classify the set of cost functions such that the mentioned cost sharing rules, namely the Shapley rule φ and both nucleolus rules ν and ν^*, coincide with average cost pricing a for every demand profile. We shall say that a cost function C satisfies the **coincidence property (CP) at** α for some $\alpha \geq 0$, if

$$a(N, C, q) = \varphi(N, C, q) \tag{1}$$

holds true for every agent set N with two agents and every demand profile $q \in \mathbb{R}_{\geq 0}^N$ such that the aggregate demand coincides with α, i.e. $q(N) = \alpha$.
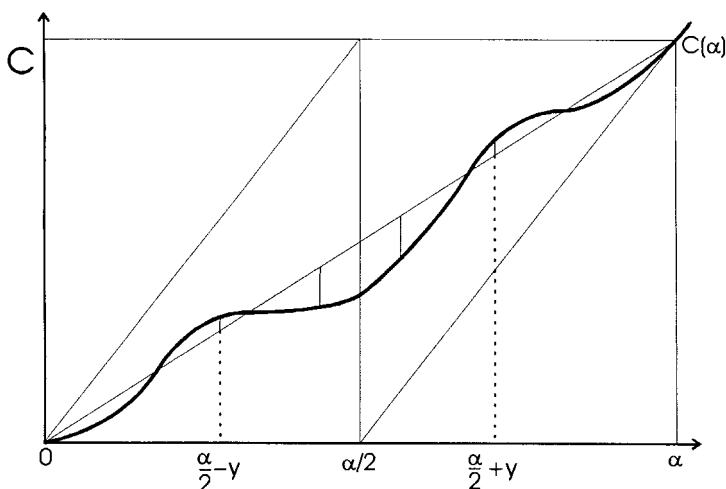
FIG. 2. C with the CP at α.

THEOREM 5.1.   (i) *A cost function satisfies CP at* α, *if and only if*

$$\alpha \cdot (C(y) - C(\alpha - y)) = (2y - \alpha) \cdot C(\alpha) \quad \text{for } 0 \le y \le \alpha. \quad (2)$$

(ii) *If C satisfies CP at* α *and* (N, C, q) *is a CSP with aggregate demand* α, *then*

$$\nu(N, C, q) = \nu^*(N, C, q) = \varphi(N, C, q) = \alpha(N, C, q).$$

The typical shape of a cost function satisfying CP is sketched in Fig. 2. Note that in assertion (ii) of this theorem $n = 2$ is **not** assumed. The nucleolus rule, the antinucleolus rule, average cost pricing, and the Shapley rule coincide for any cost sharing problem with a cost function satisfying CP at the aggregate demand of the agents.

Proof of Theorem 5.1.   Let C be any cost function. For $\alpha = 0$ both assertions are trivially satisfied, thus $\alpha > 0$ is assumed from now on. Take any y with $0 < y < \alpha$ and define $N = \{1, 2\}$, $q = (y, \alpha - y)$. Standardness of the Shapley value φ shows that

$$\varphi_1(N, C, q) = (C(q_1) + C(q(N)) - C(q_2))/2$$
$$= (C(y) + C(\alpha) - C(\alpha - y))/2.$$

Average cost pricing yields

$$a_1(N, C, q) = (q_1/q(N)) \cdot C(\alpha) = y/\alpha \cdot C(\alpha).$$

This shows that φ and a coincide on (N, C, q), if and only if

$$(C(y) + C(\alpha) - C(\alpha - y))/2 = y/\alpha \cdot C(\alpha),$$

which is equivalent to (2). For $y \in \{\alpha, 0\}$ equality (2) is trivially satisfied.

