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THE POSITIVE PREKERNEL OF A COOPERATIVE GAME

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The positive prekernel, a solution of cooperative transferable utility games, is introduced. We show that this solution inherits many properties of the prekernel and of the core, which are both sub-solutions. It coincides with its individually rational variant, the positive kernel, when applied to any zero-monotonic game. The positive (pre)kernel is a sub-solution of the reactive (pre)bargaining set. We prove that the positive prekernel on the set of games with players belonging to a universe of at least three possible members can be axiomatized by non-emptiness, anonymity, reasonableness, the weak reduced game property, the converse reduced game property, and a weak version of unanimity for two-person games.

1. Introduction

The positive prekernel is a set-valued solution of cooperative transferable utility games. Its definition is strongly related to the definition of the prekernel. A preimputation belongs to the prekernel of a game, if for distinct players i and j the maximum surplus of i over j coincides with that of j over i. The only difference that occurs in the definition of the **positive** prekernel is that the maximum surplus is replaced by its **positive** part. Therefore, the positive prekernel is a supersolution of both, the prekernel and the core, thus it is a non-empty supersolution of the core.

The core is regarded as one of the most intuitive solution concepts for cooperative games. The core elements of a game may be considered as outcomes which should not be rejected. If, on the contrary, it is recommended to reject any proposal outside the core, then, from this normative point of view, the solution should not be applied to any game with an empty core. It is the aim of the present paper to show that the positive prekernel is a non-empty extension of the core which is intuitively related to the core. In literature, "core-like" solutions like the prenucleolus [see Sobolev (1975)] and the prekernel [see Peleg (1986)] are axiomatized.

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A further solution, the intersection of the kernel and the core, has an interesting geometric characterisation [see Maschler *et al.* (1979)] and an axiomatization [see Peleg (1989)]. Therefore, it seems to be rather natural to consider a non-empty extension of the core which contains both, the core and the prekernel. Moreover, its characterisation by plausible properties (see Sec. 7) indicates the intuitive relation between the positive prekernel and the core.

The positive prekernel has all characterising properties of the prekernel except the equal treatment property. It especially satisfies the converse reduced game property and the null player property. That may be regarded as an advantage over the prebargaining sets mentioned below which do not have these properties. Moreover, it is a sub-solution of the prebargaining set and even of the reactive prebargaining set in the sense of Granot and Maschler (1997) (see Sec. 5). In the special case of the market game discussed in Maschler (1976) the positive kernel coincides with the bargaining set. Orshan[a] (1994) showed that every non-symmetric prekernel is a sub-solution of the positive prekernel.

Our main results (see Sec. 7) show that the positive prekernel has axiomatizations that are similar to an axiomatization of the core of market games [see Peleg (1989)]. The positive prekernel is the only known solution that is non-empty for every game, contains the core, and has an intuitive axiomatization.

The paper is organised as follows: In Sec. 2, the notation and some definitions are presented.

The positive prekernel of a game is not necessarily individually rational. However, in Sec. 3 it turns out that it is reasonable. A preimputation is reasonable, if it assigns to every player at least her minimal and at most her maximal marginal contribution. The first condition of this property (called reasonableness from below), which is weaker than individual rationality, implies the null player property. Moreover, it is shown that the positive kernel, the individually rational modification which resembles the relation between the kernel and the prekernel, coincides with the positive **pre**kernel on the class of zero-monotonic games.

In Sec. 4 it is shown that a preimputation belongs to the positive prekernel, if and only if there is a preimputation of the prekernel which yields the same positive part of the excess of every coalition. Like the prekernel, the positive prekernel of a game is a finite union of compact convex polytopes. As a correspondence on games with fixed set of players, it is upper hemi continuous and need not be lower hemi continuous.

In Sec. 5 it is shown that the positive prekernel coincides with the reactive bargaining set in the sense of Granot and Maschler (1997) for both the seven-person projective game and a five-person market game. Moreover, an example of a game with a non-empty core is presented, in which the positive prekernel is strictly

[a]He did not explicitly mention the positive prekernel and, as far as we know, this solution was not mentioned in literature up to now. However, M. Maschler introduced the expression "positive kernel" in discussions.

placed between the reactive prebargaining set and the union of the core and the prekernel. In this example, the reactive prebargaining set is a proper subset of the prebargaining set.

In Sec. 6 it is proved that the positive prekernel satisfies the reduced game property and its converse.

Section 7 presents two characterisations of the positive prekernel on the set of games with player set contained in some universe of at least three members. This solution concept is uniquely determined by non-emptiness, anonymity, reasonableness, the weak reduced game property, the converse reduced game property, and weak unanimity for two-person games (a solution concept satisfies this last property, if it contains the set of all imputations for every two-person game). If covariance under strategic equivalence is added, then we can replace reasonableness by some weaker property which resembles individual rationality in an obvious way.

In Sec. 8 the logical independence of the axioms of the first characterisation is proved.

2. Notation and Definitions

A *cooperative game with transferable utility*, a *game*, is a pair (N, v), where N is a finite non-void set and

$$v : 2^N \to \mathbb{R}, \quad v(\emptyset) = 0$$

is a mapping. Here $2^N = \{S \subseteq N\}$ is the set of *coalitions* of (N, v).

If (N, v) is a game, then N is the *grand* coalition or the set of *players* and v is called *coalitional function* of (N, v).

The set of *feasible payoff vectors* of (N, v) is denoted by

$$X(N, v) = \{x \in \mathbb{R}^N | x(N) \leq v(N)\},$$

whereas

$$\mathcal{I}^*(N, v) = \{x \in \mathbb{R}^N | x(N) = v(N)\}$$

is the set of *preimputations* of (N, v) (*Pareto optimal* feasible payoffs of (N, v)) and

$$\mathcal{I}(N, v) = \{x \in \mathcal{I}^*(N, v) | x_k \geq v(\{k\}) \quad \forall\, k \in N\}$$

is the set of *imputations* (*individually rational* preimputations) of (N, v). Here

$$x(S) = \Sigma_{i \in S} x_i \quad (x(\emptyset) = 0)$$

for each $x \in \mathbb{R}^N$ and $S \subseteq N$. Additionally, let x_S denote the restriction of x to S, i.e.

$$x_S = (x_i)_{i \in S} \in \mathbb{R}^S.$$

For disjoint coalitions $S, T \subseteq N$ and $x \in \mathbb{R}^N$ let $(x_S, x_T) = x_{S \cup T}$.

A *solution* σ *on* a set Γ of games is a mapping that associates with every game $(N, v) \in \Gamma$ a set $\sigma(N, v) \subseteq X(N, v)$.

If $\bar{\Gamma}$ is a subset of Γ, then the canonical restriction of a solution σ on Γ is a solution on $\bar{\Gamma}$. We say that σ is a solution on $\bar{\Gamma}$, too. If Γ is not specified, then σ is a solution on every set of games.

