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# A NOTE ON AN AXIOMATIZATION OF THE CORE OF MARKET GAMES

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As shown by Peleg (1993), the core of market games is characterized by nonemptiness, individual rationality, superadditivity, the weak reduced game property, the converse reduced game property, and weak symmetry. It was not known whether weak symmetry was logically independent. With the help of a certain transitive 4-person TU game, it is shown that weak symmetry is redundant in this result. Hence, the core on market games is axiomatized by the remaining five properties, if the universe of players contains at least four members.

In this note we solve an open problem of the theory of cooperative games that arises in a natural way in the context of the characterization of the core of market games due to Peleg (1989, 1993). Theorem 2 of Peleg (1993) shows that the core of market games is characterized by *nonemptiness* (NE), *individual rationality* (IR), *superadditivity* (SUPA), the *weak reduced game property* (WRGP), the *converse reduced game property* (CRGP), and weak symmetry (WS). (Precise definitions of properties are recalled below.) The assertion without the assumption of WS was formulated in Peleg (1989), but it turned out that WS, which is a weak variant of *anonymity*, was needed in addition in the proof of the uniqueness part. The problem whether WS is logically independent was mentioned in Peleg (1993). With the help of a "cyclic" 4-person game (its symmetry group is generated by a cyclic permutation) we show that the core of market games is characterized by NE, IR, SUPA, WRGP, and CRGP (see Theorem 3). Thus, WS is redundant.

We adopt the notation of Peleg (1989). Let $U$ be a set of players satisfying $|U| \geq 4$, let us say $M := \{1, 2, 3, 4\}$ is contained in $U$. A (cooperative TU) *game* is a pair $(N, v)$ such that $\emptyset \neq N \subseteq U$ is finite and $v$: $2^N \to \mathbb{R}$, $v(\emptyset) = 0$. For any game $(N, v)$, let

$$X^*(N, v) = \{x \in \mathbb{R}^N \mid x(N) \leq v(N)\} \text{ and } X(N, v) = \{x \in \mathbb{R}^N \mid x(N) = v(N)\}$$

denote the set of *feasible* and *Pareto optimal* feasible payoffs (*preimputations*), respectively. The *core* of $(N, v)$ is given by

$$\mathscr{C}(N, v) = \{x \in X^*(N, v) \mid x(S) \geq v(S) \ \forall S \subseteq N\}.$$

A game is *balanced* if its core is nonempty and it is *totally balanced* if every subgame is balanced. A *solution* $\sigma$ on a set $\Gamma$ of games associates with each game $(N, v) \in \Gamma$ a subset of $X^*(N, v)$. Let $\theta$ denote the set of all totally balanced games.

Let $\sigma$ be a solution on a set $\Gamma$ of games; $\sigma$ satisfies *nonemptiness* (NE), if $\sigma(N, v) \neq \emptyset$ for every $(N, v) \in \Gamma$. $\sigma$ is *covariant under strategic equivalence* (COV), if for $(N, v), (N, w) \in \Gamma$ with $w = \alpha v + \beta_*$ for some $\alpha > 0, \beta \in \mathbb{R}^N$, the equation $\sigma(N, w) = \alpha\sigma(N, v) + \beta$ holds. Here, $(N, \beta_*)$ is the *inessential (additive)* game given by $\beta_*(S) = \sum_{i \in S} \beta_i$. In this case, the games $v$ and $w$ are called *strategically equivalent*. $\sigma$ is *individually rational* (IR) if $x_i \geq v(\{i\})$ holds for every $(N, v) \in \Gamma$, for every $x \in \sigma(N, v)$, and for every $i \in N$. $\sigma$ is

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*superadditive* (SUPA) if $\sigma(N,v)+\sigma(N,w) \subseteq \sigma(N,v+w)$, when $(N,v),(N,w),(N,v+w) \in \Gamma$. Let $(N,v)$ be a game, $x \in X^*(N,v)$, and $\varnothing \neq S \subseteq N$. The *reduced game* $(S, v_{x,S})$ with respect to $S$ and $x$ is defined by $v_{x,S}(\varnothing)=0$, $v_{x,S}(S)=v(N)-x(N\setminus S)$, and $v_{x,S}(T)=\max_{Q\subseteq N\setminus S} v(T\cup Q)-x(Q)$ for every $T\neq\varnothing, S$. Note that $\sigma$ satisfies the *weak reduced game property* (WRGP) if $(S,v_{x,S})\in\Gamma$ and $x^S\in\sigma(S,v_{x,S})$ for every $(N,v)\in\Gamma$, for every $S\subseteq N$ with $1\le |S|\le 2$, and for every $x\in\sigma(N,v)$. Here, $x^S$ denotes the restriction of $x$ to $S$. Note that $\sigma$ satisfies the *converse reduced game property* (CRGP) if for every $(N,v)\in\Gamma$ with $|N|\ge 2$ the following condition is satisfied for every $x\in X(N,v)$: If, for every $S\subseteq N$ with $|S|=2$, $(S,v_{x,S})\in\Gamma$ and $x^S\in\sigma(S,v_{x,S})$, then $x\in\sigma(N,v)$.