In order to prove (ii) let (N, C, q) be any CSP such that $q(N) = \alpha$ and C satisfies CP at α. Let $x \in \mathbb{R}^N$ abbreviate $a(N, C, q)$; i.e., $x_i = (q_i/\alpha) \cdot C(\alpha)$ for $i \in N$. The explicit formula for the Shapley value (see Shapley, 1953) yields

$$\varphi_i(N, C, q)$$

$$= \sum_{i \in S \subseteq N} \frac{(s-1)! \cdot (n-s)!}{n!} (C(q(S)) - C(q(S \setminus \{i\})))$$

$$= \sum_{i \in S \subseteq N} \frac{(s-1)! \cdot (n-s)!}{n!}$$

$$\times \frac{(C(q(S)) - C(q(S \setminus \{i\})) + C(q(S^c \cup \{i\})) - C(q(S^c)))}{2}$$

(by taking "complements")

$$= \sum_{i \in S \subseteq N} \frac{(s-1)! \cdot (n-s)!}{n!}$$

$$\times \frac{(2q(S) - \alpha) + (2q(S^c \cup \{i\}) - \alpha)}{2 \cdot \alpha} \quad \text{(by (2))}$$

$$= \sum_{i \in S \subseteq N} \frac{(s-1)! \cdot (n-s)!}{n!} \frac{q_i}{\alpha} \cdot C(\alpha)$$

$$= (q_i/\alpha) \cdot C(\alpha)$$

$$\cdot \sum_{s=1}^{n} \frac{(s-1)! \cdot (n-s)!}{n!} \binom{n-1}{s-1} \quad \text{(by "counting subsets")}$$

$$= x_i.$$

For the nucleolus rules ν and ν^* Kohlberg's approach is used. At x the excess $x(S) - v(S)$ coincides with that of S^c, where $v = v^{C,q}$ (by (2)). Therefore the balancedness criteria show that both nucleolus rules coincide with average cost pricing. Q.E.D.

COROLLARY 5.2.   *A cost function satisfying CP at α is continuous, if it is restricted to $\{y \mid 0 \leq y \leq \alpha\}$.*

Proof.   This assertion can be shown with the help of (2) and monotonicity of a cost function C. If C satisfies CP at α then the slope of C is bounded from below by 0 by definition. Equality (2) shows that the slope is bounded from above by $2 \cdot C(\alpha)$, whence C is restricted to the interval $[0, \alpha]$; i.e, $(C(y) - C(z))/(y - z) \leq 2 \cdot C(\alpha)$ for $0 \leq z < y \leq \alpha$. Indeed,

$$\frac{(C(y) - C(z))}{y - z} + \frac{(C(\alpha - z) - C(\alpha - y))}{(\alpha - z) - (\alpha - y)}$$

$$= \frac{(C(y) - C(\alpha - y)) - (C(z) - C(\alpha - z))}{y - z}$$

$$= 2 \cdot C(\alpha) \quad (\text{by } (2))$$

and both summands are nonnegative.   Q.E.D.

For $\alpha \geq 0$ the proposed cost sharing rules (Shapley rule φ, both nucleolus rules ν and ν^*, and average cost pricing α) can be characterized on the set Γ of cost sharing problems with cost functions satisfying CP at α, where α is the aggregate demand. Indeed, Moulin and Shenker (1994) showed that average cost pricing is the unique CSR satisfying Pareto optimality, separable costs, and monotonicity on sets of cost sharing problems (N, C, q), which contain $(N, \tilde{C}^q, q)$ and $(N, \tilde{C}^q \wedge C, q)$, where $\tilde{C}^q$ is the "constant returns" cost function given by $\tilde{C}(y) \cdot q(N) = y \cdot C(q(N))$. Since Γ possesses this property by (2), in view of Theorem 5.1 the same result holds true for φ, ν, and ν^* on Γ. Moreover, if $|U| \geq 2$, the logical independence of PO, SC, and MON can be verified by considering the following cost sharing rules. The CSR, which assigns $\alpha_i(N, C, q) + (C(q_i) - \tilde{C}_i^q(q_i))_+$ to agent i for every CSP (N, C, q) of Γ satisfies SC and MON but is not Pareto optimal. The "equal split" solutions shows the independence of SC. Finally, the CSR which assigns

$$C(q_i) + (1/n) \cdot \left(C(q(N)) - \sum_{j \in N} C(q_j) \right)$$

to agent i for every CSP (N, C, q) of Γ satisfies PO and SC but is not monotonic.