More notation will be needed. Let (N, v) be a game and $x \in \mathbb{R}^N$. The *excess of* a coalition $S \subseteq N$ at x is the real number

$$e(S, x, v) = v(S) - x(S)\,.$$

For distinct players $i, j \in N$, let $\mathcal{T}_{ij} = \{S \subseteq N | j \notin S \ni i\}$ be the set of coalitions containing i and not containing j and let

$$s_{ij}(x, v) = \max\{e(S, x, v) | S \in \mathcal{T}_{ij}\}$$

denote the *maximum surplus of* i *over* j *at* x. The *core* of (N, v) is the set

$$\mathcal{C}(N, v) = \{x \in X(N, v) | e(S, x, v) \leq 0 \quad \forall\ S \subseteq N\}$$

of feasible payoff vectors which generate non-positive excesses. The *prekernel* of (N, v) is the set

$$\mathcal{K}^*(N, v) = \{x \in \mathcal{I}^*(N, v) | s_{ij}(x, v) \leq s_{ji}(x, v) \quad \forall i, j \in N \text{ with } i \neq j\}$$

of preimputations that *balance* the maximum surpluses of the pairs of players.

Now we are able to present the definition of the positive (pre)kernel.

Definition 2.1. The **positive prekernel** of a game (N, v) is the set

$$\mathcal{K}^*_+(N, v) = \{x \in \mathcal{I}^*(N, v) | s_{ij}(x, v) \leq (s_{ji}(x, v))_+ \quad \forall\ i, j \in N \text{ with } i \neq j\}\,,$$

where $r_+ = \max\{r, 0\}$ denotes the positive part of a real number r.

Note that the positive prekernel of a game contains both, the core and the prekernel of the game, by definition. In Sec. 5 it is shown that the inclusion may be proper.

Some intuitive and well-known properties of a solution σ on a set Γ of games are as follows:

(1) σ satisfies *anonymity* (AN), if for each $(N, v) \in \Gamma$ and each bijective mapping $\tau : N \to N'$ with $(N', \tau v) \in \Gamma$

$$\sigma(N', \tau v) = \tau(\sigma(N, v))$$

holds (where $(\tau v)(T) = v(\tau^{-1}(T))$, $\tau_j(x) = x_{\tau^{-1}j}(x \in \mathbb{R}^N,\ j \in N',\ T \subseteq N')$).
In this case (N, v) and $(N', \tau v)$ are *isomorphic* games.

(2) σ satisfies *covariance under strategic equivalence* (COV), if for $(N, v), (N, w) \in \Gamma$ with $w = \alpha v + \beta$ for some $\alpha > 0, \beta \in \mathbb{R}^N$

$$\sigma(N, w) = \alpha\sigma(N, v) + \beta$$

holds. The games v and w are called *strategically equivalent.*

(3) σ satisfies *non-emptiness* (NE), if $\sigma(N, v) \neq \emptyset$ for $(N, v) \in \Gamma$.

(4) σ satisfies *Pareto optimality* (PO), if $\sigma(N, v) \subseteq \mathcal{I}^*(N, v)$ for $(N, v) \in \Gamma$.

Remark 2.1.

(1) It is well-known [see e.g. Davis and Maschler (1965) and Peleg (1986)] that both the prekernel as well as the core (restricted, of course, to games with a non-empty core) satisfy all the above properties.
(2) The positive prekernel satisfies anonymity, covariance, non-emptiness and Pareto optimality. Indeed, AN, PO and COV are immediate consequences of the definition. The NE of the prekernel implies the NE of the positive prekernel.
(3) For every two-person game (N, v), the positive prekernel is either the prekernel, i.e. consists of the **standard solution** x^v (defined by

$$x_i^v = (v(\{i\}) - v(N \setminus \{i\}) + v(N))/2 \ \forall i \in N)$$

only, or it coincides with the core of the game.

3. Individual Rationality and the Positive Kernel

In the definition of the positive prekernel, individual rationality is not required. However, it is possible to relax individual rationality adequately. Indeed, a solution σ on a set Γ of games is said to satisfy *reasonableness from below* (REASB), if

$$x_i \geq \min\{v(S \cup \{i\}) - v(S) | S \subseteq N \setminus \{i\}\} = d_i^{\min}(N, v) \tag{3.1}$$

for $i \in N, (N, v) \in \Gamma$, and $x \in \sigma(N, v)$. We say that σ satisfies *reasonableness* (on both sides) (REAS), if in addition to Eq. (3.1), it satisfies *reasonableness from above*, i.e. if

$$x_i \leq \max\{v(S \cup \{i\}) - v(S) | S \subseteq N \setminus \{i\}\} = d_i^{\max}(N, v) \tag{3.2}$$

holds. A payoff vector that satisfies (3.1) and (3.2) is also called *reasonable*. With the help of assertion (3.2), Milnor (1952) defined his notion of reasonableness.

Lemma 3.1. *The positive prekernel satisfies* REAS.

Proof. Let $x \in \mathcal{I}^*(N, v)$ be any preimputation. If $x_k < d_k^{\min}(N, v)$ for some $k \in N$, then $e(\{k\}, x, v) > 0$ and $e(S, x, v) < e(S \cup \{k\}, x, v)$ for any coalition $S \subseteq N \setminus \{k\}$. Therefore, player k is a member of any coalition T of maximal excess (which is positive). Moreover, by Pareto optimality, the coalition T is a proper sub-coalition of N and, hence, $s_{kl}(x, v) = e(T, x, v) > (s_{lk}(x, v))_+$ for $l \in N \setminus \{k\}$. Thus the positive prekernel satisfies REASB.

In order to show the remaining property, we now assume that there is a player $l \in N$ with $x_l > d_l^{\max}(N, v)$. Then $e(N \setminus \{l\}, x, v) > 0$, because x is a feasible payoff vector, and $e(S, x, v) > e(S \cup \{l\}, x, v)$ for $S \subseteq N \setminus \{l\}$. Thus $s_{kl}(x, v) > (s_{lk}(x, v))_+$ for any member k of a coalition of maximal excess. □

Analogously to the prekernel, the positive prekernel has an individually rational variant. We recall that the *kernel* of a game (N, v) is the set

$$\mathcal{K}(N, v) = \{x \in \mathcal{I}(N, v) | s_{ij}(x, v) \leq s_{ji}(x, v) \text{ or } x_j = v(\{j\}) \quad \forall\, i, j \in N,\ i \neq j\}\,.$$

Analogously, we define the *positive kernel* of a game (N,v) to be the set

$$\mathcal{K}_+(N,v)=\{x\in\mathcal{I}(N,v)|s_{ij}(x,v)\le(s_{ji}(x,v))_+ \text{ or } x_j=v(\{j\})\quad\forall\, i,j\in N,\ i\ne j\}\,.$$

It is well-known [see Maschler *et al.* (1979)] that the prekernel and the kernel coincide, whenever the game is *zero-monotonic*. Recall that a game (N,v) is called *zero-monotonic*, if the minimal marginal contribution of each player $i\in N$ is attained by the single player coalition, i.e. if $d_i^{\min}(N,v)=v(\{i\})\ \forall\, i\in N$. Note that a *superadditive* game (N,v), i.e. a game satisfying $v(S\cup T)\le v(S)+v(T)$ whenever the coalitions S and T are disjoint, is always zero-monotonic.