REMARK 1. It is well known (see, e.g., Peleg 1989) that the core satisfies IR, SUPA, CRGP, and COV on every set $\Gamma$ of games. Moreover, it satisfies NE and WRGP on $\theta$.

The 4-person game $(M,u)$, defined by

$$u(S)=\begin{cases} 0, & \text{if } S\in\{M,\{1,2\},\{2,3\},\{3,4\},\{4,1\},\varnothing\},\\ -1, & \text{if } |S|=3,\\ -4, & \text{otherwise,}\end{cases} \tag{1}$$

will be used in the proof of Theorem 3. Note that the symmetry group of $(M,u)$ is generated by the cyclic permutation, which maps 1 to 2, 2 to 3, 3 to 4, and 4 to 1; thus the game is *transitive*. (A game is called *transitive* if its symmetry group is transitive.)

REMARK 2. (1) The core of $(M,u)$ is the line segment with the extreme points $(-1,1,-1,1)$ and $(1,-1,1,-1)$, i.e.,

$$\mathscr{C}(M,u)=\{(\gamma,-\gamma,\gamma,-\gamma)\in\mathbb{R}^M \mid -1\le\gamma\le 1\}.$$

Indeed, every member $x^\gamma := (\gamma,-\gamma,\gamma,-\gamma)\in\mathbb{R}^M$ with $-1\le\gamma\le 1$ of this line segment belongs to the core. On the other hand, every core element assigns zero to $M$, and thus, to the members of the partitions $\{\{1,2\},\{3,4\}\}$, and $\{\{2,3\},\{4,1\}\}$. Therefore, the core is contained in the line $\{x^\gamma \mid \gamma\in\mathbb{R}\}$. The facts $x^\gamma(\{1,2,3\}) < -1\ \forall\gamma<-1$ and $x^\gamma(\{2,3,4\}) < -1\ \forall\gamma>1$ show that the core has the claimed shape.

(2) Let $x=x^\gamma\in\mathscr{C}(M,u)$. Then the reduced coalitional function $w:=u_{x,\{1,2\}}$ is given by

$$\begin{aligned} w(\{1\}) &= \max_{Q\subseteq\{3,4\}} u(\{1\}\cup Q)-x(Q)=u(\{4,1\})-x_4=\gamma,\\ w(\{2\}) &= \max_{Q\subseteq\{3,4\}} u(\{2\}\cup Q)-x(Q)=u(\{2,3\})-x_3=-\gamma,\\ w(\{1,2\}) &= u(M)-x(\{3,4\})=0, \quad \text{and} \quad w(\varnothing)=0.\end{aligned}$$

Thus $(\{1,2\},w)$ is an *additive* game. Similarly, it can be shown that all 2-person reduced games are additive. Note that the restriction of $x$ to any 2-person coalition is the unique member of the core of the corresponding additive 2-person reduced game.

(3) The restrictions of the vectors $x^\gamma$ for $\gamma=1$ and $\gamma=-1$, respectively, to the coalitions $\{1,2,4\}$, $\{2,3,4\}$, and $\{1,2,3\}$, $\{1,3,4\}$, respectively, show that the 3-person subgames are balanced. All 1- and 2-person subgames are balanced as well. We conclude that $(M,u)$ is totally balanced.

THEOREM 3. *There is a unique solution on $\theta$ that satisfies NE, IR, SUPA, WRGP, and CRGP, and it is the core.*

REMARK 4. We now describe the basic idea of the following proof of Theorem 3. By a careful inspection of Peleg (1989), it remains to be shown that $\mathscr{C}(N,v)\subseteq\sigma(N,v)$ for every balanced 2-person game $(N,v)$ that is not additive and for every solution $\sigma$ satisfying the above axioms. We shall show that there exists a totally balanced 4-person game, derived from $(M,u)$, with a player set containing $N$ such that the following property is satisfied:

For every $x \in \mathscr{C}(N, v)$, there exists a member of the solution of this 4-person game such that (a) the reduced game with respect to $N$ coincides with $(N, v)$ and (b) the restriction of this member to $N$ coincides with $x$. In this step of the proof CRGP is applied to two games and SUPA is applied once.