A cost function C possesses the **global coincidence property** (**GCP**), if C satisfies CP at every $\alpha \in \mathbb{R}_{\geq 0}$. The set of cost functions satisfying GCP can be characterized. I am indebted to A. Sobolev who found the elementary proof of the following

THEOREM 5.3. *A cost function C satisfies GCP, if and only if C is a parabola or if it is linear; i.e., there are nonnegative real numbers A, B such that*

$$C(y) = A \cdot y^2 + B \cdot y^2 \quad \text{for } y \geq 0. \tag{3}$$

Proof. Every function $C: \mathbb{R}_{\geq 0} \to \mathbb{R}$ given by (3) is a cost function. The proof that C satisfies (2) for every α and $0 \leq y \leq \alpha$ is straightforward and skipped. For the converse let C be a cost function satisfying GCP. By (2) we have

$$z \cdot (C(z + x) - C(z - x)) = x \cdot C(2z) \tag{4}$$

for $0 \leq z \leq x$, which can be seen by applying (2) to $\alpha = 2z$ and $y = z + x$. Define

$$\Delta = \Delta_n = 1/2n, \quad z = k \cdot \Delta_n, \quad x = m \cdot \Delta_n, \quad y_p = C(p \cdot \Delta_n)$$

for $n, p, k, m \in \mathbb{N}$ with $m \leq k$. By (4) we come up with

$$k \cdot (y_{k+m} - y_{k-m}) = m \cdot y_{2k}. \tag{5}$$

The following recursive formulae

$$y_{2k} = k^2 \cdot (y_3 - y_2 - y_1) + k(2y_2 + y_1 - y_3) \tag{6}$$

and

$$y_{2k-1} = (k - 1)^2 \cdot (y_3 - y_2 - y_1) + (k - 1) \cdot y_2 + y_1 \tag{7}$$

are obviously valid for $k = 1, 2$. By induction on k it can be shown that (6), (7) are generally true. Indeed, for $k \geq 3$ first of all the application of (5) with $m = 1$ shows (6). Second, (5) applied to $m = k - 1$ shows (7). The necessary computations are straightforward and skipped.

With $\eta = 2y_2 + y_1 - y_3$ and $\rho = y_3 - y_2 - y_1$ we obtain

$$C(1) = C(2n \cdot \Delta) = y_{2n} = n^2 \cdot \rho + n \cdot \eta,$$

$$C(2) = C(4n \cdot \Delta) = y_{4n} = 4n^2 \cdot \rho + 2n \cdot \eta \quad \text{(both by (6))}$$

thus

$$n^2\rho = (C(2) - 2 \cdot C(1))/2$$

and

$$n\eta = (4 \cdot C(1) - C(2))/2.$$

The observation

$$C(k/n) = C(2k/2n) = C(2k \cdot \Delta) = k^2 \cdot \rho + k \cdot \eta \quad \text{(by (6))}$$
$$= (k/n)^2 \cdot n^2 \cdot \rho + (k/n) \cdot n \cdot \rho$$
$$= ((C(2) - 2 \cdot C(1))/2) \cdot (k/n)^2$$
$$+ ((4 \cdot C(1) - C(2))/2) \cdot (k/n)$$

for $k, n \in \mathbb{N}$ shows that $C(y) = Ay^2 + By$ for any rational number $y \geq 0$, where

$$A = (C(2) - 2 \cdot C(1))/2, \quad B = (4 \cdot C(1) - C(2))/2.$$

Since C is a cost function (nondecreasing and nonnegative), A and B are nonnegative. Corollary 5.2, i.e. continuity of C, shows that C possesses the desired properties.   Q.E.D.