Remark 3.1. If (N,v) is a zero-monotonic game, then its positive prekernel coincides with its positive kernel.

Proof. By REASB, $\mathcal{K}_+^*(N,v)\subseteq\mathcal{K}_+(N,v)$. To show the other inclusion, let $x\in\mathcal{K}_+(N,v)$. If $s_{ij}(x,v)>(s_{ji}(x,v))_+$, then $x_j=v(\{j\})$. Let $S\subseteq N$ be a coalition of maximal excess. Then i must be a member of S, because otherwise

$$s_{ji}(x,v)\ge e(S\cup\{j\},x,v)\ge e(S,x,v)\ge s_{ij}(x,v)\,.$$

Let $|S|$ be of maximal size. Then $S\ne N$ by Pareto optimality of x. Moreover, $x_k>d_k^{\min}(N,v)=v(\{k\})$ for every $k\in N\setminus S$ by maximality of S. Take $k\in N\setminus S$ and a coalition $T\in\mathcal{T}_{ki}$ attaining $s_{ki}(x,v)$. The observation

$$s_{ji}(x,v)\ge e(T\cup\{j\},x,v)\ge e(T,x,v)\ge s_{ik}(x,v)\ge s_{ij}(x,v)$$

establishes the required contradiction. □

4. Basic Properties

In this section, we prove that every preimputation of the positive prekernel of a game can be "supported" (in the sense of Theorem 4.1) by some member of the prekernel of the game. Moreover, we show that the positive prekernel, when restricted to the set of games with a fixed player set, is upper hemi continuous.

Theorem 4.1. *If* (N,v) *is a game, then*

$$\mathcal{K}_+^*(N,v)=\{y\in\mathbb{R}^N|\exists x\in\mathcal{K}^*(N,v)\ \text{such that}\ (e(S,x,v))_+=(e(S,y,v))_+\quad\forall S\subseteq N\}.$$

Proof.

(1) $\supseteq$: This inclusion is a direct consequence of the corresponding definitions.
(2) $\subseteq$: Let $y\in\mathcal{K}_+^*(N,v)$ and define

$$X=\{x\in\mathcal{I}^*(N,v)|(e(S,x,v))_+=(e(S,y,v))_+\quad\forall S\subseteq N\}\,.$$

With this definition we have $x\in X$, if and only if the following conditions hold for any coalition $S\subseteq N$: (a) If $e(S,y,v)>0$, then $x(S)=y(S)$. (b) If $e(S,y,v)\le 0$, then $x(S)\ge v(S)$. Therefore X is a non-empty compact convex polyhedron. It remains to show that X intersects $\mathcal{K}^*(N,v)$. Let $\mathcal{N}(N,v;X)$ denote the nucleolus of (N,v) with respect to (w.r.t.) X, i.e.

$$\mathcal{N}(N,v;X) = \{x \in X | \theta(e(S,x,v)_{S\subseteq N}) \leq_{lex} \theta(e(S,z,v)_{S\subseteq N}) \quad \forall\, z \in X\},$$

where $\theta(z) \in \mathbb{R}^{2^{|N|}}$ is the vector whose components are those of $z \in \mathbb{R}^{2^N}$ arranged in non-increasing order. The set X is non-empty, compact, and convex, thus $\mathcal{N}(N,v;X)$ consists of a unique member ν by Schmeidler (1969). In order to show that $\nu \in \mathcal{K}^*(N,v)$ let $i,j \in N,\ i \neq j$. If $s_{ij}(\nu,v) > s_{ji}(\nu,v)$, then $s_{ij}(\nu,v) \leq 0$ by the definition of X. With $0 < \epsilon \leq (s_{ij}(\nu,v) - s_{ji}(\nu,v))/2$, we define $\nu^\epsilon \in \mathbb{R}^N$ by

$$\nu_k^\epsilon = \begin{cases} \nu_i + \epsilon, & \text{if } k = i \\ \nu_j - \epsilon, & \text{if } k = j \\ \nu_k, & \text{otherwise} \end{cases}$$

and obtain $\nu^\epsilon \in X$. Moreover, the fact $\theta(e(S,\nu^\epsilon,v)_{S\subseteq N}) <_{lex} \theta(e(S,\nu,v)_{S\subseteq N})$ shows the required contradiction. □

Remark 4.1. Let (N,v) be a game and $\Gamma = \{(N,u)|(N,u) \text{ is a game}\}$ be the set of games with player set N.

(1) Then $\mathcal{K}^*_+(N,v)$ is a finite union of convex polytopes.
(2) The positive prekernel on the set Γ is upper hemi continuous.
(3) If $|N| \geq 5$, then the positive prekernel on Γ is **not** lower hemi continuous.

Proof.
ad (1): There is only a finite number of sets

$$X^x = \left\{ y \in \mathcal{I}^*(N,v) \;\middle|\; \forall S,T \subseteq N : \begin{array}{c} (e(S,x,v) \geq e(T,x,v) \geq 0 \Rightarrow \\ e(S,y,v) \geq e(T,y,v) \geq 0) \quad \text{and} \\ (e(S,x,v) \leq 0 \Rightarrow e(S,y,v) \leq 0) \end{array} \right\},$$

where $x \in \mathcal{I}^*(N,v)$. If $x \in \mathcal{K}^*_+(N,v)$ then X^x is a polytope containing x and contained in $\mathcal{K}^*_+(N,v)$.

ad (2): For any coalition $S \subseteq N$ the mapping $e(S,\cdot,\cdot) : \mathbb{R}^N \times \Gamma \to \mathbb{R},\ (x,(N,u)) \mapsto e(S,x,u)$, is continuous. For distinct players $k,l \in N$, the mapping $s_{kl}(\cdot,\cdot)$: $\mathbb{R}^N \times \Gamma \to \mathbb{R}$ is the maximum of finitely many continuous mappings, thus it is continuous. Continuity remains valid, if $s_{kl}(\cdot,\cdot)$ is replaced by its positive part.

ad (3): A careful inspection of Example 1 of Stearns (1968) shows this assertion. □

5. Inclusion in the Bargaining Set

In this section, examples are presented which demonstrate some aspects concerning the relations between the "classical" bargaining set in the sense of Aumann and Maschler (1964), the reactive bargaining set [recently discussed by Granot and

Maschler (1997)], and the positive prekernel. In order to recall the definitions of the mentioned solution concepts, let (N, v) be a game and $k, l \in N$ be distinct players. An *objection* of k against l at a payoff vector x is a pair (P, y) satisfying

$$P \in \mathcal{T}_{kl}, y \in \mathbb{R}^P, y(P) = v(P), \text{ and } y \gg x_P \text{ (i.e. } y_i > x_i \quad \forall\, i \in P).$$