PROOF. By Remark 1, the core satisfies the desired properties. Let $\sigma$ be a solution that satisfies the desired properties. By Lemma 4.2 of Peleg (1989), the solution is a subsolution of the core. Let $(N, v) \in \theta$, $x \in \mathscr{C}(N, v)$ and put $n := |N|$. It remains to show Lemma 4.10 of Peleg (1989), i.e.,

$$\mathscr{C}(N, v) \subseteq \sigma(N, v). \tag{2}$$

The cases $n = 1$ and $n \geq 3$ of the proof of Lemma 4.10 can be copied: If $n = 1$, then $x \in \sigma(N, v)$ by NE and IR. If $n \geq 3$ and (2) is shown for 2-person games, then $x^S \in \mathscr{C}(S, v_{x,S})$ and $(S, v_{x,S}) \in \theta$ for every $S \subseteq N$ with $|S| = 2$, because the core satisfies WRGP. Hence, $x^S \in \sigma(S, v_{x,S})$ for every $S \subseteq N$ with $|S| = 2$. By CRGP, $x \in \sigma(N, v)$. It remains to prove (2) for $n = 2$. Without loss of generality, we may assume $N = \{1, 2\}$. Two cases can be distinguished.

(1) If $(N, v)$ is additive, then the proof is finished by NE and IR (see Corollary 4.5 of Peleg 1989).

(2) If $(N, v) \in \theta$ is not additive, then $v(N) > v(\{1\}) + v(\{2\})$. Define $(N, w)$ by

$$w(S) = \begin{cases} -2, & \text{if } |S| = 1, \\ 0, & \text{if } S = \varnothing, N, \end{cases}$$

and observe that $v = \alpha w + \beta_*$ where $\alpha = \frac{v(N) - v(\{1\}) - v(\{2\})}{4}$ and

$$\beta = \left( \frac{v(N) + v(\{1\}) - v(\{2\})}{2}, \frac{v(N) + v(\{2\}) - v(\{1\})}{2} \right) \in \mathbb{R}^N.$$

As $\alpha > 0$, $(N, v)$ is strategically equivalent to $(N, w)$. Put $y := \frac{x - \beta}{\alpha}$ and observe that $y \in \mathscr{C}(N, w)$ by COV of the core. We first prove that $y \in \sigma(N, w)$. With $\gamma := y_1/2$, the vector $y$ can be expressed as $y = (2\gamma, -2\gamma)$. Also, $-1 \leq \gamma \leq 1$, because $y \in \mathscr{C}(N, w)$. Choose two members of $U \setminus N$, let us say 3, 4, which is possible by the assumption $|U| \geq 4$, let $(M, u)$ be defined by (1), let $\pi$ be the permutation of $M$ that exchanges 1 and 2, and let $(M, \pi u)$ be the "permuted" game (given by $\pi u(S) = u(\pi^{-1}(S))$ for all $S \subseteq M$). By Remark 2(3), $(M, u) \in \theta$, and similarly, $(M, \pi u) \in \theta$. By Remark 2(1), $x^\gamma, x^{-\gamma} \in \mathscr{C}(M, u)$, and similarly, $\pi x^{-\gamma} = (\gamma, -\gamma, -\gamma, \gamma) \in \mathscr{C}(M, \pi u)$. An application of CRGP shows that $x^\gamma \in \sigma(M, u)$ and $\pi x^{-\gamma} \in \sigma(M, \pi u)$ by Remark 2(2) and Part (1) of this proof. SUPA implies $z := (y, 0, 0) = x^\gamma + \pi x^{-\gamma} \in \sigma(M, u + \pi u)$. WRGP yields $y \in \sigma(N, (u + \pi u)_{z,N})$. The reduced coalition function $\hat{u} = (u + \pi u)_{z,N}$ coincides with $w$. Indeed, $\hat{u}(N) = (u + \pi u)(M) - z(\{3, 4\}) = 0 = w(N)$ by definition of the reduced game. Moreover, the unique 2-person coalitions $S$ with $u(S) = (\pi u)(S) = 0$ are $\{1, 2\}$ and $\{3, 4\}$, thus $(u + \pi u)(\{i, j\}) = -4$ for $i \in N$ and $j \in M \setminus N$. Hence,

$$\hat{u}(\{i\}) = (u + \pi u)(\{i, 3, 4\}) - z(\{3, 4\}) = -2 = w(\{i\}) \quad \text{for } i = 1, 2,$$

thus $y = z^N \in \sigma(N, w)$.

In order to show $x \in \sigma(N, v)$, we can proceed similarly. Only the coalition functions $u$ and $\pi u$ are replaced by $\alpha u + (\frac{1}{2}(\beta, 0, 0)_*$ and $\alpha(\pi u) + (\frac{1}{2}(\beta, 0, 0))_*$. The proof is finished by COV of the core. $\square$

Examples 5.1, 5.3, 5.4, 5.5 of Peleg (1989) show that each of the axioms NE, SUPA, WRGP, CRGP is logically independent of the remaining axioms. Example 5.2 contains a misprint. The solution, which differs from the core only inasmuch as it assigns to any 1-person game the set of all feasible payoffs, satisfies all axioms of Theorem 3, except IR.

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