Remark 5.4. If (N, C, q) is a cost sharing problem for which average cost pricing a, the nucleolus rule ν, the anti-nucleolus rule ν^*, and the Shapley rule φ coincide, then the Shapley value $\overline{\varphi}$, the prenucleolus $\overline{\nu}$, and the antinucleolus $\overline{\nu}^*$ coincide when applied to the induced game $v^{C,q}$ (by definition). Conversely, if (N, v) is a game satisfying $\overline{\varphi}(N, v) = \overline{\nu}(N, v) = \overline{\nu}^*(N, v)$, then it can be shown that there is a cost sharing problem (N, C, q) such that its induced game is strategically equivalent to (N, v) and average cost pricing coincides with the Shapley rule (hence with the nucleolus rules) when applied to (N, C, q). Indeed, we can assume that $\overline{\varphi}(N, v) = \overline{\nu}(N, v) = \overline{\nu}^*(N, v) = (0, \dots, 0)$ by $\overline{COV}$. Let q and C be chosen as in the proof of Lemma 1.6. This lemma and the definition of a directly show that (N, C, q) possesses the desired properties. In view of these facts a possible characterization of all cost sharing problems, for which average cost pricing, the nucleolus rules, and the Shapley rule coincide, **requests and implies** a characterization of all *TU* games, for which the Shapley value, the prenucleolus, and the antinucleolus coincide. Of course, this author cannot supply any characterization of the foregoing type.

6. SUMMARIZING DIAGRAMS

Table I shows the properties of the cost sharing rules discussed in this paper. A "$-$" means that the corresponding CSR does not satisfy this property, at least on the set of all cost sharing problems with the agent set contained in some universe U of more than two members. The symbols

TABLE I
Properties

	a	φ	ν	ν^*
PO	+	+	+	+
RAN	+	+	+	+
SC	+	+	+	+
ET	+	+	+	+
NP	+	+	+	+
AN	+	+	+	+
"common properties"				
	a	φ	ν	ν^*
MON	+	−	−	−
ADD	+	+	−	−
MMADD	+	+	−	−
CON	+	+	−	−
CONS	+	−	+	−
SCONS	−	−	−	−
DCONS	+	−	−	+
COV	−	+	+	+

"+" and "$\oplus$" mean that the corresponding property is satisfied or belongs to the axiomatization respectively.

With a finite subset N of U such that $n \geq 3$ and $\alpha > 0$ the following three tables summarize the presented axiomatizations on Γ_N, $\Gamma(U)$, $\Gamma^*(U)$, and on the set Γ of cost sharing problems (N, C, q) with cost functions C

TABLE II
On Γ_N

On Γ_N	φ		a
	2.1	2.2	M–S
ET	$\oplus$	$\oplus$	+
NP	$\oplus$	$\oplus$	+
ADD	$\oplus$	+	+
MMADD	+	$\oplus$	+
COV	$\oplus$	$\oplus$	−
SC	+	+	$\oplus$
PO	+	+	$\oplus$
MON	−	−	$\oplus$

TABLE III
On $\Gamma(U)$

On $\Gamma(U)$	ν		ν^*	φ
	33.3	3.4	4.2	2.5
ET	$\oplus$	$\oplus$	$\oplus$	$\oplus$
CON	$-$	$-$	$-$	$\oplus$
CONS	$+$	$\oplus$	$-$	$-$
SCONS	$\oplus$	$-$	$-$	$-$
DCONS	$-$	$-$	$\oplus$	$-$
COV	$\oplus$	$\oplus$	$\oplus$	$\oplus$
	on $\Gamma^*(U)$		U infinite	

satisfying the coincidence property at α, where $\alpha = q(N)$ is the aggregate demand of the agents. The abbreviation "M-S" indicates that this axiomatization is due to Moulin and Shenker (1994). The numbers refer to the theorems or to the sections where these results are proved. Table II treats cost sharing rules on cost sharing problems Γ_N with a fixed agent set.

Although Theorems 2.1 and 2.2 also hold for $\Gamma(U)$, Table III neither mentions this fact nor repeats the characterization of average cost pricing. These axiomatizations are strongly based on "varying sets of agents," i.e. on consistency principles (see Sections 2–4).

Table IV presents the common axiomatizations on the set Γ of cost sharing problems with cost function satisfying the coincidence property at the aggregate demand of the agents.

ACKNOWLEDGMENT

TABLE IV
Coincidence

On Γ	φ	ν	ν^*	a
		Section 5		M-S
PO	$\oplus$	$\oplus$	$\oplus$	$\oplus$
SC	$\oplus$	$\oplus$	$\oplus$	$\oplus$
MON	$\oplus$	$\oplus$	$\oplus$	$\oplus$

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