A *counter objection* to the objection (P, y) of k against l is a pair (Q, z) satisfying

$$Q \in \mathcal{T}_{lk}, z \in \mathbb{R}^Q, z(Q) = v(Q), z \geq x_Q, \text{ and } z_{P\cap Q} \geq y_{P\cap Q}.$$

The *prebargaining set* $\mathcal{M}^*(N, v)$ is the set of all preimputations such that any objection can be countered (i.e., no player has a *justified* objection against any other player). The *reactive prebargaining set* $\mathcal{M}_r^*(N, v)$ is the set of all preimputations with the following property: For every pair (k, l) of distinct players, there is a coalition $Q \in \mathcal{T}_{lk}$ such that any objection of k against l can be countered by using the coalition Q (i.e. player k does not have a *justified* objection against l *in the sense of the reactive bargaining set*). Thus, the reactive prebargaining set of a game is contained in the prebargaining set. Moreover, the *bargaining set* $\mathcal{M}(N, v)$ and the *reactive bargaining set* $\mathcal{M}_r(N, v)$ arise from the corresponding sets $\mathcal{M}^*(N, v)$ and $\mathcal{M}_r^*(N, v)$ by intersecting these sets with the set $\mathcal{I}(N, v)$ of imputations.

Remark 5.1. The positive (pre)kernel is a sub-solution[b] of the reactive (pre)bargaining set.

Proof. Let (N, v) be a game, $x \in \mathcal{K}_+^*(N, v)$ and k, l be distinct players in N. Then player k has an objection against player l at x, if and only if $s_{kl}(x, v) > 0$. Such an objection can be countered by any coalition attaining $s_{lk}(x, v)$, because $s_{kl}(x, v) = s_{lk}(x, v)$. The same argument shows that $\mathcal{K}_+(N, v)$ is contained in $\mathcal{M}_r(N, v)$. □

We start with two examples discussed in Secs. 3 and 4 of Granot and Maschler (1997).

Example 5.1. For the seven-person projective game $(\{1, \ldots, 7\}, v)$, which is defined by

$$v(S) = \begin{cases} 1, & \text{if } S \in \mathcal{T} \\ 0, & \text{otherwise} \end{cases},$$

where $\mathcal{T}$ is the set of all coalitions that contain one of the following three-person coalitions

$$\{1, 2, 4\}, \{2, 3, 5\}, \{3, 4, 6\}, \{4, 5, 7\}, \{1, 5, 6\}, \{2, 6, 7\} \text{ or } \{1, 3, 7\},$$

the reactive bargaining set coincides with the (pre)kernel of the game. This game is zero-monotonic, thus its positive (pre)kernel coincides with its reactive bargaining set by Remark 3.1.

[b]The solution σ^1 is a sub-solution of σ^2, if both are defined on a set Γ of games and $\sigma^1(N, v) \subseteq \sigma^2(N, v)\ \forall\ (N, v) \in \Gamma$.

Example 5.2. For the five-person market game $(\{1,\ldots,5\},v)$, defined by

$$v(S) = \min\{|S \cap \{1,2\}|,\ a|S \cap \{3,4,5\}|\}\,,$$

where $a \geq 0$, Granot and Maschler (1997) showed that the reactive bargaining set is the union of the core and of the kernel of the game. Again by zero-monotonicity the positive (pre)kernel coincides with the reactive bargaining set in this case.

The following example shows the existence of games for which the core is non-empty and for which the union of the core and the prekernel is a proper subset of the positive prekernel, which itself is a proper subset of the reactive prebargaining set. Moreover, the reactive prebargaining set is a proper subset of the bargaining set.

Example 5.3. Let $N = P \cup Q \cup R$, where $P = \{1,2\}$, $Q = \{3\}$, and $R = \{4,5,6,7,8\}$. Let (N,v) be defined by

$$v(S) = \begin{cases} 0, & \text{if } S = \emptyset \text{ or } S = N \\ 2, & \text{if } |S \cap R| = 3 \text{ and } S \cap (P \cup Q) \in \{P, Q\} \\ -1, & \text{if } S = \{i\} \cup \{j\} \text{ for some } i \in P,\ j \in R \\ -2, & \text{if } S = Q \cup T \text{ for some } T \subseteq R \text{ with } |T| \ = 2 \\ -5, & \text{if } S = \{i\} \text{ for some } i \in P \\ -40, & \text{otherwise} \end{cases}.$$

(1) Claim: $\mathcal{C}(N,v) \neq \emptyset$

As the reader may check, $(-5,-5,-10,4,4,4,4,4)$ is in the core of the game. (Indeed, the core is a singleton.)

(2) Claim: The union of the core and the prekernel is a proper subset of the positive kernel.

In order to show this claim observe that $x^1 = (-1,1,0,0,0,0,0,0)$ is not a member of the core (because $e(\{3,4,5,6\},x^1,v) = 2$) and not a member of the prekernel (because $s_{12}(x,v) > s_{21}(x,v)$). However, a coalition S satisfies $e(S,x^1,v) > 0$, iff $e(S,x^1,v) = v(S) = 2$. Therefore the two "types" of coalitions of positive excess balance the maximum surplus of distinct players i,j satisfying $\{i,j\} \neq P$. Pareto optimality of x^1 together with the fact that a coalition of positive excess either contains P or does not intersect P shows this claim.

(3) Claim: The reactive prebargaining set $\mathcal{M}^*_r(N,v)$ is a proper subset of the prebargaining set $\mathcal{M}^*$.

Let $x^2 = (0,0,0,0,0,0,-2/3,2/3)$. First, we show that x^2 does not belong to the reactive prebargaining set $\mathcal{M}^*_r(N,v)$ by verifying that player 7 has a justified objection against player 8 in the sense of the reactive bargaining set.

Precisely the coalitions $S = P \cup T \cup \{8\}$ and $S = Q \cup T \cup \{8\}$, where $T \subseteq \{4,5,6\}$ with $|T| \ = 2$, are the coalitions with non-negative excess containing 8 and not containing 7. Of course, player 7 can take every player $i \in T$ to define a justified objection against S by using the coalition $(N \setminus S) \cup \{i\}$.

In order to show that $x^2 \in \mathcal{M}^*(N,v)$, note that $(s_{ij}(x^2,v))_+ = (s_{ji}(x^2,v))_+$ for distinct players with $i,j \notin \{7,8\}$. Of course, $s_{7j}(x^2,v) \geq s_{j7}(x^2,v)$ for $j \neq 7$ and $s_{i8}(x^2,v) \geq s_{8i}(x^2,v)$ for $i \neq 8$. It remains to show that there is no player $i \neq 8$ who has a justified objection against 8 and that 7 does not have a justified objection against any player $j \neq 7$ in the sense of the prebargaining set. Every objection (S,y) of a player $i \neq 8$ against 8 using a coalition S **not** containing player 7 can be countered by the coalition $(N \setminus S) \cup \{k\}$, which has the same excess as S, where $k \in S \cap \{4,5,6\} \setminus \{i\}$. If $7 \in S$, then $e(S,x^2,v) = 8/3$. Nevertheless, there are at least two distinct players $k,l \in R \setminus \{i\}$, thus one of them improves by at most 4/3, let us say $y_k \leq x_k^2 + 4/3$. The excess of $(N \setminus S) \cup \{k\}$ is 4/3, thus this coalition can be used to counterobject. A similar argument shows that 7 does not possess a justified objection.

(4) Claim: The positive prekernel is a proper subset of the reactive prebargaining set.

Let $x^3 = (-10,-10,-20,8,8,8,8,8)$. Then

$$e(S,x^3,v) \begin{cases} = \begin{cases} 1, & \text{if } S = \{i\} \cup \{j\} \text{ for some } i \in P, j \in R \\ 2, & \text{if } S = Q \cup T \text{ for some } T \subseteq R,\ |T| = 2 \\ 5, & \text{if } S = \{i\} \text{ for some } i \in P \end{cases} \\ \leq 0, & \text{otherwise} \end{cases} \tag{5.1}$$

holds true. Players inside P (or R) do not possess justified objections in the sense of the reactive bargaining set against each other, because they are interchangeable and they are treated equally. Moreover, for $l \in N \setminus P$ there is a coalition of the "second type" [coalitions that occur in the second row of (5.1)] which contains l. This coalition can be used to counter any objection of any player $k \in P$ against l. Every objection against k can be countered using $\{k\}$. Every objection of 3 against some player $l \in R$ can be countered by using the coalition $\{1,l\}$. Finally, every coalition of the second type that does not contain l can be used to counter any objection of l against 3.

6. Reduced Game Properties

We recall the definitions of the reduced game [see Davis and Maschler (1965)], of the reduced game property and its converse [see Sobolev (1975) and Peleg (1986)].

Definition 6.1. Let (N,v) be a game, let $\emptyset \neq S \subseteq N$, and $x \in X(N,v)$. The **reduced game** w.r.t. S and x is the game $(S, v^{S,x})$ defined by

$$v^{S,x}(T) = \begin{cases} 0, & \text{if } T = \emptyset \\ v(N) - x(N \setminus S), & \text{if } T = S \\ \max\{v(T \cup Q) - x(Q) \mid Q \subseteq N \setminus S\}, & \text{otherwise} \end{cases}.$$

Definition 6.2. Let σ be a solution on a set Γ of games. Then σ satisfies the

(1) **reduced game property (RGP)**, if the following condition holds:
If $(N,v) \in \Gamma$, $\emptyset \neq S \subseteq N$, and $x \in \sigma(N,v)$, then $(S, v^{S,x}) \in \Gamma$ and $x_S \in \sigma(S, v^{S,x})$.

(2) **weak reduced game property (WRGP)**, if the following condition holds:
If $(N,v) \in \Gamma$, $\emptyset \neq S \subseteq N$, $|S| \leq 2$, and $x \in \sigma(N,v)$, then $(S, v^{S,x}) \in \Gamma$ and $x_S \in \sigma(S, v^{S,x})$.

(3) **converse reduced game property (CRGP)**, if the following condition holds:
If $(N,v) \in \Gamma$ is a game with at least two players, if $x \in \mathcal{I}^*(N,v)$, and if for every $S \subseteq N$ with two members $(S, v^{S,x}) \in \Gamma$ and $x_S \in \sigma(S, v^{S,x})$, then $x \in \sigma(N,v)$.

Note that Definition 6.2(2) is due to Peleg (1989) and that RGP implies WRGP. Furthermore, note that the prekernel and the core satisfy CRGP and RGP, if the set Γ of games is rich enough. The following lemmata show that the same properties hold in the case of the positive prekernel. If U is a set (the universe of players), then let Γ_U denote the set of games with player set contained in U.

Lemma 6.1. *The positive prekernel on* Γ_U *satisfies* RGP.

Proof. If $(N,v) \in \Gamma_U$, $x \in \mathcal{K}_+^*(N,v)$, and $\emptyset \neq S \subseteq N$, then $(S, v^{S,x})$ is a game, thus it is a game of Γ_U. Let $i, j \in S$, $i \neq j$. Then Definition 6.1 implies

$$s_{ij}(x_S, v^{S,x}) = s_{ij}(x,v)\,, \tag{6.1}$$

thus the positive prekernel satisfies RGP. □

Lemma 6.2. *The positive prekernel on* Γ_U *satisfies* CRGP.

Proof. Let (N,v) be a game and $x \in \mathcal{I}^*(N,v)$ be a preimputation. If $x \notin \mathcal{K}_+^*(N,v)$, then distinct players $i, j \in N$ exist such that $s_{ij}(x,v) > (s_{ji}(x,v))_+$, thus Eq. (6.1) implies that $x_{\{i,j\}} \notin \mathcal{K}_+^*(\{i,j\}, v^{\{i,j\}})$. □

7. A Characterisation of the Positive Prekernel

In this section, we shall assume that the universe U of players contains at least three members. We recall Peleg's (1989) notion of *unanimity for two-person games* (UTPG). A solution σ on a set Γ of games satisfies UTPG, if

$$\sigma(N,v) = \{x \in \mathcal{I}^*(N,v) | x_i \geq v(\{i\})\ \forall i \in N\}$$

holds true for every two-person game $(N,v) \in \Gamma$. This property, together with WRGP, CRGP, and individual rationality ($x \in X(N,v)$ is *individually rational*, if $x_i \geq v(\{i\})$ for every $i \in N$) can be used to axiomatize the core of the set of markets games with player set in U [see, Peleg (1989)]. If Γ contains a two-person game with an empty core, then there is no solution satisfying NE and UTPG. A weaker property will be used.

Definition 7.1. A solution σ on a set Γ of games satisfies **weak unanimity for two-person games (WUTPG)**, if

$$\sigma(N,v) \supseteq \{x \in \mathcal{I}^*(N,v) | x_i \geq v(\{i\}) \quad \forall\, i \in N\}$$

holds true for every two-person game $(N,v) \in \Gamma$.

Now we present the main result of this section.

Theorem 7.1. *The positive prekernel is the unique solution on* Γ_U *that satisfies* NE, AN, REAS, WRGP, CRGP *and* WUTPG.

The following lemmata are useful in the proof of Theorem 7.1.

Lemma 7.1. *Let* σ^1, σ^2 *be solutions on* Γ_U. *If* σ^1 *satisfies* WRGP, σ^2 *satisfies* CRGP, *and if* $\sigma^1(N,v) \subseteq \sigma^2(N,v)$ *for every game* $(N,v) \in \Gamma$ *with at most two persons, then* σ^1 *is a sub-solution of* σ^2.

Proof. It suffices to show $\sigma^1(N,v) \subseteq \sigma^2(N,v)$ $\forall (N,v) \in \Gamma$ with $|N| \geq 3$. If $x \in \sigma^1(N,v)$, then $x_S \in \sigma^1(S, v^{S,x})$ for every coalition $\emptyset \neq S \subseteq N$ with $|S| \leq 2$ by WRGP of σ^1. Therefore $x_S \in \sigma^2(S, v^{S,x})$ for these coalitions by the assumption, thus $x \in \sigma^2(N,v)$ by CRGP of σ^2. □

Lemma 7.2. *If* σ *is a solution on* Γ_U *that satisfies* NE, AN, REAS, WRGP *and* CRGP, *then* σ *is a sub-solution of the positive prekernel.*

Proof. By REASB (only Condition (3.1) is needed here), σ is Pareto optimal on one-person games. WRGP, applied to one-person reduced games, implies that σ satisfies PO. In view of Lemma 7.1 applied to $\sigma^1 = \sigma$ and $\sigma^2 = \mathcal{K}^*_+$, it suffices to show that $\sigma(N,v) \subseteq \mathcal{K}^*_+(N,v)$ for every two-person game (N,v) with $N \subseteq U$. If $\mathcal{C}(N,v) \neq \emptyset$, then it coincides with the core [see Remark 2.1 (3)], thus $\sigma(N,v) \subseteq \mathcal{K}^*_+(N,v)$ by REASB and PO in this case. Let $(N,v) \in \Gamma_U$ with $|N| = 2$ and $\mathcal{C}(N,v) = \emptyset$. Let $x = x^v \in \mathbb{R}^N$ denote the standard solution. We have to show that $\sigma(N,v) = \{x\}$.

Claim 1. $x \in \sigma(N,v)$.

Assume, on the contrary, $x \notin \sigma(N,v)$. Take $* \in U \setminus N$ (which is possible by $|U| \geq 3$), let $N = \{i,j\}$, and define $(N \cup \{*\}, w)$ by

$$w(S) = \begin{cases} 0, & \text{if } S = \emptyset \\ v(N), & \text{if } S = N \cup \{*\} \\ x(S \cap N) + \alpha, & \text{otherwise} \end{cases},$$

where $\alpha = e(\{1\}, x, v) = e(\{2\}, x, v) > 0$. With $y = (x, 0) \in \mathbb{R}^{N \cup \{*\}}$, we obtain $w(S) = y(S) + \alpha\ \forall S \neq \emptyset, N \cup \{*\}$ and $s_{kl}(y,w) = \alpha$ for distinct players $k, l \in N \cup \{*\}$, thus y is an element of the prekernel of w. (Indeed, it is well-known [see, e.g., Sudhölter (1993)] that the prekernel of a three-person game is a singleton, thus Theorem 4.1 also shows that y is the unique element of the positive prekernel.)

The reduced game $(N, w^{N,y})$ coincides with (N, v), thus $y \notin \sigma(N \cup \{*\}, w)$. Take $z \in \sigma(N \cup \{*\}, w)$ (which is possible by NE). The components of $y - z$ can be arranged in non-increasing order, i.e. there is a bijective mapping $\theta : \{1,2,3\} \to N \cup \{*\}$ such that

$$y_{\theta(1)} - z_{\theta(1)} \geq y_{\theta(2)} - z_{\theta(2)} \geq y_{\theta(3)} - z_{\theta(3)} .$$

We assume for simplicity reasons that $N \cup \{*\} = \{1,2,3\}$ and $\theta = id$, thus

$$y_1 - z_1 \geq y_2 - z_2 \geq y_3 - z_3 . \tag{7.1}$$

By REASB $y_i - \alpha \leq z_i \ \forall i = 1,2,3$. By PO $z_1 < y_1, z_3 > y_3$ and one of the inequalities of (7.1) is strict. Two cases may occur.

(1) $y_1 - z_1 > y_2 - z_2$

If $y_2 \geq z_2$, then define $(\{1,2,3\}, u)$ by

$$u(S) = \begin{cases} w(\{2,3\}) - z_2, & \text{if } S = \{3\} \\ v(N) - z_1 - 1, & \text{if } S = \{2,3\} \\ w(S), & \text{otherwise} \end{cases}$$

and observe that $u^{S,z} = w^{S,z}$ for every proper non-void sub-coalition $\emptyset \neq S \subseteq N \cup \{*\},\ S \neq N \cup \{*\}$, because

$$e(\{2,3\}, z, w) < e(\{2\}, z, w),\ e(\{3\}, z, w) \leq e(\{2,3\}, z, w) < e(\{1,3\}, z, w)$$

and $v(N) - z_1 - 1 < w(\{2,3\})$. By CRGP $z \in \sigma(\{1,2,3\}, u)$. However,

$$\begin{aligned} &d_1^{\min}(\{1,2,3\}, u) \\ &= \min_{S \subseteq \{2,3\}} u(S \cup \{1\}) - u(S) \\ &= \min\{u(\{1\}), u(\{1,2\}) - u(\{2\}), u(\{1,3\}) - u(\{3\}), u(\{1,2,3\}) - u(\{2,3\})\} \\ &= \min\{w(\{1\}), w(\{1,2\}) - w(\{2\}), w(\{1,3\}) - w(\{2,3\}) + z_2, z_1 + 1\} \\ &= \min\{\alpha + y_1, y_1, y_1 - y_2 + z_2, z_1 + 1\} > z_1 , \end{aligned}$$

which yields a contradiction to REASB.

If $y_2 \leq z_2$, then define $(\{1,2,3\}, u)$ by

$$u(S) = \begin{cases} q, & \text{if } S = \{2,3\} \\ w(S), & \text{otherwise} \end{cases} ,$$

where $q < \min\{w(\{3\}) + z_2,\ z_2 + z_3\}$, and observe that $u^{S,z} = w^{S,z}$ for every proper non-void sub-coalition $\emptyset \neq S \subseteq N \cup \{*\},\ S \neq N \cup \{*\}$, because

$$e(\{2,3\}, z, w) < e(\{2\}, z, w),\ e(\{2,3\}, z, w) \leq e(\{3\}, z, w) .$$

By CRGP $z \in \sigma(\{1,2,3\},u)$. However,

$$d_1^{\min}(\{1,2,3\},u) = \min\{\alpha + y_1, y_1, v(N) - q\} > z_1\,,$$

which yields a contradiction to REASB.

(2) $y_1 - z_1 = y_2 - z_2$

This implies $z_2 < y_2$. Therefore $s_{31}(z,w)$ is attained by $\{2,3\}$ and $s_{32}(z,w)$ is attained by $\{1,3\}$. Define $(\{1,2,3\},u)$ by

$$u(S) = \begin{cases} w(S), & \text{if } S \neq \{3\} \\ z_3 - 1, & \text{if } S = \{3\} \end{cases}$$

and observe that $u^{S,z} = w^{S,z}$ for every proper non-void sub-coalition $\emptyset \neq S \subseteq N \cup \{*\}$, $S \neq N \cup \{*\}$. By CRGP $z \in \sigma(\{1,2,3\},u)$. However,

$$d_3^{\max}(\{1,2,3\},u) = \max_{S \subseteq \{1,2\}} u(S \cup \{3\}) - u(S) = \max\{z_3 - 1, y_3, y_3 - \alpha\} < z_3\,,$$

which yields a contradiction to condition (3.2) of REAS.

Claim 2. $\sigma(N,v) = \{x\}$.

Assume, on the contrary, there exists $y \in \sigma(N,v) \setminus \{x\}$. Assume without loss of generality $N = \{1,2\}$, $3 \in U$, and $y_1 < x_1$, $y_2 > x_2$. Define $(\{1,2,3\},w)$ by $w(\{1\}) = w(\{2\}) = v(\{2\})$, $w(\{1,2\}) = v(N) + v(\{2\}) - v(\{1\})$, and $w(S \cup \{3\}) = w(S) + d$, for $S \subseteq N$, where $d = v(\{1\}) - v(\{2\}) + y_2 > y_1$. Moreover, define $z \in \mathbb{R}^{N \cup \{3\}}$ by $z_1 = z_2 = y_2$ and $z_3 = y_1$. Then

$$w^{N,z}(S) = \begin{cases} 0, & \text{if } S = \emptyset \\ 2y_2, & \text{if } S = N \\ \max\{v(\{2\}), v(\{1\}) + y_2 - y_1\}, & \text{otherwise} \end{cases},$$

thus

$$w^{N,z}(N) = v(N) + y_2 - y_1 < v(\{2\}) + v(\{1\}) + y_2 - y_1 \leq w^{N,z}(\{1\}) + w^{N,z}(\{2\})\,.$$

This last observation shows that $\mathcal{C}(N,w^{N,z}) = \emptyset$ holds true. Moreover, the fact that $w^{N,z}(\{1\}) = w^{N,z}(\{2\})$ holds shows that z_N is the standard solution of $(N,w^{N,z})$, thus $z_N \in \sigma(N,w^{N,z})$ is valid by Claim 1. The other two-person reduced games $(\{k,3\},w^{\{k,3\},z})$, where $k = 1,2$, are isomorphic to (N,v). Indeed, $w^{\{k,3\},z}(\{3\}) = \max\{d, v(\{2\}) + d - y_2\} = v(\{1\}), w^{\{k,3\},z}(\{k\}) = \max\{v(\{2\}), v(N) + v(\{2\}) - v(\{1\}) - y_2\} = v(\{2\})$, and $w^{\{k,3\},z}(\{k,3\}) = v(N)$, thus the bijection $\tau : N \to \{k,3\}$, defined by $\tau(1) = 3$ and $\tau(2) = k$, satisfies $\tau v = w^{\{k,3\},z}$. Therefore $z_S \in \sigma(S,w^{S,z})$ for every two-person sub-coalition of $\{1,2,3\}$ by AN. Applying CRGP yields $z \in \sigma(\{1,2,3\},w)$, but player 3 is *inessential*[c] of worth d, where

$$d_3^{\min}(\{1,2,3\},w) = d = d_3^{\max}(\{1,2,3\},w)\,,$$

thus $z_3 < d$ establishes a contradiction to REASB. □

[c]A player k in a game (N,v) is *inessential* of worth d, if $v(S \cup \{k\}) - v(S) = d \;\forall\; S \subseteq N \setminus \{k\}$.

Corollary 7.1. *The positive prekernel is the maximum solution on* Γ_U *that satisfies* NE, AN, REAS, WRGP *and* CRGP.

Proof. The positive prekernel satisfies the required properties by Remark 2.1, Lemma 6.1, and Lemma 6.2. Lemma 7.2 completes the proof. □

Proof of Theorem 7.1. By Remark 2.1, the positive prekernel satisfies WUTPG. Corollary 7.1 and Lemma 7.1 complete the proof. □

The core on the set of market games contained in Γ_U is the unique solution that satisfies individual rationality (IR), WRGP, CRGP and UTPG [see Peleg (1989)]. We used WRGP, CRGP and WUTPG, a property that is weaker than UTPG, in our characterisation. In some sense, REAS replaces IR. However, it is possible to replace REAS by REASB and COV in Theorem 7.1. That means that the positive prekernel can be characterised by weakening the axioms for the core and adding some "standard axioms".

Theorem 7.2. *The positive prekernel is the unique solution on* Γ_U *that satisfies* NE, AN, COV, REASB, WRGP, CRGP *and* WUTPG.

Proof. Only the uniqueness part has to be shown. In the proof of Lemma 7.2, condition (3.2) of REAS is only used once, namely in Claim 1, part (2). This case leads to a contradiction, because $z_{\{1,2\}}$ is the standard solution of the game $(\{1,2\}, w^{\{1,2\},z})$ with an empty core. This game is isomorphic to a game that is strategically equivalent to (N,v), thus WRGP, AN and COV establish a contradiction. □

8. On the Independence of the Axioms

The following examples show that the properties used in Lemma 7.2 and Theorem 7.1 are logically independent. We start showing that these results are not valid, if $|U| = 2$.

Example 8.1. If $|U| = 2$, then define

$$\sigma^0(N,v) = \{x \in \mathcal{I}^*(N,v) | x \text{ is reasonable}\} \quad \forall\ (N,v) \in \Gamma_U\,.$$

Then σ^0 satisfies NE, AN, REAS, COV, RGP, CRGP and WUTPG, but it is not a sub-solution of the positive prekernel.

From now on we assume that the universe U of players contains at least three members.

Example 8.2. The solution σ^1 on Γ_U is defined by distinguishing cases.

(1) If $N = \{i\}$, then $\sigma^1(N,v) = \mathcal{I}^*(N,v) = \{v(\{i\})\}$.

(2) If $|N| = 2, N = \{i, j\}$, then

$$\sigma^1(N,v) = \begin{cases} \mathcal{C}(N,v), & \text{if } \mathcal{C}(N,v) \neq \emptyset \\ \{x \in \mathcal{I}^*(N,v) | x_i = v(\{i\}) \text{ or } x_j = v(\{j\})\}, & \text{otherwise} \end{cases}$$

assigns to every two-person game with an empty core the extreme points of its set of reasonable preimputations.

(3) $\sigma^1(N,v) = \{x \in \mathcal{I}^*(N,v) | x_S \in \sigma^1(S, v^{S,x}) \ \forall\ S \subseteq N, |S| = 2\}$, if $|N| > 2$.

This solution satisfies AN, COV and CRGP by definition. The core is a subsolution of σ^1, thus it satisfies WUTPG. Moreover, σ^1 satisfies RGP by the *transitivity of reducing*:

$$v^{T,x} = (v^{S,x})^{T,x_S} \quad \forall\ (N,v) \in \Gamma_U, \emptyset \neq T \subseteq S \subseteq N,\ x \in X(N,v)$$

σ^1 does not satisfy NE (even if $|U| = 3$), because its application to the game $(N \cup \{*\}, w)$ of Claim 1 of the proof of Lemma 7.2 yields the empty set.

Claim. σ^1 satisfies REAS.

Assume, on the contrary, there is a game (N,v) and $x \in \sigma^1(N,v)$ that does not satisfy reasonableness. Then there is a player $i \in N$ such that (3.1) or (3.2) of REAS is violated at i.

(1) $x_i < d_i^{\min}(N,v)$.
Then $s_{ij}(x,v) > 0 \ \forall\ j \in N \setminus \{i\}$, thus $s_{ji}(x,v) = 0$ by definition of σ^1. Take $S \subseteq N \setminus \{i\}$ which is maximal (under $\subseteq$) such that $e(S,x,v) = 0$. Then $S \neq N \setminus \{i\}$. Take $T \subseteq N \setminus \{i\}$ attaining $0 = s_{ji}(x,v)$ for some $j \in N \setminus (S \cup \{i\})$. By maximality of S there exists $k \in S \setminus T$. Therefore

$$s_{kj}(x,v) \geq e(S \cup \{i\}, x, v) > 0 \text{ and } s_{jk}(x,v) \geq e(T \cup \{i\}, x, v) > 0\,,$$

a contradiction.

(2) $x_i > d_i^{\max}(N,v)$.
Then $s_{ji}(x,v) > 0\ \forall j \in N \setminus \{i\}$, thus $s_{ij}(x,v) = 0$ by definition of σ^1. Take $S \subseteq N \setminus \{i\}$ which is minimal such that $e(S \cup \{i\}, x, v) = 0$. Then $S \neq \emptyset$. Take $j \in S$ and $T \subseteq N \setminus \{i\}$ such that $s_{ij}(x,v)$ is attained by $T \cup \{i\}$. By minimality of S, we have $T \setminus S \neq \emptyset$. Let $k \in T \setminus S$. Then

$$s_{kj}(x,v) \geq e(T,x,v) > 0 \text{ and } s_{jk}(x,v) \geq e(S,x,v) > 0\,,$$

a contradiction.

Example 8.3. In order to show that AN is independent, we proceed similarly to Example 8.2 by defining σ^2. The only difference in the definition occurs for two-person games (N,v) with an empty core. Choose two different players, let us say 1 and 2, of U and define

$$\sigma^2(N,v) = \begin{cases} \{x \in \mathcal{I}^*(N,v) | x_1 = v(\{1\}) \text{ or } x_2 = v(\{2\}) \text{ or } x = x^v\}, & \text{if } N = \{1,2\} \\ x^v, & \text{otherwise} \end{cases},$$

where x^v is the standard solution of (N, v). As in the last example it is easy to verify that σ^2 satisfies NE, COV, RGP, CRGP, WUTPG. It does not satisfy AN.

Claim. σ^2 satisfies REAS.

By construction, $x \in \sigma^2(N, v)$ implies

$$(s_{kl}(x, v))_+ = (s_{lk}(x, v))_+ \quad \forall\, k, l \in N \text{ with } k \neq l,\ \{k, l\} \neq \{1, 2\}\,. \tag{8.1}$$

Therefore $x \in \mathcal{K}_+^*(N, v)$ for every N satisfying $\{1, 2\} \not\subseteq N$. Assume, on the contrary, $x \in \sigma^2(N, v)$ is not reasonable, thus $x \notin \mathcal{K}_+^*(N, v)$ and $|N| > 2$. Property (8.1) implies that (3.1) and (3.2) of REAS are satisfied for $i \in N \setminus \{1, 2\}$. Moreover, $s_{12}(x, v) > 0$ and $s_{21}(x, v) = 0$ can be assumed (otherwise exchange 1 and 2). Hence

$$x_2 > d_2^{\max}(N, v) \text{ or } x_1 < d_1^{\min}(N, v)\,.$$

Two cases may occur:

(1) $x_2 > d_2^{\max}(N, v)$.
Then

$$s_{i2}(x, v) = s_{2i}(x, v) \quad \forall\, i \in N \setminus \{1, 2\}\,. \tag{8.2}$$

If $S \subseteq N$ has maximal excess, then $S \subseteq N \setminus \{2\}$, thus $S = \{1\}$ by (8.2). This observation contradicts $|N| > 2$.

(2) $x_1 < d_1^{\min}(N, v)$.
The fact that $s_{1i}(x, v) = s_{i1}(x, v) > 0$ for $i \neq 1, 2$ implies that $N \setminus \{2\}$ is the unique coalition of maximal excess. Take $j \in N \setminus \{1, 2\}$ and observe that $s_{j2}(x, v) = s_{2j}(x, v) > 0$ yields a contradiction.

Example 8.4. The solution σ^3, defined by

$$\sigma^3(N, v) = \{x \in X(N, v) | s_{ij}(x, v) \leq (s_{ji}(x, v))_+ \quad \forall\, i, j \in N \text{ with } i \neq j\}\,,$$

satisfies AN and COV by definition and contains the positive prekernel, thus it satisfies NE and WUTPG. If $\mathcal{K}_+^*(N, v)$ is replaced by $\sigma^3(N, v)$, then the proofs of Lemmata 6.1 and 6.2 also show that σ^3 satisfies RGP and CRGP. In the proof of reasonableness from above (see Lemma 3.1) Pareto optimality is not used, thus σ^3 shows that REASB is logically independent and that REAS cannot be relaxed to reasonableness from above.

Example 8.5. The solution σ^4 is defined inductively on $|N|$. If $|N| \leq 2$, then $\sigma^4(N, v) = \mathcal{K}_+^*(N, v)$. If $|N| > 2$, then two cases are distinguished.

(1) If (N, v) does not contain inessential players, then $\sigma^4(N, v)$ is the set of all reasonable preimputations of (N, v).
(2) If $i \in N$ is an inessential player of (N, v), then

$$\sigma^4(N, v) = \{x \in \mathcal{I}^*(N, v) | x_{N \setminus \{i\}} \in \sigma^4(N \setminus \{i\}, v_{N \setminus \{i\}})\}\,.$$

An inductive argument shows that σ^4 is well-defined. In view of the fact that $\mathcal{K}_+^*$ is a sub-solution of σ^4 which coincides on games with at most two persons, σ^4 satisfies NE, WUTPG, and CRGP. COV, AN are satisfied by definition and, thus, WRGP is violated.

Example 8.6. Define
$\sigma^5(N,v) = \{x \in \mathcal{I}^*(N,v) | x_S$ is reasonable for $(S, v^{S,x})$ $\forall \emptyset \neq S \subseteq N\}$. Then σ^5 satisfies NE, AN, REAS, COV and WUTPG. Moreover, RGP is a consequence of the transitivity of reducing. Of course σ^5 violates CRGP, because it is a proper sub-solution of the positive prekernel.

Example 8.7. The prekernel shows the logical independence of WUTPG in Lemma 7.1 and Theorems 7.1 and 7.2.

Remark 8.1.

(1) Note that we do not know whether Theorem 7.2 remains valid, if COV is dropped as a condition. Moreover, we do not know, whether REAS can be replaced by REASB in Theorem 7.1 or Lemma 7.2.
(2) A careful inspection of the proofs of Sec. 7 shows that AN and COV are only applied to two-person games. Moreover, REAS(B) and NE are only applied to games of at most three persons (cp. Examples 8.2, 8.3 and 8.4).

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